A connection between Abel and ${}_pF_q$ hypergeometric differential equations

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Abstract

In a recent paper, a new 3-parameter class of Abel type equations, so-called AIR, all of whose members can be mapped into Riccati equations, is shown. Most of the Abel equations with solution presented in the literature belong to the AIR class. Three canonical forms were shown to generate this class, according to the roots of a cubic. In this paper, a connection between those canonical forms and the differential equations for the hypergeometric functions ${}_2F_1$, ${}_1F_1$ and ${}_0F_1$ is unveiled. This connection provides a closed form ${}_pF_q$ solution for all Abel equations of the AIR class.

Introduction

Abel-type ordinary differential equations (ODE), in either their second kind form [1],

$$y' = \frac{f_3\, y^3 + f_2\, y^2 + f_1\, y + f_0}{g_1\, y + g_0}, \tag{1}$$

where the $\{f_i,\ g_i\}$ are arbitrary functions of x, or their first kind form obtained taking $\{g_1 = 0,\, g_0 = 1\}$, appear frequently in physical applications [1, 2, 3]. This has for a long time motivated their study.

After pioneering work on their solutions by Abel [4], the basis of today's solving approach for these equations was set by Liouville [5, 6] and Appel [7], using classical invariant theory. In brief, through transformations of the form

$$\{x = F(t), \quad y = \frac{P_1(t)\, u + Q_1(t)}{P_2(t)\, u + Q_2(t)}\}, \tag{2}$$

where $\{F,\, P_1,\, P_2,\, Q_1,\, Q_2\}$ are arbitrary analytic functions restricted only by $P_1\, Q_2 - P_2\, Q_1 \neq 0$, one can define an *Abel class* of equations, all of whose members can be mapped between themselves by means of (2). Then, a given Abel equation can be tackled by formulating an equivalence problem between itself and a representative of that class, for instance one whose solution is known.

A key ingredient in such an approach is the number of Abel classes that possess a member - herein called "solvable equation" - whose solution is known. Through the class transformations (2), solvable equations generate "solvable classes". In a recent work [8] (2000), it has been shown that a large number of solvable Abel equations scattered in the literature, including those presented by Abel, Liouville and Appel, are all members of one or another of only four 1-parameter and seven 0-parameter Abel solvable classes. In [9] (2003), three new rather general Abel classes were presented, so-called AIL, AIR and AIA, respectively depending on 2, 3 and 4 parameters, and those solvable classes collected in [8] were in turn all shown to be particular cases of just these three. Apart from the generalizing aspect of these new multi-parameter classes, an important feature of AIL and AIR is that all their members can respectively be mapped into first order linear and Riccati type equations; that is, they are "solvable" (AIL) or linearizable (AIR).

The presentation of these multi-parameter classes in [9], however, didn't include a closed form solution for any member of the AIR class. A solution was not known at that time. The authors acknowledged this in [9] and instead presented a partial classification of the members of AIR as generated from three canonical forms, related to the roots of a cubic. To know the closed form solutions for these canonical forms, however, is important, because Riccati equations are not "solvable" in general; only with these solutions at hands can we think of the multi-parameter AIR class as solvable.

In this paper, it is shown that a surprisingly simple connection exists between the three AIR canonical forms and the differential equations satisfied by the hypergeometric functions ${}_2\mathrm{F}_1$, 1F1 and 0F1. This connection provides ${}_p\mathrm{F}_q$ closed form solutions for all members of the AIR class, and permits formulating an alternative approach for resolving the membership problem.

1 The Abel Inverse Riccati - AIR - class

Recalling the material presented in [9], a multi-parameter class of Abel equations, all of whose members can be transformed into Riccati equations, can be obtained departing from the general form of a Riccati equation,

$$y' = f(x)\, y^2 + g(x)\, y + h(x)\,, \tag{3}$$

and applying to it the *inverse* transformation $\{x \leftrightarrow y\}$[1], resulting in

$$y' = \frac{1}{f(y)\, x^2 + g(y)\, x + h(y)} \tag{4}$$

When all of f, h and g are of the form

$$y \rightarrow \frac{s\, y + r}{a_0 + a_1\, y + a_2\, y^2 + a_3\, y^3} \tag{5}$$

where $\{a_i,\, s,\, r\}$ are constants with respect to x, (4) has the form

$$y' = \frac{a_3\, y^3 + a_2\, y^2 + a_1\, y + a_0\, y}{(s_0 + s_1\, x + s_2\, x^2)\, y + r_0 + r_1\, x + r_2\, x^2} \tag{6}$$

for some new constants $\{a_i,\, s_i,\, r_i\}$. This equation is of Abel 2nd kind type, can be transformed into a Riccati equation by means of $\{x \leftrightarrow y\}$, and is a representative of the most general Abel ODE class - so-called Abel Inverse Riccati (AIR) - all of whose members are linearizable [9]. In [9] it is shown that the 1-parameter classes presented by Abel [4], Liouville [6] and Appell [7], as well as most of the solvable examples found in the literature[2], are all members of this AIR class, generated from (6) by applying the transformation (2).

1.1 Six-parameter AIR canonical forms

An important property of (6) is that its rational structure with respect to x and y is invariant under Möbius (linear fractional) changes of x and y. The problem of solving an arbitrary Abel equation can then be formulated in two steps: bring the given equation to the AIR form (6) using the class transformations (2), then use Möbius transformations to reduce the AIR form to a canonical form we expect to be able to solve.

With that in mind, in [9], Möbius transformations of y were used to transform (6) into three different canonical forms, and in doing so, the number of parameters entering (6) was reduced from ten to six. No solutions to these canonical forms were known when [9] was written.

[1] By $\{x \leftrightarrow y\}$ we mean changing variables $\{x = u(t),\, y(x) = t\}$ followed by renaming $\{u \rightarrow y,\, t \rightarrow x\}$.

[2] For a collection of these see [8].

To construct these canonical forms of AIR with six parameters, following [9], the numerator of (6) is written in terms of its roots ρ_i

$$y' = \frac{(y-\rho_0)(y-\rho_1)(y-\rho_2)}{(a_2\,x^2+a_1\,x+a_0)\,y+b_2\,x^2+b_1\,x+b_0} \tag{7}$$

Then, using Möbius transformations for the dependent variable

$$y \rightarrow \frac{p+q\,y}{r+s\,y} \tag{8}$$

where $\{p,q,r,s\}$ are any constants satisfying $p\,s-r\,q \neq 0$, (7) can be transformed into

$$y' = \frac{P\,(y)}{(a_2\,x^2+a_1\,x+a_0)\,y+b_2\,x^2+b_1\,x+b_0} \tag{9}$$

for some new constants $\{a_i\,b_i\}$, with $P(y)=1$, $P(y)=y$ or $P(y)=y\,(y-1)$, respectively according to whether in (7) there are only one, two or three distinct roots ρ_i. These three canonical forms are shown in [9] depending on six parameters, as in (9).

1.2 Three-parameter AIR canonical forms

Apart from the Möbius transformations of y used in [9], Möbius transformations of the independent variable,

$$x \rightarrow \frac{p+q\,x}{r+s\,x}, \tag{10}$$

also leave the structure of (6) invariant, and so can be used to further transform the three canonical forms represented by (9). In doing so, following [10], the number of parameters can be reduced from six to three. For that purpose, in (9), if any of $\{a_1,a_2\}$ are different from zero, a transformation $x \rightarrow 1/x+\kappa$, with κ being any non-zero root of $a_2\,\kappa^2-a_1\,\kappa+a_0=0$, will cancel the $a_2\,x^2$ term in the denominator, taking the equation to a form with only five parameters[3]. At this point, two different cases can happen:

<u>Case A:</u> $a_1=0$, so applying the transformation $x \rightarrow 1/x+\kappa$ just mentioned, (9) will be of the form

$$y' = \frac{P\,(y)}{a_0\,y+b_2\,x^2+b_1\,x+b_0} \tag{11}$$

for some new constants $\{a_0,\,b_i\}$.

<u>Case B:</u> $a_0=0$, hence applying that same transformation $x \rightarrow 1/x+\kappa$, (9) will be of the form

$$y' = \frac{P\,(y)}{a_1\,x\,y+b_2\,x^2+b_1\,x+b_0} \tag{12}$$

for some new constants $\{a_i\,b_i\}$. When both $a_1 \neq 0$ and $a_0 \neq 0$, a transformation of the form $x \rightarrow x+a_0/a_1$ will transform (9) into an equation of the form (12).

Summarizing, in all cases (9) can be transformed into an equation of the form (11) or (12), with only four parameters.

Finally, scaling $x \rightarrow \kappa\,x$ with $\kappa^2=1/b_2$ maps both cases (11) and (12) into equations depending on only three parameters[4], of the form

$$y' = \frac{P(y)}{\tilde{a}\,y+(x-b)(x-c)} \tag{13}$$

where b and c are constants, and in case A: $\tilde{a}$ is a constant, while in case B: $\tilde{a}=a\,x$, and a is a constant.

[3] If in (9) $a_1=0$ and $a_2=0$, then this equation already depends on only four parameters.

[4] If $b_2=0$, (11) and (12) depend on only three parameters; applying $\{x \leftrightarrow y\}$ one directly obtains a first order linear ODE.

1.3 Three, two, one and zero-parameter AIR canonical forms

Following [9] and [10] we have arrived at the three canonical forms of AIR implicit in (13), according to whether $P(y) = y\,(y-1)$, $P(y) = y$ or $P(y) = 1$. Each of these canonical forms splits into two cases, according to whether or not $\tilde{a}$ is constant, and each of these six equations seem to depend on three parameters $\{a, b, c\}$. Although that is the case when $P(y) = y\,(y-1)$ and $\tilde{a}$ is non-constant, we will show below that the cases when $\tilde{a}$ is constant or $P(y) = y$, and the case $P(y) = 1$, respectively depend on only two and one parameters at most.

Establishing this fact is of relevance since the classification of second order ${}_p\mathrm{F}_q$ hypergeometric equations also splits into three canonical forms, which are the equations satisfied by the ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$ and ${}_0\mathrm{F}_1$ functions, and these equations respectively depend on three, two and one parameters. After the number of parameters of the AIR canonical forms is precisely determined, one can see that these classifications of Abel and ${}_p\mathrm{F}_q$ equations are connected; this connection is discussed in the next section.

The parameters when the three roots ρ_i in (7) are different

Considering first the case where, in (13), $P(y) = y\,(y-1)$, if $\tilde{a} \equiv a\,x$, (13) depends on three parameters. If $\tilde{a} \equiv a$ is a constant, however, a translation $x \rightarrow x + b$ followed by renaming $c \rightarrow c + b$ transforms (13) into the representative of a 2-parameter class

$$y' = \frac{y\,(y-1)}{a\,y + x\,(x-c)} \tag{14}$$

The parameters when only two roots ρ_i in (7) are different

This case corresponds to (13) at $P(y) = y$. By scaling $y \rightarrow \kappa\,y$, the equation transforms into

$$y' = \frac{y}{\tilde{a}\,\kappa\,y + (x-b)(x-c)} \tag{15}$$

Choosing $\kappa = 1/a$, the factor $\tilde{a}\,\kappa$ is either equal to 1 or to x and in this way equation (13) is reduced to an equation representative of a 2-parameter class. When, in (15), $\tilde{a}\,\kappa = 1$, a translation $x \rightarrow x + c$, followed by $c \rightarrow c + b$, transforms the equation into

$$y' = \frac{y}{y + x\,(x-c)}, \tag{16}$$

This is a canonical form representative of a 1-parameter class, which happens to be equivalent to the 1-parameter Abel class presented by Liouville in [6], shown in [8] as equation number 37.

The parameters when the three roots ρ_i in (7) are equal

This is equation (13) at $P(y) = 1$; starting with the case of non-constant $\tilde{a} \equiv a\,x$,

$$y' = \frac{1}{a\,x\,y + (x-b)(x-c)},$$

using $y \rightarrow y/a + (b+c)/a$, this equation is transformed into

$$y' = \frac{a}{x\,y + x^2 + b\,c}$$

Scaling the variables $\{x \rightarrow x\sqrt{a},\, y \rightarrow y\sqrt{a}\}$ further transforms the equation into

$$y' = \frac{a}{a\,x\,y + a\,x^2 + b\,c} \tag{17}$$

If $c \neq 0$, redefining $b \rightarrow a\,b/c$, (17) is reduced to the representative of a 1-parameter class

$$y' = \frac{1}{x\,y + x^2 + b} \tag{18}$$

This equation happens to be equivalent to the 1-parameter Abel class presented by Appel in [7] and shown in [8] as equation number 58. If, in (17), $c = 0$, then a is a factor which cancels, leading to the representative of a 0-parameter class,

$$y' = \frac{1}{y + x^2}, \tag{19}$$

equivalent to one presented by Liouville in [6], shown in [8] as equation number 35.

The last case to consider occurs when, in (13) at $P(y) = 1$, $\tilde{a} \equiv a$ is a constant:

$$y' = \frac{1}{a\,y + (x-b)(x-c)} \tag{20}$$

If $b + c \neq 0$, using $\{x \rightarrow -(b+c)\,x,\; y \rightarrow -y/(b+c) - b\,c/a\}$ followed by redefining $a \rightarrow -a\,(b+c)^3$, this equation transforms into

$$y' = \frac{1}{a\,y + x^2 + x}$$

A further Möbius transformation $\{x \rightarrow x\sqrt[3]{a} - 1/2,\; y \rightarrow 1/(4\,a) + y/\sqrt[3]{a}\}$ takes this equation into (19). If, in (20), $b + c = 0$, using $\{x \rightarrow a^{1/2}\,x,\; y \rightarrow y/a^{1/2} + b^2/a\}$ followed by redefining $a \rightarrow 1/a^2$, equation (20) is transformed into

$$y' = \frac{1}{a\,y + x^2}$$

A further scaling $\{x \rightarrow t\sqrt[3]{a},\; y \rightarrow y/\sqrt[3]{a}\}$ takes this equation also into (19).

1.4 Classification summary

Using Möbius transformations of y and x, (6) is reduced to:

1. The representative of a 3-parameter AIR subclass, equation (13) at $\{P(y) = y\,(y-1),\; \tilde{a} \equiv a\,x\}$. This case occurs when the three roots ρ_i entering (7) are different.

2. The representative of a 2-parameter AIR subclass, equation (14), occurring when the three roots ρ_i entering (7) are different and in (13) $\tilde{a} \equiv a$ is constant.

3. The representative of a 2-parameter AIR subclass, equation (15) at $\kappa = 1/a$, corresponding to the case where only two of the roots ρ_i entering (7) are different and in (13) $\tilde{a} \equiv a\,x$.

4. The representative of a 1-parameter AIR subclass, equation (16), corresponding to the case where only two of the roots ρ_i entering (7) are different and in (13) $\tilde{a} \equiv a$ is constant.

5. The representative of a 1-parameter AIR subclass, equation (18), corresponding to the case where the three roots ρ_i entering (7) are equal and in (13) $\tilde{a} \equiv a\,x$.

6. The representative of a 0-parameter AIR subclasse, equation (19), happening when the three roots ρ_i entering (7) are equal and in (13) $\tilde{a} \equiv a$ is constant.

Möbius transformations map members of each of these six classes into members of the same class, and the class representatives depending on one or zero parameters (cases 4, 5 and 6) are equivalent to equations whose solutions were presented by Liouville [6] and Appel [7].

2 The connection with ${}_p\mathrm{F}_q$ second order hypergeometric equations

It is remarkable that the classification of the AIR class can be done in terms of the multiplicity of the roots of a cubic and resulting in canonical forms depending on three or less class parameters. A similar classification is used for second order linear equations admitting ${}_p\mathrm{F}_q$ hypergeometric solutions [11]. In that case, equations with three regular singular points admit ${}_2\mathrm{F}_1$ hypergeometric solutions, depending on three class parameters, and equations with one regular and one irregular singular points admit ${}_1\mathrm{F}_1$ hypergeometric solutions, depending on two class parameters. A special case of the latter happens when the parameters entering ${}_1\mathrm{F}_1(a;\, b;\, x)$ are related by $a = b/2$, in which case ${}_1\mathrm{F}_1$ can be re-expressed in terms of the ${}_0\mathrm{F}_1$ function through a quadratic transformation. That case represents a third ${}_p\mathrm{F}_q$ class depending on only one parameter.

A connection between this ${}_p\mathrm{F}_q$ classification and that for the Abel AIR equation summarized in sec. 1.4 exists. This connection provides ${}_2\mathrm{F}_1$ solutions when the three roots ρ_i of the cubic in (7) are different (cases 1. and 2. of sec. 1.4), ${}_1\mathrm{F}_1$ solutions when only two roots ρ_i in (7) are different (cases 3. and 4. in sec. 1.4), and ${}_1\mathrm{F}_1$ or ${}_0\mathrm{F}_1$ solutions when the three roots ρ_i in (7) are equal (case 5 and 6 in sec. 1.4). The formulas relating these canonical forms of AIR to the three ${}_p\mathrm{F}_q$ equations, and so relating the corresponding solutions, can be derived as follows.

2.1 ${}_2\mathrm{F}_1$ solutions for the AIR equation (7) having three different roots ρ_i

The Gauss or ${}_2\mathrm{F}_1$ hypergeometric linear equation,

$$y'' = \frac{(\alpha+\beta+1)\,x - \gamma}{x\,(1-x)}\,y' + \frac{\alpha\,\beta}{x\,(1-x)}\,y, \tag{21}$$

has for solution

$$y = {}_2\mathrm{F}_1(\alpha, \beta;\, \gamma;\, x)\,C_1 + x^{1-\gamma}{}_2\mathrm{F}_1(\alpha-\gamma+1, \beta-\gamma+1;\, 2-\gamma;\, x)\,C_2 \tag{22}$$

where C_1 and C_2 are arbitrary constants. As with all second order linear equations [1], using the change of variables $y \to \exp(-\int y/x\,dx)$, (21) can be transformed into a Riccati type equation

$$y' = \frac{y^2}{x} + \frac{((\alpha+\beta)\,x - \gamma + 1)}{x\,(1-x)}\,y - \frac{\alpha\,\beta}{1-x} \tag{23}$$

Applying the $\{x \leftrightarrow y\}$ transformation, the following Abel equation results

$$y' = \frac{y\,(y-1)}{(x^2 - (\alpha+\beta)\,x + \alpha\,\beta)\,y - x^2 + (\gamma-1)\,x} \tag{24}$$

This equation is of the canonical form (9), with $P(y) = y\,(y-1)$, so it corresponds to the case where the three roots ρ_i in (7) are different. Using Möbius transformations of x, equations of this form can be transformed into the 3-parameter canonical form (13) of AIR as shown in sec. 1.2. The cases $\tilde{a} \equiv a\,x$ and $\tilde{a} \equiv a$ of (13) are both included in (24), the latter corresponding to $\alpha + \beta = 0$, and so resulting in an equation depending only on two parameters. These are the cases 1. and 2. of the classification section 1.4.

By construction, the solution to (24) is obtained changing $\{x \leftrightarrow y\}$ in the solution of (23), which, in turn, is equal to $-y'\,x/y$ with y given in (22).

2.2 ${}_1\mathrm{F}_1$ solutions for the AIR equation (7) having two different roots ρ_i

The confluent ${}_1\mathrm{F}_1$ hypergeometric equation,

$$y'' = \frac{(x-\beta)}{x}\,y' + \frac{\alpha}{x}\,y, \tag{25}$$

has for solution

$$y = C_1\, \mathrm{M}(\alpha, \beta, x) + C_2\, \mathrm{U}(\alpha, \beta, x) \tag{26}$$

where C_1 and C_2 are arbitrary constants and M and U are the Kummer functions[5] [12]. Using the same two changes of variables which transform the ${}_2\mathrm{F}_1$ equation (21) into the Abel form (24), equation (25) is transformed into the Abel equation

$$y' = \frac{y}{(x-\alpha)\, y + x\,(x-\beta+1)} \tag{27}$$

which is of the canonical form (9), with $P(y) = y$, and so it corresponds to the case where two of the three roots ρ_i in (7) are equal. As shown in the previous section, equations of this form can be reduced to the 2-parameter canonical form (15) of AIR. Concretely, changing $x \to x+\alpha$ followed by renaming $\beta \to \beta+\alpha+1$ and $\alpha \to -\alpha$, (27) transforms into

$$y' = \frac{y}{x\, y + (x-\alpha)\,(x-\beta)} \tag{28}$$

This equation has the form (15) for $\tilde{a}\,\kappa = x$, hence this is case 3 of the classification section 1.4. A solution to this canonical form (28),

$$C_1 + \frac{x\, \mathrm{M}(-\beta, 1+\alpha-\beta, y) - \beta\, \mathrm{M}(1-\beta, 1+\alpha-\beta, y)}{x\, \mathrm{U}(-\beta, 1+\alpha-\beta, y) + \alpha\, \beta\, \mathrm{U}(1-\beta, 1+\alpha-\beta, y)} = 0, \tag{29}$$

is obtained applying to the solution of (27) the same transformations used to map (27) into (28). A solution for (27) is obtained from the solution (26) as explained in the ${}_2\mathrm{F}_1$ case

If, instead of departing from the general form (25) for ${}_1\mathrm{F}_1$, one departs from the particular ${}_1\mathrm{F}_1$ equation admitting ${}_0\mathrm{F}_1$ solutions, that is,

$$y'' = -\frac{c}{x}\, y' + \frac{y}{x}, \tag{30}$$

and applies the same two changes of variables used to transform the ${}_2\mathrm{F}_1$ and ${}_1\mathrm{F}_1$ equations into Abel forms, followed by changing $y \to -y$ and renaming $c \to c+1$, the resulting Abel equation is identical to the AIR canonical form (16). So the ${}_0\mathrm{F}_1$ equation can be associated to case 4 of sec. 1.4. The solution to (16) can then be expressed directly in terms of the Bessel functions of the first and second kind, $\mathrm{J_c}(x)$ and $\mathrm{Y_c}(x)$, as

$$C_1 - \frac{x\, \mathrm{J_c}\left(2\sqrt{y}\right) - \mathrm{J_{c+1}}\left(2\sqrt{y}\right)\sqrt{y}}{-x\, \mathrm{Y_c}\left(2\sqrt{y}\right) + \mathrm{Y_{c+1}}\left(2\sqrt{y}\right)\sqrt{y}} = 0 \tag{31}$$

2.3 ${}_1\mathrm{F}_1$ and ${}_0\mathrm{F}_1$ solutions for the AIR equation (7) having three equal roots ρ_i

Equation (17), presented as case 5 in sec. 1.4, is the 1-parameter canonical form of AIR equivalent to the one presented by Appell in [7]. Its connection with the ${}_1\mathrm{F}_1$ equation can be derived as follows. Applying $\{x \leftrightarrow y\}$ to the AIR form (17), in order to obtain its Riccati form, then changing $y \to -y'/y$, we obtain the second order linear form

$$y'' = x\, y' - b\, y \tag{32}$$

This equation, in turn, can be obtained from the ${}_1\mathrm{F}_1$ equation by changing $\{x \to x^2/2,\ y \to y/x\}$ in (25) and evaluating its parameters at $\{\alpha = 1/2 - b/2,\ \beta = 3/2\}$. So applying the same transformation to the solution (26) of the ${}_1\mathrm{F}_1$ equation we obtain the solution to (32); applying to this result the inverse of the transformations used to map (17) into (32), we obtain the solution of the AIR form (17) as

$$C_1 + \frac{2\,(1-b)\mathrm{M}((3-b)/2,\ 3/2,\ y^2/2) + 2\,\mathrm{M}((1-b)/2,\ 3/2,\ y^2/2)\,(b + x\, y)}{b\,(b-1)\mathrm{U}((3-b)/2,\ 3/2,\ y^2/2) + 2\,\mathrm{U}((1-b)/2,\ 3/2,\ y^2/2)\,(b + x\, y)} \tag{33}$$

[5] $\mathrm{M}(\alpha, \beta, x) = {}_1\mathrm{F}_1(\alpha;\, \beta;\, x)$.

Finally, for the 0-parameter AIR canonical form (19), that is, case 6 of the summary in sec. 1.4, a solution can be obtained as in the previous case, applying to (19) $\{x \leftrightarrow y\}$, followed by $y \rightarrow -y'/y$, resulting in its second order linear form. Resolving an equivalence between this linear form and the ${}_0\mathrm{F}_1$ equation (30), then reversing the transformations used, a solution for (19), in terms of the Airy functions[6] $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$, is

$$C_1 + \frac{x\,\mathrm{Bi}(-y) - \mathrm{Bi}'(-y)}{x\,\mathrm{Ai}(-y) - \mathrm{Ai}'(-y1)} = 0 \tag{34}$$

3 Conclusion

This paper presented a complete classification of the Abel Inverse Riccati (AIR) class of equations, which is the most general Abel class all of whose members are linearizable [9]. With this classification at hands, a direct relation between the canonical forms of AIR presented in sec. 1 and the second order linear equations for the hypergeometric functions ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$ and ${}_0\mathrm{F}_1$ was established in sec. 2.

The first important consequence of this connection is that it makes the whole AIR class of Abel equations solvable: through the class transformations (2), the connections to ${}_p\mathrm{F}_q$ hypergeometric equations provide a ${}_p\mathrm{F}_q$ closed form solution to any Abel equation member of the AIR class, as shown in sec. 2.

This connection also permits tackling the membership problem with respect to AIR using a different and simpler approach than the traditional one. Let us recall that the traditional approach consists of formulating an equivalence between a given Abel equation and each of the canonical forms of AIR. When the equivalence is possible, this approach also requires computing the values of the class parameters for which the equivalence exists. Even with the powerful computers currently available and using the most modern symbolic algebra packages, such an approach is unrealistic: when the number of class parameters is greater than one, the computation involves composed multivariable resultants, resulting in untractable expression swell [8].

An alternative approach, exploring the results of sec. 1, consists of splitting the equivalence process into two steps. In the first step, one attempts an equivalence to the AIR "form" (6) (symmetry and integrating factor techniques can be of use for this purpose), not requiring the computation of the value of the class parameters. In a second step, one formulates the tractable and relatively easy problem of an equivalence under Möbius transformations of x and y, between the AIR form (6) obtained for the given equation and each of the canonical forms of AIR summarized in sec. 1.4. This second step leads to the values of the class parameters resolving the equivalence and in that way to a solution for the problem. An implementation of these ideas using symbolic algebra software is currently under development.

Finally, the connection with ${}_p\mathrm{F}_q$ linear equations shown in sec. 2 indicates that other connections between ${}_p\mathrm{F}_q$, Elliptic and Heun type functions is possible; work on this topic is in progress.

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[6] $\mathrm{Ai}(z) = \frac{{}_0\mathrm{F}_1(\ ;2/3;1/9\,z^3)\ \sqrt[3]{3}}{3\,\Gamma(2/3)} - \frac{z\ {}_0\mathrm{F}_1(\ ;4/3;1/9\,z^3)\,\Gamma(2/3)\ \sqrt[6]{3}}{2\,\pi}$

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