A Generalization of the Differential Gosper's Algorithm

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ABSTRACT
Consider the linear ordinary differential equation

$$\sum_{k=0}^{d} p_k(x)\,\partial_x^k z(x) = t(x) \tag{1}$$

where $p_k(x)$ are polynomials for $0 \le k \le d$, $p_d \ne 0$, and $t(x)$ is an exponential function. Under a constraint on p_d, we present an efficient method for computing exponential solutions of (1). The method is based on differential Gosper forms of rational functions and their properties.

This work is the differential analogue of the work by Petkovšek [10].

1. PRELIMINARIES
Let $\mathbb{F}$ be a field of characteristic 0, and ∂_x denote the usual derivation w.r.t. x. A nonzero function $T(x)$ is exponential over $\mathbb{F}$ if there are coprime nonzero polynomials $p, q \in \mathbb{F}[x]$, q monic, such that $q(x)\,\partial_x T(x) = p(x)\,T(x)$. The rational function $\partial_x T(x)/T(x) = p/q$ is unique, and is called the *certificate* of T.

A representation of a rational function $R \in \mathbb{F}(x)$ in the form

$$R(x) = K(x) + \frac{\partial_x S(x)}{S(x)} \tag{2}$$

where $K, S \in \mathbb{F}(x)$ satisfy certain conditions provides a key step in a number of algorithms related to exponential functions [4, 7]. In this paper, the use of a particular canonical form of the form (2), known as the differential Gosper-Petkovšek form (or the GP'-form), leads to an efficient method for finding exponential solutions $z(x)$ of (1).

Let $r \in \mathbb{F}(x)$ be the certificate of the exponential function $t(x)$ in (1). The general approach is to write the GP'-form of r in two different forms. The uniqueness of GP'-form then allows one to equate the components of these two forms, and the problem of finding exponential solutions of (1) is reduced to finding *polynomial* solutions of a linear ordinary differential equation of order d (which is the order of the differential equation (1)), and with polynomial coefficients. This provides an efficiency improvement over the known method, called the *rational* method, where the problem is reduced to finding *rational* solutions of a linear ordinary differential equation also of order d and with polynomial coefficients.

Note that this general approach is used by Petkovšek [10], by Paule and Strehl [9] in deriving the Gosper's ansatz (shift case). It is also used by Petkovšek [10] in the algorithm which finds hypergeometric solutions z_n of the recurrence $\sum_{k=0}^{d} p_k(n) z_{n+k} = t_n$ where t_n is a nonzero sequence, and $p_k \in \mathbb{F}[n]$, for $0 \le k \le d$, with the *additional restriction* that p_0 and p_d are constant. In this respect, our work can be considered as the differential analogue of Petkovšek's work.

The structure of the paper is as follows. In Section 2, we give an overview of differential Gosper forms (or G'-forms) and of the GP'-form of a rational function, describe their construction, and state an important common property of G'-forms. In Section 3, under a restriction on the polynomial $p_d(x)$ in (1), we show a reduction of finding exponential solutions $z(x)$ of (1) to the problem of finding polynomial solutions of a linear differential equation with polynomial coefficients. In Section 4, we compare our method with the rational method. A Maple implementation, and experiments on the two methods are presented in Section 5.

Throughout the paper, $\mathbb{F}$ is a field of characteristic zero; $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and nonnegative integers, respectively. For $p, q \in \mathbb{F}[x]$, we write $p \perp q$ to indicate that the polynomials p and q are coprime. For $R \in \mathbb{F}(x)$, $\mathrm{num}(R)$ and $\mathrm{den}(R)$ denote the numerator and the denominator of R, respectively.

2. DIFFERENTIAL GOSPER FORM

Following [6], we call an ordered pair $(a,b) \in \mathbb{F}[x]^2$ *weakly-normalized* if $b \perp (a - i(\partial_x b))$ for all $i \in \mathbb{N}$. A rational function R in $\mathbb{F}(x)$ is weakly-normalized if $(\mathrm{num}(R), \mathrm{den}(R))$ is weakly-normalized.

DEFINITION 1. *Let $R \in \mathbb{F}(x)$. If there are $a,b,c \in \mathbb{F}[x]$, b,c monic, such that (i) $R = \frac{a}{b} + \frac{\partial_x c}{c}$, (ii) the pair (a,b) is weakly-normalized, then (a,b,c) is a differential Gosper-form, or a G'-form, of R. If, in addition, (iii) $b \perp c$, then (a,b,c) is a differential Gosper-Petkovšek form, or a GP'-form, of R. We call a/b and c the* kernel *and the* shell *of a G'-form (a,b,c), respectively.*

Every nonzero rational function has a G'-form and a unique GP'-form. See [4, 8] for proofs and algorithms for constructing them. A G'-form of a rational function R can also be constructed from a classification and distribution of the simple fractions in the irreducible partial fraction decomposition of R. This construction is very similar to that of DRNF's of R [7, Section 2], and can be summarized as follows.

Consider the irreducible partial fraction decomposition

$$R = \sum_i \frac{u_i(x)}{v_i(x)}. \tag{3}$$

Each simple fraction u_i/v_i in (3) belongs to one of the following three classes: (I) $u_i/v_i = m_i\,(\partial_x v_i)/v_i,\ m_i \in \mathbb{N}\setminus\{0\},\ v_i^2$ does not divide $\mathrm{den}(R)$; (II) $u_i/v_i = m_i\,(\partial_x v_i)/v_i,\ m_i \in \mathbb{N}\setminus\{0\},\ v_i^2$ divides $\mathrm{den}(R)$; (III) u_i/v_i is not a logarithmic derivative of any polynomial. Let (a,b,c) be a G'-form of R. Then the simple fractions in class (I) appear in the irreducible partial fraction decomposition of $(\partial_x c)/c$, not in the irreducible partial fraction decomposition of a/b; the simple fractions in class (III) appear in the irreducible partial fraction decomposition of a/b, not in the irreducible partial fraction decomposition of $(\partial_x c)/c$; the simple fractions in class (II) can appear in the irreducible partial fraction decomposition of either a/b or $(\partial_x c)/c$. This construction leads to the following lemma.

LEMMA 1. *For $R \in \mathbb{F}(x)$, if (a_1,b_1,c_1) and (a_2,b_2,c_2) are two G'-forms of R, then $b_1 = b_2$.*

Note that if all simple fractions in class (II) are moved to the kernel a/b, then the constructed G'-form is the GP'-form.

EXAMPLE 1. Consider the rational function

$$R = \frac{4}{x-2} + \frac{4}{x+1} - \frac{3}{(x+1)^2} - \frac{9}{(x-1)^2} + \frac{9x^2+12}{x^3+4x-2} + \frac{1}{(x^3+4x-2)^2}.$$

The simple fractions of R are classified as follows:

$$\text{(I)}\ \ u_1 = \frac{4}{x-2}, \qquad \text{(II)}\ \ v_1 = \frac{4}{x+1},\ v_2 = \frac{9x^2+12}{x^3+4x-2},$$

$$\text{(III)}\ w_1 = -\frac{9}{(x-1)^2}, w_2 = -\frac{3}{(x+1)^2}, w_3 = \frac{1}{(x^3+4x-2)^2}.$$

Now we construct four different G'-forms of R.

The first G'-form is constructed by moving both simple fractions in class (II) to the shell: $\big(\,\mathrm{num}(w_1+w_2+w_3),$

$$\mathrm{den}(w_1+w_2+w_3),\ \mathrm{den}(u_1)^4\mathrm{den}(v_1)^4\mathrm{den}(v_2)^3\big).$$

The second G'-form is constructed by moving both simple fractions in class (II) to the kernel (this G'-form is also the GP'-form of R): $\big(\,\mathrm{num}(w_1+w_2+w_3+v_1+v_2),$

$$\mathrm{den}(w_1+w_2+w_3+v_1+v_2),\ \mathrm{den}(u_1)^4\big).$$

The third G'-form is constructed by moving v_1 to the shell and v_2 to the kernel: $\big(\,\mathrm{num}(w_1+w_2+w_3+v_2),$

$$\mathrm{den}(w_1+w_2+w_3+v_2),\ \mathrm{den}(u_1)^4\mathrm{den}(v_1)^4\big).$$

Finally, the fourth G'-form is constructed by moving v_2 to the shell and v_1 to the kernel: $\big(\,\mathrm{num}(w_1+w_2+w_3+v_1),$

$$\mathrm{den}(w_1+w_2+w_3+v_1),\ \mathrm{den}(u_1)^4\mathrm{den}(v_2)^3\big).$$

3. GOSPER'S ALGORITHM FOR EQUATIONS OF ARBITRARY ORDER

Consider a differential equation of the form (1). For the remainder of the paper, we denote $r(x)$ and $R(x)$ the certificates of $t(x)$ and $z(x)$, respectively, (a,b,c) the GP'-form of $r(x)$, and (A,B,C) the GP'-form of $R(x)$.

LEMMA 2.

$$r(x) = R(x) + \frac{\partial_x M}{M} \ \ \textit{where}\ M = \sum_{k=0}^{d} p_k(x)\frac{\partial_x^k z(x)}{z(x)}. \tag{4}$$

Proof: Since $\partial_x\left(\frac{z(x)}{t(x)}\right) = R(x)\frac{z(x)}{t(x)} - r(x)\frac{z(x)}{t(x)}$,

$$r(x) = R(x) - \frac{\partial_x(z(x)/t(x))}{(z(x)/t(x))}. \tag{5}$$

The substitution of $z/t = 1/M$ into (5) yields (4). ∎

The following lemma shows an explicit form of M.

LEMMA 3. *For an indeterminate P, let $P_0 = 1$, $P_i = P_{i-1}\,P + \partial_x P_{i-1}$, for $i \in \mathbb{N}\setminus\{0\}$. Then*

$$M = \frac{1}{C}\left(\sum_{k=0}^{d} p_k\left(\sum_{j=0}^{k}\binom{k}{j}\left(\partial_x^j C\right)\left(\frac{A}{B}\right)_{k-j}\right)\right). \tag{6}$$

Proof: Since (A,B,C) is the GP'-form of R,

$$z(x) = C\ \underbrace{\exp\left(\int \frac{A}{B}\,dx\right)}_{H}.$$

Hence, the lemma is proven if we can show that

$$\partial_x^k z(x) = \partial_x^k(CH) = H\left(\sum_{j=0}^{k}\binom{k}{j}\left(\partial_x^j C\right)\left(\frac{A}{B}\right)_{k-j}\right). \tag{7}$$

The validity of (7) can be obtained by induction. It is clear that (7) holds for $k=0$. Assume that it holds for k. Then

$$\begin{aligned}\partial_x^{k+1}(CH) &= \partial_x\left(\partial_x^k(CH)\right)\\ &= \partial_x\left(H\left(\sum_{j=0}^{k}\binom{k}{j}\left(\partial_x^j C\right)\left(\frac{A}{B}\right)_{k-j}\right)\right)\end{aligned}\quad(8)$$

By applying the chain rule to (8), by making use of the fact that $\partial_x H = H\,(A/B)$, and by applying a change of summation index, one obtains

$$\begin{aligned}\partial_x^{k+1}(CH) &= H\left(\sum_{j=0}^{k}\binom{k}{j}\left(\partial_x^j C\right)\left(\frac{A}{B}\right)_{k+1-j}\right)+\\ &\quad H\left(\sum_{j=1}^{k+1}\binom{k}{j-1}\left(\partial_x^j C\right)\left(\frac{A}{B}\right)_{k+1-j}\right)\\ &= H\left(\sum_{j=0}^{k+1}\binom{k+1}{j}\left(\partial_x^j C\right)\left(\frac{A}{B}\right)_{k+1-j}\right).\end{aligned}$$

■

Let $L=\sum_{k=0}^{d}p_k(x)\partial_x^k$ be the differential operator corresponding to the differential equation (1). Let S be the symmetric product of L and $\partial_x + A(x)/B(x)$. Then $M = S(C(x))$. Note that the sequence $(P_i)_{i\geq 0}$ in Lemma 3 is used to defined the associated Riccati equation [5].

The following theorem follows from Lemmas 2 and 3.

Theorem 1. *Set*

$$B_k=\sum_{j=0}^{k}\binom{k}{j}\left(\partial_x^j C\right)\left(\frac{A}{B}\right)_{k-j},\quad 0\leq k\leq d. \qquad (9)$$

Then

$$r(x)=R(x)+\frac{\partial_x M}{M},\quad M=\frac{1}{C}\left(\sum_{k=0}^{d}p_k B_k\right). \qquad (10)$$

Recall that (a,b,c) is the GP'-form of $r(x)$. We now attempt to transform the right hand side of the first relation in (10) into another GP'-form of $r(x)$, equate the corresponding components, and solve for the unknown polynomials A,B,C. The last step is equivalent to finding polynomial solutions of a linear differential equation of order d and with polynomial coefficients.

Lemma 4. $\mathrm{den}\left(\left(\frac{A}{B}\right)_i\right)=B^i$, $B\perp\mathrm{num}\left(\left(\frac{A}{B}\right)_i\right)$, $i\in\mathbb{N}$.

Proof: It is clear that the two relations in Lemma 4 hold for $i=0$. Assume that they hold for i. Let $\left(\frac{A}{B}\right)_i = \frac{f}{B^i}$, $f\in\mathbb{F}[x]$, $B\perp f$. By definition,

$$\begin{aligned}\left(\frac{A}{B}\right)_{i+1} &= \left(\frac{A}{B}\right)_i\left(\frac{A}{B}\right)+\partial_x\left(\frac{A}{B}\right)_i\\ &= \frac{(\partial_x f)B+f(A-i\,(\partial_x B))}{B^{i+1}}.\end{aligned}$$

Since $B\perp f$ and since (A,B) is weakly-normalized, $B\perp\mathrm{num}\left(\left(\frac{A}{B}\right)_{i+1}\right)$, and $\mathrm{den}\left(\left(\frac{A}{B}\right)_{i+1}\right)=B^{i+1}$. ■

The following lemma reveals some insight into the structure of M in (10).

Lemma 5. $\mathrm{den}(B_k)=B^k$*,* $B\perp\mathrm{num}(B_k)$*,* $0\leq k\leq d$*.*

Proof: Expand B_k in (9) as

$$B_k = C\left(\frac{A}{B}\right)_k + k\,(\partial_x C)\left(\frac{A}{B}\right)_{k-1}+\cdots+\left(\partial_x^k C\right).$$

By Lemma 4, $\left(\frac{A}{B}\right)_i=\frac{f_i}{B^i}$, $f_i\in\mathbb{F}[x]$, $f_i\perp B$. Hence,

$$B_k=\frac{Cf_k+k\,(\partial_x C)\,f_{k-1}B+\cdots+\left(\partial_x^k C\right)B^k}{B^k}. \qquad (11)$$

Since (A,B,C) is the GP'-form of $R(x)$, $B\perp C$; and since $B\perp f_k$ (Lemma 4), $B\perp\mathrm{num}(B_k)$, $\mathrm{den}(B_k)=B^k$. ■

The structure of M leads to the following G'-form of $r(x)$.

Lemma 6. *The triple* $(A-d(\partial_x B),\,B,\,\mathrm{num}(M))$ *is a* G'*-form of the certificate* $r(x)$ *of the right hand side* $t(x)$ *in (1).*

Proof: By Lemma 5, write M in (10) as

$$M=\frac{p_0\ \mathrm{num}(B_0)\ B^d+\cdots+p_d\ \mathrm{num}(B_d)}{B^d C}. \qquad (12)$$

It follows from (10), (12) and Theorem 1 that

$$\begin{aligned}r(x) &= \frac{A}{B}+\frac{\partial_x C}{C}+\frac{\partial_x(\mathrm{num}(M)/(B^d C))}{(\mathrm{num}(M)/(B^d C))}\\ &= \frac{(A-d(\partial_x B))}{B}+\frac{\partial_x\mathrm{num}(M)}{\mathrm{num}(M)}.\end{aligned}\qquad (13)$$

Since the pair (A,B) is weakly-normalized, the pair $(A-d(\partial_x B),\,B)$ is also weakly-normalized for any $d\in\mathbb{N}$. ■

Note that both (a,b,c) and $(A-d(\partial_x B),\,B,\,\mathrm{num}(M))$ are G'-forms of $r(x)$. Hence, it follows from Lemma 1 that

Corollary 1. B*=*b*.*

The following is the main theorem of the paper.

Theorem 2. *Consider a differential equation of the form (1). Let* (a,b,c) *be the* GP'*-form of the certificate of the exponential right hand side* $t(x)$*. If* $b\perp p_d$*, then finding exponential solutions* $z(x)$ *of (1) is reduced to finding polynomial solutions* $C(x)$ *of the differential equation of order* d *and with polynomial coefficients*

$$\mathrm{num}(M)|_{(A,B)=(a+d(\partial_x b),b)}=c, \qquad (14)$$

which can be explicitly written as

$$\mathrm{num}\left(\sum_{k=0}^{d}p_k\left(\sum_{j=0}^{k}\binom{k}{j}\left(\frac{a}{b}+d\frac{\partial_x b}{b}\right)_{k-j}\partial_x^j C\right)\right)=c. \qquad (15)$$

Proof: It follows from (12) that

$$\operatorname{num}(M) = \left(\sum_{k=0}^{d-1} p_k \operatorname{num}(B_k)\, B^{d-k}\right) + p_d \operatorname{num}(B_d), \quad (16)$$

where the B_k's are defined in (9). If $b \perp p_d$, then it follows from (16), the second relation of Lemma 5, and Corollary 1 that $B \perp \operatorname{num}(M)$. Hence, $(A - d(\partial_x B), B, \operatorname{num}(M))$ is the GP'-form of $r(x)$. Since GP'-form is canonical, and since (a, b, c) is another GP'-form of $r(x)$, we have $A = a + d(\partial_x b)$, $B = b$, $\operatorname{num}(M) = c$, and the relation (14) holds. The explicit form (15) is readily derived from (9), (10) and (14).■

It follows from (1) and (4) that $z/t = 1/M$. Hence, the exponential solution corresponding to a polynomial solution C of (15) is

$$\frac{B^d\, C}{\operatorname{num}(M)|_{(A,B,C)}}\, t(x). \quad (17)$$

Note that for the special case of (1) where $d = 1$, $p_1 = 1$, and $p_0 = 0$, i.e., (1) becomes $\partial_x z(x) = t(x)$, the differential equation (15) becomes

$$b\ (\partial_x\, C) + (a + \partial_x\, b)\, C = c,$$

which is the "key equation"($G8$) in the differential Gosper's algorithm for exponential indefinite integration [4]. This helps derive the differential Gosper's ansatz (($G7$) in [4]), and also helps justify the title of this paper.

EXAMPLE 2. Consider the differential equation

$$\overbrace{(x^3{+}2x^2{+}2x{+}1)}^{p_2}\,\partial_x^2 z(x) - (x^4{+}2x^3{+}2x^2{+}2x{-}1)\partial_x z(x) -$$
$$(x^3{+}x^2{+}3x{+}1)z(x) = t(x), \quad (18)$$

$$t(x) = -\frac{x^7{+}8x^6{+}29x^5{+}65x^4{+}101x^3{+}107x^2{+}64x{+}15}{(x{+}2)^4} \times \exp\left(\frac{1}{x{+}2}\right).$$

The GP'-form of the certificate of $t(x)$ is $(a, b, c) =$

$$\Big(-4x - 9,\ \underbrace{(x+2)^2}_{b},\ (x+1)(x^6 + 7x^5 + 22x^4 + 43x^3 + 58x^2 + 49x + 15)\Big).$$

Since $p_2 \perp b$, Theorem 2 is applicable. The value of the pair (A, B) in (14) is $(A, B) = \big(-1, (x+2)^2\big)$, and the differential equation (15) has $C = -1$ as the only polynomial solution. By (17), the only exponential solution of (18) is $\exp\left(\frac{1}{x+2}\right)$.

4. A COMPARISON

By using our *polynomial* method, the problem is reduced to finding polynomial solutions $C \in \mathbb{F}[x]$ of the linear differential equation (15). Let $L = \sum_{k=0}^{d} p_k(x)\partial_x^k$ be the differential operator corresponding to (1), $(a, b, c) \in \mathbb{F}[x]^3$ be the GP'-form of the certificate r of the exponential right hand side t. Let S be the symmetric product of L and $(\partial_x + r(x))$. By using the rational method, the problem of computing exponential solutions of (1) is reduced to finding rational solutions $f \in \mathbb{F}(x)$ of $S(f(x)) = 1$[1], which can be explicitly written as

$$\operatorname{num}\left(\sum_{k=0}^{d} p_k \left(\sum_{j=0}^{k} \binom{k}{j} \left(\underbrace{\frac{a}{b} + \frac{\partial_x c}{c}}_{r}\right)_{k-j} \partial_x^j f\right)\right) = (bc)^d. \quad (19)$$

Note the similarity of the two forms (19) and (15), and the difference in the right hand sides. (The rewrite of r in terms of its GP'-form is for comparing (19) with (15).)

In order to reduce the problem of finding rational solutions f of (19) to finding polynomial solutions of a linear differential equation with polynomial coefficients and polynomial right hand side, one first computes a universal denominator $u \in \mathbb{F}[x]$, i.e., a polynomial which is divisible by the denominator of any rational solution of (19). This can be attained by the use of balanced factorization [2]. Then the substitution of f by $\tilde{f}/u$ in (19) where $\tilde{f} \in \mathbb{F}[x]$ is unknown, and the transformation into a linear differential equation with polynomial coefficients can increase the degrees of the polynomial coefficients of the differential equation and also the degree of the polynomial right hand side dramatically. The higher the order of the input differential equation is, the more dramatic the increase of the degrees of these polynomials is. This explains the efficiency gain by using our method provided that it is applicable. For instance, if one uses the method of undetermined coefficients for finding polynomial solutions [1], the size of the linear system of algebraic equations would be smaller using our method.

In the case where our method is not applicable, one reverts to the rational method. The additional cost is that of computing the GP'-form of the rational certificate of the exponential right hand side t in (1), a rather negligible cost in the comparison with the cost of finding rational solutions.

5. IMPLEMENTATION, EXPERIMENTS

We implemented the rational method and the polynomial method in the computer algebra system Maple. Both programs are for finding a particular exponential solution of (1). The program for the rational method, called `rat_esols` (or simply R) is built on top of the function `DEtools[ratsols]` (see [2] for the main reference), and our program, called `esols` (or simply P) is built on top of the function `DEtools[polysols]` (see [1] for the main reference).

In the following two experiments, we compare the resource requirements of R and P[2]. In the first experiment, the order of the input differential equation is fixed, while the degrees of the polynomials $r_1, r_2 \in \mathbb{Z}[x]$ in the exponential solution $\exp(r_1/r_2 + \partial_x r_3/r_3)$ vary, $r_3 \in \mathbb{Z}[x]$, $\deg r_3 = 2$. In the second experiment, the degrees of r_1 and r_2 are fixed, while the order of the input differential equation varies.

Experiment 1. The first experiment consists of four sets of

[1]The author would like to thank Mark v. Hoeij for his explanation on how the rational method works.

[2]All the reported timings were obtained on a 2.8Ghz Intel P4 Xeon with 4Gb RAM.

tests. Each set consists of ten randomly-generated linear ordinary differential equations with polynomial coefficients and exponential right hand sides. Each element in a set is of the form

$$L \bullet z(x) = t(x), \quad L \in \mathbb{Z}[x, \partial_x], \tag{20}$$

$L = \operatorname{lclm}(L_1, L_2)$, $L_i = p_{i_1} \partial_x + p_{i_0}$, $p_{i_j} \in \mathbb{Z}[x]$, $2 \le \deg p_{i_j} \le 3$, and each p_{i_j} has at most two monomials. The right hand side exponential $t(x)$ is such that the randomly-generated exponential function $\exp(r_1/r_2 + \partial_x r_3/r_3)$ is a solution of (20). Table 1 shows the average resource requirements for different values of $\deg r_1$ and $\deg r_2$.

Table 1: First experiment: resource requirements

$\deg r_1$	$\deg r_2$	Timing (seconds)		Memory (kilobytes)	
		R	P	R	P
1	2	53.32	0.31	653,077	9,034
2	3	151.23	0.38	2,689,441	12,379
3	4	367.83	0.51	8,222,518	14,649
4	5	860.42	0.67	24,088,511	21,065

Experiment 2. The second experiment consists of three sets of tests. Each set consists of ten randomly-generated equations each of which is of the form (20) where $L = \operatorname{lclm}(L_1, L_2, \ldots, L_n)$, $L_i = p_{i_1} \partial_x + p_{i_0}$, $p_{i_j} \in \mathbb{Z}[x]$, $\deg p_{i_j} = 1$, each p_{i_j} has only one monomial, and the integer coefficients of p_{i_j}, r_1, r_2, r_3 are in the interval $[-5, 5]$. These restrictions are necessary because of the "expression-swell" problem in the rational method. The right hand side exponential $t(x)$ is such that the randomly-generated exponential function $\exp(r_1/r_2 + \partial_x r_3/r_3)$, $r_1, r_2, r_3 \in \mathbb{Z}[x]$, $\deg r_1 = 1$, $\deg r_2 = 2$, $\deg r_3 = 2$, is a solution of (20). Table 2 shows the average resource requirements for different values of n. Note that the polynomial method is not applicable to one element in the first set and one element in the third set (and hence the rational method is used). This explains the "high" average resource requirements for these two sets.

Table 2: Second experiment: resource requirements

n	Timing (seconds)		Memory (kilobytes)	
	R	P	R	P
2	3.70	0.31	90,059	9,057
3	72.47	0.27	731,833	7,740
4	2,563.9	5.27	6,819,517	79,833

The Maple source code, the help page, and details of the experiments reported in this paper are available, and can be downloaded from

`http://www.cecm.sfu.ca/~hle/maple/esols/`.

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