New closed form solutions in terms of ${}_pF_q$ for families of the General, Confluent and Bi-Confluent Heun differential equations

E.S. Cheb-Terrab[a,b]

[a] *CECM, Department of Mathematics*
Simon Fraser University, Vancouver, British Columbia, Canada.

[b] *Maplesoft, Waterloo Maple Inc.*

Abstract

In a recent paper, the canonical forms of a new multi-parameter class of Abel differential equations, so-called AIR, all of whose members can be mapped into Riccati equations, were shown to be related to the differential equations for the hypergeometric ${}_2F_1$, ${}_1F_1$ and ${}_0F_1$ functions. In this paper, a connection between the AIR canonical forms and the Heun General (GHE), Confluent (CHE) and Biconfluent (BHE) equations is presented. This connection exists after fixing the value of one of the Heun parameters and expressing another one in terms of those remaining. The resulting GHE, CHE and BHE families respectively depend on four, three and two irreducible parameters. This connection provides closed form solutions in terms of ${}_pF_q$ functions for these Heun equation families, shows that the problems formulated in terms of Abel AIR equations can also be formulated in terms of these linear GHE, CHE and BHE equations, and suggests a mechanism for relating linear equations with N and N-1 singularities.

Introduction

The Heun equation [1] is a second order linear equation of the form

$$y'' + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right) y' + \frac{(\alpha\,\beta\,x - q)}{x\,(x-1)\,(x-a)}\,y = 0, \tag{1}$$

where $\{\alpha, \beta, \gamma, \delta, \epsilon, a, q\}$ are constant with respect to x, are related by $\gamma+\delta+\epsilon = \alpha+\beta+1$, and $a \neq 0, a \neq 1$. This equation has four regular singular points at $\{0, 1, a, \infty\}$. Through confluence processes, equation (1), herein called the General Heun Equation (GHE), transforms into four other multi-parameter equations [2], so-called Confluent (CHE), Biconfluent (BHE), Doubleconfluent (DHE) and Triconfluent (THE). Following [2], through transformations of the form $y \to P(x)\,y$, these five equations can be written in normal form[1] and expressed in terms of arbitrary constants $\{a,\ A, B, C, D, E, F\}$; for the 6-parameter GHE (1) we have

$$y'' + \left(\frac{A}{x} + \frac{B}{x-1} - \frac{A+B}{x-a} + \frac{D}{x^2} + \frac{E}{(x-1)^2} + \frac{F}{(x-a)^2}\right) y = 0 \tag{2}$$

The 5-parameter CHE,

$$y'' + \left(A + \frac{B}{x} + \frac{C}{x-1} + \frac{D}{x^2} + \frac{E}{(x-1)^2}\right) y = 0, \tag{3}$$

has two regular singularities at $\{0, 1\}$ and one irregular singularity at ∞. The 4-parameter BHE,

[1] A second order linear ODE is in *normal form* when the coefficient of y' is equal to zero.

$$y'' + \left(-x^2 + B\,x + C + \frac{D}{x} + \frac{E}{x^2}\right) y = 0, \tag{4}$$

has one regular singularity at 0 and one irregular singularity at ∞. The 4-parameter DHE,

$$y'' + \left(A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3} + \frac{A}{x^4}\right) y = 0, \tag{5}$$

has two irregular singularities at $\{0, \infty\}$. The 3-parameter THE, with one irregular singularity at ∞, is

$$y'' + \left(-\frac{9}{4}\,x^4 + C\,x^2 + D\,x + E\right) y = 0 \tag{6}$$

Eq.(1), originally studied by Heun as a generalization of Gauss' hypergeometric (${}_p\mathrm{F}_q$) equation, as well as these related confluent families represented by (3–6), appear in applications in varied areas[2]. As a sample of recent related works, in [3] quase-normal modes of near extremal black branes are found solving a singular boundary value problem for (1); in [4], hyper-spherical harmonics, with applications in three-body systems, are developed in connection with the solutions of (1); in [5], a method of calculation of propagators for the case of a massive spin 3/2 field, for arbitrary space-time dimensions and mass, is developed in terms of the solutions of (1); in [6], parametric resonance after inflation is discussed in connection with the solutions of a particular form of (1). The separation of variables for the Schrödinger equation in a large number of problems results in Heun type equations too, typically for the radial coordinate, and also non-linear formulations involving Painlevé type equations [7] can be derived from Heun equations regarded as quantum Hamiltonians. A number of traditional equations of mathematical physics, as for instance the Lamé, spheroidal wave, and Mathieu equations, are also particular cases of Heun equations.

The solutions for these five Heun equations are the subject of current study [8, 9, 10, 11, 12]. In this paper, exact closed form solutions for non trivial families of the GHE, CHE and BHE equations, are derived. These solutions are expressed in terms of exponentials of integrals involving the ${}_2\mathrm{F}_1$ and ${}_1\mathrm{F}_1$ functions. As we shall see, this type of solution exists provided that one of the parameters in (2), (3) and (4) is fixed and one other varies as a function of the remaining ones.

The transformations presented below, relating Heun to ${}_p\mathrm{F}_q$ differential equations, were derived exploring a connection between Heun, ${}_p\mathrm{F}_q$ and Abel (first order) equations. This connection with Abel differential equations is important in itself since these equations also appear frequently in applications.

In sec. 1, some results of [13] and [14] are reviewed and a hitherto unknown connection between Heun and Abel equations is made explicit. In sec. 2, 3 and 4, the restrictions that this connection implies on the parameters entering the BHE, CHE and GHE equations are derived, and it is shown how a sequence mapping Heun $\rightarrow$ Abel $\rightarrow$ ${}_p\mathrm{F}_q$ equations can be composed to obtain transformations mapping Heun $\rightarrow$ ${}_p\mathrm{F}_q$ equations. When the aforementioned restriction on the Heun parameters holds, these transformations lead to explicit closed form solutions, shown below, expressed in terms of ${}_2\mathrm{F}_1$ and ${}_1\mathrm{F}_1$ functions, for the GHE (2), CHE (3) and BHE (4). The formulas relating the normal and canonical forms for these equations are included in an Appendix for completeness. In sec. 5, a discussion around these results is developed.

1 A connection between Heun, Abel and ${}_p\mathrm{F}_q$ differential equations

The transformations being presented, relating Heun and ${}_p\mathrm{F}_q$ hypergeometric equations, were obtained by composing transformations which map Heun, Riccati, Abel and ${}_p\mathrm{F}_q$ equations according to the sequence

$$Heun \rightarrow Riccati \rightarrow Abel \rightarrow Abel_{canonical} \rightarrow Riccati \rightarrow {}_p\mathrm{F}_q$$

Although it is possible to shortcut the step which goes through Abel equations, these equations are important by themselves, they appear frequently in applications (see for instance [15, 16, 17, 18]), and it is through

[2] For a list of applications of Heun's equations compiled in 1995 see p.340 of [1].

work related to them that the $Heun \leftrightarrow {}_p\mathrm{F}_q$ relation being presented became evident. Hence it is interesting to keep this $Heun \leftrightarrow Abel$ connection visible.

Abel equations of the second kind [16] are equations of the form

$$y' = \frac{f_3(x)\, y^3 + f_2(x)\, y^2 + f_1(x)\, y + f_0(x)}{g_1(x)\, y + g_0(x)}, \tag{7}$$

where the $\{f_i,\ g_i\}$ are arbitrary functions and either $f_3(x) \neq 0$ or $g_1(x) \neq 0$. In [13] it is shown that, departing from a Riccati type equation,

$$y' = h_2(x)\, y^2 + h_1(x)\, y + h_0(x), \tag{8}$$

by suitably restricting the form of the mappings h_i and making use of the *inverse* transformation $\{x \leftrightarrow y\}$[3], one can construct an Abel equation,

$$y' = \frac{(y-\rho_1)(y-\rho_2)(y-\rho_3)}{(s_2\, x^2 + s_1\, x + s_0)\, y + r_2\, x^2 + r_1\, x + r_0} \tag{9}$$

where the $\{s_i, r_i, \rho_i\}$ are constants, and $s_2 \neq 0$ or $r_2 \neq 0$. This Abel equation is representative of a multi-parameter class all of whose members can be transformed into Riccati equations (8) using $\{x \leftrightarrow y\}$, and from there into second order linear equations using the *Riccati* $\rightarrow$ *linear* mapping [16]

$$y \rightarrow -\frac{y'}{h_2(x)\, y} \tag{10}$$

The equations of this Abel class, so-called in [13] "Abel Inverse Riccati" (AIR), are then generated from (9) by applying to it class transformations of the form

$$\{x \rightarrow F(x), \quad y \rightarrow \frac{P_1(x)\, y + Q_1(x)}{P_2(x)\, y + Q_2(x)}\}, \tag{11}$$

where $\{F,\, P_1,\, P_2,\, Q_1,\, Q_2\}$ are arbitrary mappings with $F' \neq 0,\ P_1\, Q_2 - P_2\, Q_1 \neq 0$. The relevance of the AIR class can be inferred from the fact that most of the Abel solvable equations found in the literature[4] are shown in [13] to be particular members of AIR.

An important property of (9) is that its connection with second order linear equations, that is, its "Inverse Riccati" character, is invariant under Möbius (linear fractional) changes of x and y. This property is used in [14] to accomplish a full classification of (9) in terms of six canonical forms. For that purpose, through Möbius changes of y, (9) is first transformed into

$$y' = \frac{P\,(y)}{(s_2\, x^2 + s_1\, x + s_0)\, y + r_2\, x^2 + r_1\, x + r_0} \tag{12}$$

for some new constants $\{s_i\, r_i\}$, with $P(y)$ equal to $y\,(y-1)$, y or 1, respectively according to whether in (9) there are three, two or only one distinct roots ρ_i. As shown in [14], each of these three cases splits further into two subcases, and the six resulting canonical forms are solvable in terms of ${}_2\mathrm{F}_1$, ${}_1\mathrm{F}_1$ and ${}_0\mathrm{F}_1$ functions; in this way closed form ${}_p\mathrm{F}_q$ solutions can be constructed for the whole AIR class.

The key observation now is that the AIR equation (12) is also connected in a surprisingly simple manner to the Heun family of equations. As we shall see, by applying to (12) the transformation $\{x \leftrightarrow y\}$, one obtains a Riccati equation, and by transforming it further into a second order linear equation using (10), one directly obtains the GHE, CHE or BHE Heun equations (with some restrictions on the parameters), respectively according to the three possible values of $P(y)$. Since the AIR (12) admits solutions expressible using ${}_p\mathrm{F}_q$ functions for the three possible values of $P(y)$, the GHE, CHE and BHE Heun families which can respectively be derived from (12) also admit closed form solutions expressible in terms of these ${}_p\mathrm{F}_q$ functions.

[3]By $\{x \leftrightarrow y\}$ we mean changing variables $\{x = u(t),\, y(x) = t\}$ followed by renaming $\{u = y,\, t = x\}$.

[4]For a collection of these see [19].

2 Closed form solutions for a subfamily of the BHE

Considering first the simplest case, where the three roots ρ_i in (9) are equal, in (12) we have $P(y) = 1$ and so the AIR equation becomes

$$y' = \frac{1}{(s_2\,x^2 + s_1\,x + s_0)\,y + r_2\,x^2 + r_1\,x + r_0} \tag{13}$$

Recalling that either $s_2 \neq 0$ or $r_2 \neq 0$, changing variables using $\{x \leftrightarrow y\}$ we obtain the Riccati form

$$y' = (s_2\,x + r_2)\;y^2 + (s_1\,x + r_1)\;y + s_0\,x + r_0 \tag{14}$$

Using (10), this equation is transformed into the second order linear equation

$$y'' = \frac{\left(s_2\,s_1 x^2 + (s_2 r_1 + s_1 r_2)\,x + s_2 + r_2 r_1\right)}{s_2\,x + r_2}\,y' - \left(s_2 s_0 x^2 + (s_2 r_0 + s_0 r_2)\,x + r_2 r_0\right) y \tag{15}$$

This equation has one regular singularity at $-r_2/s_2$, one irregular singularity at ∞, and, by rewriting it in normal form, it is straightforward to verify that it is the BHE equation (4) with one of its four parameters fixed and two other ones interrelated. For that purpose, we note first that the case $s_2 = 0$ presents no interest since it directly simplifies (15) to a ${}_p\mathrm{F}_q$ equation. Assuming $s_2 \neq 0$ in (12), we take $s_2 = 1$ without loss of generality. To have the regular singularity of (15) located at 0, it suffices to take $r_2 = 0$, and, taking $s_0 = s_1^2/4 - 1$, the coefficient of x^2 in the normal form of (15) will be as in (4). In summary, using $y \to \sqrt{x}\exp((x\,(s_1\,x + 2\,r_1)\,/4\,)\,y$ to rewrite (15) in normal form at $\{s_2 = 1, r_2 = 0, s_0 = s_1^2/4\; - 1\}$, the equation becomes

$$y'' = \left(x^2 + \left(\frac{s_1 r_1}{2} - r_0\right) x + \frac{{r_1}^2}{4} + \frac{r_1}{2\,x} + \frac{3}{4\,x^2}\right) y \tag{16}$$

which is the BHE (4) at $\{B = r_0 - s_1 r_1/2,\, D = -r_1/2,\, C = -D^2,\, E = -3/4\}$.

The relevance of this result is in that, on the one hand, (16) is a non-trivial 2-parameter form of the BHE for which solutions are not known in general; on the other hand, as shown in [14], the Abel equation (13), from which (16) is derived, can always be solved in terms of ${}_1\mathrm{F}_1$ and ${}_0\mathrm{F}_1$ (Kummer and Bessel) hypergeometric functions. Therefore, a closed form solution for the BHE (16) can also be expressed using these ${}_p\mathrm{F}_q$ functions.

To separate the Liouvillian from the Non-Liouvillian solutions of (16), the parameters $\{r_0, r_1\}$ are redefined in terms of new parameters $\{\sigma,\,\tau\}$ according to[5]

$$r_0 = -2\,\sigma + s_1\tau, \quad r_1 = 2\,\tau \tag{17}$$

With this choice, the BHE (16) becomes the 2-parameter equation

$$y'' = \left(x^2 + 2\,\sigma\,x + \tau^2 + \frac{\tau}{x} + \frac{3}{4\,x^2}\right) y, \tag{18}$$

and (13) at $\{s_2 = 1, r_2 = 0, s_0 = s_1^2/4\; - 1\}$ becomes

$$y' = \frac{1}{(x^2 + s_1 x + (s_1 + 2)\,(s_1 - 2)\,/4)\;y + 2\,\tau\,x - 2\,\sigma + s_1\tau} \tag{19}$$

[5] The motivation for this particular choice of $\{\sigma, \tau\}$ becomes clear below, in connection with the form of equation (24).

2.1 Liouvillian solutions for the BHE (18) when $\sigma = \pm\tau$

As explained in sec. 2.2, when $\sigma = \pm\tau$, the BHE (18) can be obtained from an equation of the form $y'' + J(x)\,y' = 0$ through a Liouvillian transformation and so it admits Liouvillian solutions, computable using Kovacic's method [23]. For $\sigma = \tau$, a general solution for (18) is

$$y = \frac{\mathrm{e}^{-x(x+2\,\tau)/2}}{\sqrt{x}}\,C_1 + \frac{\sqrt{\pi}\,\mathrm{e}^{x(x+2\,\tau)/2} - \pi\,\tau\,\mathrm{erfi}(x+\tau)\,\mathrm{e}^{-x(x+2\,\tau)/2-\tau^2}}{\sqrt{x}}\,C_2 \tag{20}$$

where erfi is the imaginary error function [24]. For $\sigma = -\tau$, a general solution is

$$y = \frac{\mathrm{e}^{x(x-2\,\tau)/2}}{\sqrt{x}}\,C_1 + \frac{\sqrt{\pi}\,\mathrm{e}^{-x(x-2\,\tau)/2} - \pi\,\tau\,\mathrm{erf}(x-\tau)\,\mathrm{e}^{x(x-2\,\tau)/2+\tau^2}}{\sqrt{x}}\,C_2 \tag{21}$$

where erf is the error function [24].

2.2 A solution in terms of ${}_1\mathrm{F}_1$ functions for the BHE (18) when $\sigma^2 \neq \tau^2$

A transformation relating (18) to a ${}_1\mathrm{F}_1$ differential equation, providing a solution for (18) when $\sigma^2 \neq \tau^2$, is constructed by composing three transformations: the one which maps the BHE (18) into the AIR equation (19); one which maps (19) into an AIR equation admitting ${}_1\mathrm{F}_1$ solutions; finally, one which maps that AIR equation into a ${}_1\mathrm{F}_1$ equation.

Reversing the transformations used to derive (16) from (13), the transformation mapping the Heun equation (18) into the Abel AIR (19) is

$$\left\{ x \rightarrow y, \quad y \rightarrow \frac{\mathrm{e}^{-\left(\int x\,y\,y'\,dx + \frac{s_1}{4}y^2 + \tau\,y\right)}}{\sqrt{y}} \right\} \tag{22}$$

According to [14], the transformation mapping (19) into a canonical form of AIR admitting a ${}_p\mathrm{F}_q$ solution is

$$\left\{ x \rightarrow \left(\frac{\sqrt{2}\,x}{2\,(\tau+\sigma)} + \frac{1}{2} \right)^{-1} - \frac{s_1}{2} - 1, \quad y \rightarrow \frac{\sqrt{2}\,y}{2} - \sigma \right\} \tag{23}$$

The resulting AIR canonical form is

$$y' = \frac{1}{x\,y + x^2 + (\sigma^2 - \tau^2)\,/2} \tag{24}$$

This form turns evident the motivation for introducing $\{\sigma, \tau\}$ according to (17). For $\sigma = \pm\tau$, the independent term in the denominator of (24) cancels, and hence, when transforming this equation into a second order linear equation, we will obtain one of the form $y'' + J(x)\,y' = 0$ with rational $J(x)$, admitting a constant for solution. Since the transformation of such an equation into a normal form like (18) is Liouvillian, the normal form of the equation will admit Liouvillian solutions.

For the purpose of relating the Heun equation (18) to a ${}_p\mathrm{F}_q$ equation, however, a simpler derivation is possible if instead of using (23) we use

$$x \rightarrow \frac{1}{x} - \frac{s_1}{2} - 1, \tag{25}$$

not resulting in a canonical form of AIR, but still leading to an AIR equation admitting ${}_p\mathrm{F}_q$ solutions

$$y' = \frac{1}{(2\,x-1)\,y + 2\,(\sigma\ +\tau)\,x^2 - 2\,\tau\,x} \tag{26}$$

Following [14], this equation is transformed into a Riccati equation, then into a linear second order one using

$$\left\{x \to y, \quad y \to -\frac{y'}{2\,(\tau+\sigma)\,y}\right\} \tag{27}$$

leading to

$$y'' = 2\,(x-\tau)\,y' + 2\,(\tau+\sigma)\,x\,y \tag{28}$$

Finally, using $\{x \to (x+\sigma)^2\,,\; y \to ((x+\sigma)\,\mathrm{e}^{-x(\tau+\sigma)})^{-1}\,y\}$, equation (28) is obtained from the confluent ${}_1\mathrm{F}_1$ hypergeometric equation

$$x\,y'' + (\nu - x)\,y' - \mu\,y = 0 \tag{29}$$

at $\{\mu = (2+\tau^2-\sigma^2)/4,\;\nu = 3/2\}$, from where the solution to (28), in terms of the Kummer M and U functions[6] [24], is

$$y = \mathrm{e}^{-x(\tau+\sigma)}\left(\mathrm{M}\left(\frac{\tau^2-\sigma^2}{4}, \frac{1}{2}, (x+\sigma)^2\right)C_1 + \mathrm{U}\left(\frac{\tau^2-\sigma^2}{4}, \frac{1}{2}, (x+\sigma)^2\right)C_2\right) \tag{30}$$

Summarizing, departing from the Heun Biconfluent equation (18) we have arrived at the ${}_1\mathrm{F}_1$ equation (28) with solution (30) through a process of the form

$$\mathit{Heun} \to \mathit{Abel} \to \mathit{Abel}_{{}_1F_1\,solvable} \to {}_1\mathrm{F}_1$$

The three transformations used, (22), (25) and (27), can be combined into a single transformation, mapping the BHE (18) into the ${}_1\mathrm{F}_1$ (28) in one step, of the form

$$y \to \frac{1}{\sqrt{x}}\,\exp\left(\frac{x^2}{2} - \tau x + 2\,(\tau+\sigma)\int \frac{x\,y}{y'}\,dx\right) \tag{31}$$

Therefore, a closed form solution for the BHE (18) when $\sigma^2 \neq \tau^2$ is just this transformation (31), where in its "right-hand-side" the value of y is given by (30).

3 Closed form solutions for a subfamily of the CHE

A confluent family of Heun equations can be derived from (9) when, among the three roots ρ_i, only two are different. Hence, in (12) we have $P(y) = y$ and the starting AIR equation is

$$y' = \frac{y}{(s_2\,x^2 + s_1\,x + s_0)\,y + r_2\,x^2 + r_1\,x + r_0} \tag{32}$$

As in the previous section, changing variables using $\{x \leftrightarrow y\}$, to obtain a Riccati type equation, then using (10), we obtain the linear equation

$$y'' = \frac{\left(s_2 s_1 x^2 + (s_1 r_2 + s_2 r_1)\,x - r_2 + r_2 r_1\right)}{x\,(s_2 x + r_2)}\,y' - \frac{\left(s_2 s_0 x^2 + (s_2 r_0 + s_0 r_2)\,x + r_2 r_0\right)}{x^2}\,y \tag{33}$$

This equation has two regular singularities at $\{0, -r_2/s_2\}$ and one irregular singularity at ∞, and by rewriting it in normal form, its confluent Heun type becomes evident. As shown below, in (33), the implicit restrictions to the most general case (3) consist of having one of the five parameters, E, fixed and another one, B, being a function of the remaining three.

[6] An equivalent form of this solution in terms of ${}_1\mathrm{F}_1$ functions is

$$y = \mathrm{e}^{-x(\tau+\sigma)}\left({}_1\mathrm{F}_1\left((\tau^2-\sigma^2)/4;\ 1/2;\ (x+\sigma)^2\right)C_1 + (x+\sigma)\,{}_1\mathrm{F}_1\left(1/2 + (\tau^2-\sigma^2)/4;\ 3/2;\ (x+\sigma)^2\right)C_2\right)$$

.

To derive the relation between the parameters of the CHE equations (3) and (33), and then a solution for (33) in the non trivial cases, we start by noting that, when $s_2 = 0$, (33) simplifies to a ${}_p\mathrm{F}_q$ equation; the interesting case is $s_2 \neq 0$, which is equivalent to taking $s_2 = 1$ in (32). The regular singularities of (33) are fixed at $\{0, 1\}$ by taking $r_2 = -1$, and the term independent of x in the normal form of (33) is fixed to be A, as in (3), by taking $s_0 = s_1^2/4 + A$. In summary, rewriting (15) in normal form, using $y \to x^{(r_1-1)/2}\sqrt{x-1}\exp(s_1\,x/2)\,y$, at $\{s_2 = 1, r_2 = -1, s_0 = s_1^2/4 + A\}$, the equation becomes

$$y'' = \left(-A + \frac{{s_1}^2/2 + s_1(r_1-1) - r_1 - 2\,r_0 + 2A + 1}{2\,x} + \frac{s_1 + r_1 - 1}{2\,(x-1)} + \frac{{r_1}^2 + 4\,r_0 - 1}{4\,x^2} + \frac{3}{4\,(x-1)^2}\right) y \quad (34)$$

which is the CHE (3) at

$$B = -A - D - C^2, \quad C = \frac{1 - s_1 - r_1}{2}, \quad D = \frac{1 - {r_1}^2}{4} - r_0, \quad E = -\frac{3}{4} \quad (35)$$

As shown in [14], the Abel AIR equation (32) can always be solved in terms of ${}_1\mathrm{F}_1$ hypergeometric functions, from where a closed form solution expressed using ${}_1\mathrm{F}_1$ can also be constructed for the CHE (34). To separate a set of the Liouvillian solutions of (34) from the generally non-Liouvillian ones, and to achieve simpler notation, the parameters $\{r_0, r_1\}$ are redefined in terms of new parameters $\{\sigma, \tau\}$ according to $\{r_0 = (1-2\sigma)\,p^2 - {s_1}^2/4 + s_1\tau p,\ r_1 = 2\,\tau p - s_1\}$. Introducing also $A = -p^2$ (see [1] p.94), (34) becomes

$$y'' = \left(p^2 + \frac{2\,(\sigma-1)p^2 - \tau\,p + 1/2}{x} + \frac{\tau p - 1/2}{x-1} + \frac{(\tau^2 - 2\,\sigma + 1)\,p^2 - 1/4}{x^2} + \frac{3}{4\,(x-1)^2}\right) y \quad (36)$$

that is, a 3-parameter equation, and the AIR equation (32) becomes

$$y' = \frac{y}{(x^2 + s_1 x - p^2 + {s_1}^2/4)\,y - x^2 + (2\,\tau p - s_1)\,x + (1 - 2\,\sigma)\,p^2 + s_1\tau p - {s_1}^2/4} \quad (37)$$

3.1 Liouvillian solutions for the CHE (36) when $\sigma = \pm\tau$

When $\sigma = \pm\tau$, Liouvillian solutions for the CHE (36) can be computed using Kovacic's algorithm; for $\sigma = \tau$,

$$y = \frac{x^{(1-\tau)p+1/2}\,\mathrm{e}^{-p\,x}}{\sqrt{x-1}}\left(C_1 + \left(\Gamma(2\,(\tau-1)\,p+1,\ -2\,p\,x) + 2\,p\,\Gamma(2\,(\tau-1)\,p,\ -2\,p\,x)\right) C_2\right) \quad (38)$$

where Γ (of two arguments - see (6.5.3) in [24]) is the incomplete gamma function. For $\sigma = -\tau$,

$$y = \frac{x^{-(1+\tau)p+1/2}\,\mathrm{e}^{p\,x}}{\sqrt{x-1}}\left(C_1 + \left(2\,p\,\Gamma\,(2\,(\tau+1)\,p,\ 2\,p\,x) - \Gamma\,(2\,(\tau+1)\,p+1,\ 2\,p\,x)\right) C_2\right) \quad (39)$$

3.2 A solution in terms of ${}_1\mathrm{F}_1$ functions for the CHE (36) when $\sigma^2 \neq \tau^2$

As in the previous section, a solution to (36) when $\sigma^2 \neq \tau^2$ is constructed by composing three transformations: the one which maps the CHE (36) into the AIR (37); one which maps (37) into an AIR equation admitting ${}_1\mathrm{F}_1$ solutions; finally, one which maps that AIR equation into a ${}_1\mathrm{F}_1$ equation.

Reversing the transformations used to derive (34) from (32), the transformation mapping the Heun equation (34) into the Abel AIR (37) is

$$\left\{x \to y,\quad y \to \exp\left(-\int \frac{\left((x + s_1/2)\,y^2 - (2\,x - \tau\,p + s_1)\,y + x - \tau\,p + (s_1+1)/2\right) y'}{y\,(y-1)}dx\right)\right\} \quad (40)$$

According to [14], it is possible to construct a transformation mapping (37) into a canonical form of AIR admitting a ${}_p\mathrm{F}_q$ solution. However, as in the BHE case, simpler expressions result if we transform (37) into an non-canonical AIR equation. The transformation used for that purpose is

$$x \rightarrow \frac{1}{x} - \frac{s_1}{2} - p \tag{41}$$

mapping (37) into the Abel equation

$$y' = \frac{y}{(2\,p\,x - 1)\,y + 2\,p^2\,(\tau + \sigma)\,x^2 - 2\,(\tau + 1)\,p\,x + 1} \tag{42}$$

Following [14] this AIR equation is transformed into a ${}_p\mathrm{F}_q$ one by combining the $\{x \leftrightarrow y\}$ transformation with the transformation (10) mapping Riccati into a second order linear equations; the combination being

$$\left\{x \rightarrow -\frac{x\,y'}{2\,(\tau + \sigma)\,p^2\,y} \quad y \rightarrow x\right\} \tag{43}$$

leading to

$$y'' = \frac{2\,(x - \tau - 1)\,p - 1}{x}\,y' + \frac{2\,(\tau + \sigma)\,p^2\,(x - 1)}{x^2}\,y \tag{44}$$

Finally, using $\{x \rightarrow 2\,p\,x,\, y \rightarrow x^{\left(1+\tau-\sqrt{1-2\,\sigma+\tau^2}\right)p}y\}$, equation (44) is obtained from the confluent ${}_1\mathrm{F}_1$ hypergeometric equation (29) at $\{\mu = \left(\sigma - 1 + \sqrt{1 - 2\,\sigma + \tau^2}\right)p,\ \nu = 1 + 2\,\sqrt{1 - 2\,\sigma + \tau^2}\,p\}$, from where the solution to (44), in terms of the Whittaker $\mathbf{M}$ and $\mathbf{W}$ functions[7] [24], is

$$y = \frac{\mathrm{e}^{p\,x}}{x^{(\tau+1)p+1/2}} \tag{45}$$

$$\left(\mathbf{M}\left(\frac{1}{2} + (1 - \sigma)\,p,\ \sqrt{1 - 2\,\sigma + \tau^2}\,p,\ 2\,p\,x\right)C_1 + \mathbf{W}\left(\frac{1}{2} + (1 - \sigma)\,p,\ \sqrt{1 - 2\,\sigma + \tau^2}\,p,\ 2\,p\,x\right)C_2\right)$$

Summarizing, departing from the Heun confluent equation (36) we have arrived at the ${}_1\mathrm{F}_1$ equation (44) with solution (45) through a process of the form $Heun \rightarrow Abel \rightarrow Abel_{{}_1F_1 solvable} \rightarrow {}_1\mathrm{F}_1$. The three transformations (40), (41) and (43) can be combined into one transformation,

$$y \rightarrow \frac{x^{-(\tau+1)p+1/2}}{\sqrt{x - 1}}\ \exp\left(p\,x + 2\,p^2\,(\tau + \sigma)\int \frac{(x - 1)\,y}{x^2 y'}\,dx\right), \tag{46}$$

which maps the CHE (36) into the ${}_1\mathrm{F}_1$ equation (44) in one step. A closed form solution for the CHE (36) when $\sigma^2 \neq \tau^2$ is then given by (46), where in its "right-hand-side" the value of y is given by (45).

4 Closed form solutions for a subfamily of the GHE

Solutions for the GHE (3) are obtained from (12) by taking $P(y) = y\,(y - 1)$, that is, departing from

$$y' = \frac{y\,(y - 1)}{(s_2\,x^2 + s_1\,x + s_0)\,y + r_2\,x^2 + r_1\,x + r_0} \tag{47}$$

The steps to construct these solutions are the same as those of the previous sections. Using

$$\{x \rightarrow \frac{x\,(1 - x)\,y'}{(s_2 x + r_2)\,y},\ y \rightarrow x\} \tag{48}$$

the AIR equation (47) is transformed into the linear ODE

[7] $\mathbf{M}\,(a, b, z) = z^{b+1/2}\,\mathrm{e}^{-z/2}\,{}_1\mathrm{F}_1(1/2 - a + b;\ 1 + 2\,b;\ z)$.

$$y'' = \frac{\left(s_2\,(s_1-1)\,x^2+((s_1-2)\,r_2+s_2\,r_1)\,x+r_2\,(1+r_1)\right)}{x\,(s_2\,x+r_2)\,\,(x-1)}\,y' - \frac{(s_0\,x+r_0)\,(s_2\,x+r_2)}{x^2\,(x-1)^2}\,y \tag{49}$$

This is a Heun equation of the form (1), with four regular singularities at $\{0, 1, -r_2/s_2, \infty\}$. For $s_2 = 0$, (49) simplifies to a Gauss equation with ${}_2\mathrm{F}_1$ solutions. When $s_2 \neq 0$, in (47) one can take $s_2 = 1$, and, putting $r_2 = -a$, the singularities of (49) are fixed at $\{0, 1, a, \infty\}$, resulting in a non-trivial Heun family depending on four parameters. The relation between the Abel parameters $\{r_i, s_i\}$ and the Heun parameters $\{A, B, D, E, F\}$ is obtained by rewriting (49) in normal form and comparing coefficients, leading to

$$\begin{aligned}
A &= r_0-(2\,r_0+s_0)\,a-\frac{1}{2}\left((r_1+1)\left(s_1+\frac{1}{a}\right)+{r_1}^2-1\right),\\
B &= \frac{r_1+s_1a+1-2\left(A\,a^2+(1-A)\,a\right)}{2\,a\,(a-1)},\\
D &= \frac{1-{r_1}^2}{4}-r_0\,a,\\
E &= (1-a)\left((A+B)^2\,a-(A+B)\,(A+B-1)+\frac{D-A}{a}-\frac{D}{a^2}\right),\\
F &= -\frac{3}{4}
\end{aligned} \tag{50}$$

So, with respect to the most general case (2), the restriction in the GHE (49) under consideration consists of fixing one of the six Heun parameters, F, and expressing another one, E, as a function of those remaining. To derive a solution to (49), the equation is first written in normal form and the parameters $\{s_0, r_0, r_1\}$ are redefined in terms of new parameters $\{\Delta, \sigma, \tau\}$ using

$$s_0 = \frac{{s_1}^2}{4}-\Delta^2, \qquad r_0 = \left(\Delta^2-\frac{{s_1}^2}{4}\right)a-2\,\sigma\,\Delta+s_1\tau, \qquad r_1 = 2\,\tau-a\,s_1 \tag{51}$$

aiming at making explicit that there are only four independent parameters and at separating non-Liouvillian from Liouvillian solutions which will happen at $\sigma = \pm\tau$. The new parameter Δ is related to the Heun parameter A through s_0 and the second equation in (50), and is introduced here to avoid square roots in the transformation formulas[8]. With this notation, the AIR (47) becomes

$$y' = \frac{y\,(y-1)}{(x^2+s_1x+{s_1}^2/4-\Delta^2)\,y-a\,x^2+(2\,\tau-a\,s_1)\,x-({s_1}^2/4-\Delta^2)\,a-2\,\sigma\,\Delta+s_1\tau} \tag{52}$$

and the Heun equation (49) in normal form, at $\{s_2 = 1, r_2 = -a\}$, directly appears expressed in terms of the four irreducible parameters $\{a, \Delta, \sigma, \tau\}$ as

$$\begin{aligned}
y'' = \Bigg(&\frac{2\,a^2\,(a-1)\,\Delta^2-2\sigma\,a\,(2\,a-1)\,\Delta+\left(2\,\tau^2-1/2\right)a+\tau+1/2}{a\,x}\\
&-\frac{2\left(a\,(a-1)^2\,\Delta^2-\sigma\,\,(2\,a-1)\,(a-1)\,\Delta+(\tau-1/2)\,((\tau+1/2)\,a-\tau)\right)}{(a-1)\,(x-1)}+\frac{\tau-a+1/2}{a\,(a-1)\,(x-a)}\\
&+\frac{a^2\Delta^2-2\,a\,\sigma\,\Delta+\tau^2-1/4}{x^2}+\frac{(a-1)^2\,\Delta^2-2\,\sigma\,\,(a-1)\,\Delta+\tau^2-1/4}{(x-1)^2}+\frac{3}{4\,\,(x-a)^2}\Bigg)\,y
\end{aligned} \tag{53}$$

[8]The use of $p^2 = -A$ in sec. 3 brings the same advantage.

4.1 Liouvillian solutions for the GHE (53) when $\sigma = \pm\tau$

When $\sigma = \pm\tau$, Liouvillian solutions for the GHE (53) can be computed using Kovacic's method; for $\sigma = \tau$,

$$y = \frac{x^{\tau - a\Delta + 1/2}\,(x-1)^{(a-1)\Delta - \tau + 1/2}}{\sqrt{a-x}} \tag{54}$$
$$\left(C_1 + \left(\mathbf{B}_x\left(1 + 2\left(a\,\Delta - \tau\right), 2\left(\left(1-a\right)\Delta + \tau\right)\right) - a\,\mathbf{B}_x\left(2\left(a\,\Delta - \tau\right), 2\left(\left(1-a\right)\Delta + \tau\right)\right)\right) C_2\right)$$

where $\mathbf{B}$ (of three arguments - see (6.6.1) in [24]) is the incomplete beta function. A general solution for (53) at $\sigma = -\tau$ is

$$y = \frac{x^{\tau + a\Delta + 1/2}\,(x-1)^{(1-a)\Delta - \tau + 1/2}}{\sqrt{a-x}} \tag{55}$$
$$\left(C_1 + C_2\left(\mathbf{B}_x\left(1 - 2\left(a\,\Delta + \tau\right), 2\left(\left(a-1\right)\Delta + \tau\right)\right) - a\,\mathbf{B}_x\left(-2\left(a\,\Delta + \tau\right), 2\left(\left(a-1\right)\Delta + \tau\right)\right)\right)\right)$$

4.2 A solution in terms of ${}_2\mathrm{F}_1$ functions for the GHE (53) when $\sigma^2 \neq \tau^2$

As in the case of the BHE and CHE equations, a solution for the GHE (53) when $\sigma^2 \neq \tau^2$ is constructed by composing a transformation mapping (53) into the Abel AIR (52) with a transformation mapping (52) into a second order linear equation admitting hypergeometric solutions, in this case of ${}_2\mathrm{F}_1$ type.

The derivation of results below follows the same path shown in the previous sections. Summarizing, the transformation mapping the GHE (53) being solved into the AIR (52) is

$$\left\{x \rightarrow y,\; y \rightarrow \mathrm{e}^{\int \frac{\left((x + (s_1 - 1)/2)y^2 + ((1 - 2\,x - s_1)a + \tau)y + ((s_1/2 + x)a - \tau - 1/2)a\right)y'}{y(y-1)(a-y)}\,dx}\right\} \tag{56}$$

The transformation mapping the AIR (52) into a ${}_p\mathrm{F}_q$ second order linear equation is

$$\left\{x \rightarrow -2\,\frac{\Delta\,(\tau + \sigma)\,y}{x\,(x-1)\,y'} - \frac{s_1}{2} - \Delta,\quad y \rightarrow x\right\} \tag{57}$$

and the resulting equation, of ${}_2\mathrm{F}_1$ type, is

$$y'' = \frac{\left(2\,(\Delta - 1)\,x + 1 - 2\,(a\,\Delta - \tau)\right)}{x\,(x-1)}\,y' + \frac{2\,(x-a)\,\Delta\,(\tau + \sigma)}{x^2\,(x-1)^2}\,y \tag{58}$$

Combining the transformations (56) and (57), a transformation mapping (53) into (58) in one step is

$$y \rightarrow \frac{x^{a\,\Delta + \tau + 1/2}\,(x-1)^{(1-a)\Delta - \tau + 1/2}}{\sqrt{a-x}}\,\exp\left(2\,\Delta\,(\tau + \sigma)\int \frac{(x-a)\,y}{(x-1)^2\,x^2\,y'}\,dx\right) \tag{59}$$

Hence, the solution to the Heun equation (53) to which this section is dedicated is given by the expression above, where, in the "right-hand-side", the value of y is given by the solution to (58), that is,

$$y = x^{a\Delta + \tau - \mathrm{T}}\,(x-1)^{(1-a)\,\Delta + \Sigma - \tau} \tag{60}$$
$$\left({}_2\mathrm{F}_1(\Sigma + \Delta - \mathrm{T}, \Sigma - \Delta + 1 - \mathrm{T};\, 1 - 2\,\mathrm{T};\, x)\,C_1 + x^{2\,\mathrm{T}}\,{}_2\mathrm{F}_1(\Sigma + \Delta + \mathrm{T}, \Sigma - \Delta + 1 + \mathrm{T};\, 1 + 2\,\mathrm{T};\, x)\,C_2\right)$$

where, to make the structure of this solution visible, instead of $\{\sigma, \tau\}$ we are using

$$\Sigma = \left((a-1)^2\,\Delta^2 - 2\,(a-1)\,\sigma\,\Delta + \tau^2\right)^{1/2},\quad \mathrm{T} = \left(a^2\Delta^2 - 2\,a\,\sigma\,\Delta + \tau^2\right)^{1/2} \tag{61}$$

This solution (60) in turn is computed noting that (58) is obtained by changing variables

$$y \to x^{\mathrm{T}-a\,\Delta-\tau}\,(x-1)^{\tau-\Sigma+(a-1)\Delta}\,y \tag{62}$$

in Gauss' ${}_2\mathrm{F}_1$ equation

$$\left(x^2-x\right)y''+\left((\mu+\nu+1)\,x-\rho\right)y'+\mu\,\nu\,y=0 \tag{63}$$

taken at $\{\mu=\Sigma+\Delta-\mathrm{T},\ \nu=\Sigma-\Delta+1-\mathrm{T},\ \rho=1-2\,\mathrm{T}\}$. Hence, the same transformation (62) maps the solution of Gauss' equation into (60).

5 Discussion

In sec. 2, 3 and 4, new solutions in terms of ${}_p\mathrm{F}_q$ functions were derived for families of the Heun equations BHE, CHE and GHE. The approach links linear equations with four regular singularites (and related confluent cases) to linear equations with three regular singularities (and related confluent cases), by linking both types of linear equations to the canonical forms of the Abel AIR class of non-linear first order equations. The link $AIR \leftrightarrow {}_p\mathrm{F}_q$ is developed in [14] and the link $AIR \leftrightarrow Heun$ is shown in sec. 1.

Besides the presentation in sec. 2, 3 and 4, the existence of a connection between the "Heun and related confluent equations" on the one hand and the "AIR (9) and the different possible multiplicities of its roots ρ_i" on the other hand, can be seen more straightforwardly by transforming not (12) but (9) into a linear equation[9], resulting in

$$y''=\left(\frac{1}{x-a}+\frac{R_3}{x-\rho_3}+\frac{R_2}{x-\rho_2}+\frac{R_1}{x-\rho_1}\right)y'+\frac{(s_0x+r_0)\,(a-x)}{(x-\rho_3)^2\,(x-\rho_2)^2\,(x-\rho_1)^2}\,y \tag{64}$$

This is a Heun equation with its four regular singularites at $\{\rho_1,\rho_2,\rho_3,a\}$, where $R_1+R_2+R_3=-3$,

$$R_2=\frac{{\rho_2}^2-(s_1+\rho_3+\rho_1)\,\rho_2+\rho_3\rho_1-r_1}{(\rho_1-\rho_2)\,(\rho_2-\rho_3)} \tag{65}$$

and $\{R_1,R_3\}$ are obtained from R_2 multiplying by -1 and respectively swapping $\rho_2\leftrightarrow\rho_1$ and $\rho_2\leftrightarrow\rho_3$. Through the confluence processes which coalesce singularities in (1), generating the CHE and BHE confluent equations, one coalesces the corresponding singularities ρ_i of (64), generating the same type of confluent equations, and that is equivalent to having multiple roots ρ_i in the AIR (9).

By rewriting (64) in normal form (see (53)), the number of irreducible parameters involved is shown to be four instead of six as in (2). That explains the restrictions on the Heun parameters of the BHE, CHE and GHE families discussed in the previous sections. In the three cases, one parameter appears fixed and another one is dependent of those remaining.

Different than the BHE, CHE and GHE cases, in the case of the DHE (5) and THE (6) the approach considered in this paper does not lead to new solutions. That can be seen by applying to (64) the DHE and THE confluence processes [2], in both cases arriving at equations already of confluent ${}_p\mathrm{F}_q$ type. That status of things is somewhat expected: the AIR class is generated from the three canonical forms (12) and these are, in their general form, already linked to GHE, CHE and BHE families.

Independent of the possibility, here developed, of expressing the solutions to the BHE (18), CHE (36) and GHE (53) normal forms *without* introducing "Heun functions", these functions have been developed consistenly during the last years and will most certainly form part of the standard mathematical language in the near future. That can be inferred from the relevance of Heun equations in applications. In this framework, the results of this paper could be seen as the identification of multi-parameter special cases of Heun functions of the BHE, CHE and GHE types, respectively admitting the integral representations (31), (46) and (59). The mathematical properties as well as the relevance of these special cases in concrete applications still require further investigation.

[9] For that purpose, apply first $\{x\leftrightarrow y\}$ to (9) at $\{s_2=1, r_2=-a\}$, then apply (10) to the resulting Riccati equation.

This link between Heun and ${}_p\mathrm{F}_q$ second order linear equations, established here by way of Abel non-linear equations of first order, seems to be the simplest case of a link between linear equations with N and N-1 singularities, through "Abel AIR like" equations where the numerator of their right-hand-side have degree N-1 at most[10]. For example, if instead of (9) we depart from

$$y' = \frac{(y-\rho_1)\,(y-\rho_2)\,(y-\rho_3)\,(y-\rho_4)}{(s_2\,x^2+s_1\,x+s_0)\,y+r_2\,x^2+r_1\,x+r_0} \tag{66}$$

that is, an equation with structure similar to the AIR (9) but where the numerator of the right-hand-side is of degree four, then by applying $\{x \leftrightarrow y\}$ to obtain a Riccati equation and transforming the latter into a second order linear equation, we obtain an equation similar to (64) but with five regular singular points,

$$y'' = \left(\frac{1}{x-a}+\frac{R_4}{x-\rho_4}+\frac{R_3}{x-\rho_3}+\frac{R_2}{x-\rho_2}+\frac{R_1}{x-\rho_1}\right)y' + \frac{(s_0x+r_0)\,(a-x)}{(x-\rho_4)^2\,(x-\rho_3)^2\,(x-\rho_2)^2\,(x-\rho_1)^2}\,y, \tag{67}$$

which, together with its confluent cases, can be linked through (66) to the Heun equation and confluent cases, using the same approach presented in the previous sections.

In an analogous way, n^{th} order ($n>2$) linear equations in $y(x)$ can also be reduced to "Riccati like" non-linear equations of order $n-1$, due to their invariance under scalings of y. It is therefore reasonable to expect that a link equivalent to the one developed in this work also exists between linear equations with N and N-1 singularities in the n^{th} ($n>2$) order case.

6 Appendix

In sec. 2, 3 and 4, solutions were derived for the equations in *normal form* BHE (18), CHE (36) and GHE (53). Given a second order linear ODE

$$y''+c_1y'+c_0y=0 \tag{68}$$

where the $c_i \equiv c_i(x)$, the corresponding normal form,

$$y''-\left(({c_1}^2+2\,{c_1}')/4-c_0\right)y=0 \tag{69}$$

is obtained by changing $y \rightarrow \exp(-\int c_1, dx/2)\,y$.

Regarding Heun equations, one advantage of the normal form is that the general or confluent type of the equation is evident in the partial fraction decomposition of the coefficient of y (see eqs. (2) to (6)). Also, two different equations related by $y \rightarrow P(x)\,y$ have the same normal form and recognizing this equivalence is relevant for computational purposes. On the other hand, for different reasons, special functions are frequently defined as solutions to equations in *canonical form*. This appendix relates the normal and canonical forms of the BHE, CHE and GHE, expressed in terms of irreducible parameters[11], following the notation of [2], thus permitting a simple translation of the results presented.

The Biconfluent Heun equation

The BHE canonical form is given in terms of four constant parameters $\{\alpha,\beta,\gamma,\delta\}$ by

$$y''+\left(\frac{1+\alpha}{x}-\beta-2\,x\right)y'+\left(\gamma-\alpha-2-\frac{\delta+(1+\alpha)\,\beta}{2\,x}\right)y=0; \tag{70}$$

The BHE in normal form (4) in terms of four parameters $\{B,C,D,E\}$ is

[10] For Abel equations of the first kind, $N=3$; for Abel equations of the second kind like (7), $N\leq 3$.
[11] A few of the equations shown in the paper are repeated here for ease of reading.

$$y'' + \left(-x^2 + B\,x + C + \frac{D}{x} + \frac{E}{x^2}\right) y = 0 \tag{71}$$

The BHE normal form (18) solved in sec. 2, there written in terms of two parameters $\{\sigma, \tau\}$, is

$$y'' - \left(x^2 + 2\,\sigma\,x + \tau^2 + \frac{\tau}{x} + \frac{3}{4\,x^2}\right) y = 0 \tag{72}$$

The parameters $\{B, C, D, E\}$ in (71) are related to $\{\sigma, \tau\}$ by

$$B = -2\,\sigma, \quad C = -D^2, \quad D = -\tau, \quad E = -3/4 \tag{73}$$

The parameters $\{\alpha, \beta, \gamma, \delta\}$ in (70) are related to $\{B, C, D, E\}$ by

$$\alpha^2 = -4\,E + 1, \quad \beta = -B, \quad \gamma = B^2/4 + C, \quad \delta = -2\,D \tag{74}$$

So the parameters $\{\alpha, \beta, \gamma, \delta\}$ are related to $\{\sigma, \tau\}$ by

$$\alpha^2 = 4, \quad \beta = 2\,\sigma, \quad \gamma = \sigma^2 - \tau^2, \quad \delta = 2\,\tau \tag{75}$$

At these values of $\{\alpha, \beta, \gamma, \delta\}$, for $\sigma = \pm\tau$, (70) admits Liovillian solutions, and for $\sigma^2 \neq \tau^2$ the solution is obtained from (31).

The Confluent Heun equation

The CHE canonical form is given in terms of five constant parameters $\{\alpha, \beta, \gamma, \delta, \eta\}$ by

$$y'' + \left(\alpha + \frac{\beta + 1}{x} + \frac{\gamma - 1}{x - 1}\right) y' + \frac{(2\,\delta + \alpha\,(\beta + \gamma + 2))\,x + 2\,\eta + \beta + (\gamma - \alpha)\,(\beta + 1)}{2\,x\,(x - 1)}\, y = 0 \tag{76}$$

The CHE in normal form (3) in terms of five parameters $\{A, B, C, D, E\}$ is

$$y'' + \left(A + \frac{B}{x} + \frac{C}{x - 1} + \frac{D}{x^2} + \frac{E}{(x - 1)^2}\right) y = 0 \tag{77}$$

The CHE normal form (36) solved in sec. 3, there written in terms of three parameters $\{p, \sigma, \tau\}$, is

$$y'' - \left(p^2 + \frac{2\,(\sigma - 1)p^2 - \tau\,p + 1/2}{x} + \frac{\tau p - 1/2}{x - 1} + \frac{(\tau^2 - 2\,\sigma + 1)\,p^2 - 1/4}{x^2} + \frac{3}{4\,(x - 1)^2}\right) y = 0 \tag{78}$$

The parameters $\{A, B, C, D, E\}$ in (77) are related to $\{p, \sigma, \tau\}$ by

$$A = -p^2, \quad B = 2\,(1 - \sigma)\,p^2 + \tau\,p - \frac{1}{2}, \quad C = \frac{1}{2} - \tau\,p, \quad D = \frac{1}{4} + \left(2\,\sigma - \tau^2 - 1\right) p^2, \quad E = -3/4, \tag{79}$$

The parameters $\{\alpha, \beta, \gamma, \delta, \eta\}$ in (76) are related to $\{A, B, C, D, E\}$ by

$$\alpha^2 = -4\,A, \quad \beta^2 = -4\,D + 1, \quad \gamma^2 = 4\,\gamma - 4\,E - 3 \quad \delta = C + B - \alpha, \quad \eta = -\frac{1}{2} - B - \beta \tag{80}$$

So the relation between $\{\alpha, \beta, \gamma, \delta, \eta\}$ and $\{p, \sigma, \tau\}$ is

$$\alpha^2 = 4\,p^2, \quad \beta^2 = 4\,\left(1 - 2\,\sigma + \tau^2\right) p^2 \quad \gamma^2 = 4\,\gamma, \quad \delta = 2\,(1 - \sigma)\,p^2 - \alpha, \quad \eta = (2\,\sigma - 2)\,p^2 - \tau\,p - \beta \tag{81}$$

At these values of $\{\alpha, \beta, \gamma, \delta, \eta\}$, the CHE (76) admits Liouvillian solutions for $\sigma = \pm\tau$, and for $\sigma^2 \neq \tau^2$ the solution is obtained from (46).

The General Heun equation

Following [2] the GHE canonical form is written in terms of seven constant parameters $\{a, \alpha, \beta, \gamma, \delta, \eta, h\}$ as

$$y'' + \left(\frac{\alpha}{x} + \frac{\beta}{x-1} + \frac{\gamma}{x-a}\right) y' + \frac{\delta\,\eta\,(x-h)}{x\,(x-1)\,(x-a)}\, y = 0 \tag{82}$$

where $\alpha + \beta + \gamma = \delta + \eta + 1$ and $a \neq 0, a \neq 1$. The GHE in normal form (2) in terms of six parameters $\{a, A, B, D, E, F\}$ is

$$y'' + \left(\frac{A}{x} + \frac{B}{x-1} - \frac{A+B}{x-a} + \frac{D}{x^2} + \frac{E}{(x-1)^2} + \frac{F}{(x-a)^2}\right) y = 0 \tag{83}$$

The GHE normal form solved in sec. 4 (see (53)), written in terms of four parameters $\{a, \sigma, \tau, \Delta\}$, is

$$\begin{aligned} y'' = \Bigg(& \frac{2\,a^2\,(a-1)\,\Delta^2 - 2\sigma\,a\,(2\,a-1)\,\Delta + \left(2\,\tau^2 - 1/2\right) a + \tau + 1/2}{a\,x} \\ & - \frac{2\left(a\,(a-1)^2\,\Delta^2 - \sigma\,\,(2\,a-1)\,(a-1)\,\Delta + (\tau - 1/2)\,((\tau + 1/2)\,a - \tau)\right)}{(a-1)\,(x-1)} + \frac{\tau - a + 1/2}{a\,(a-1)\,(x-a)} \\ & + \frac{a^2\Delta^2 - 2\,a\,\sigma\,\Delta + \tau^2 - 1/4}{x^2} + \frac{(a-1)^2\,\Delta^2 - 2\,\sigma\,\,(a-1)\,\Delta + \tau^2 - 1/4}{(x-1)^2} + \frac{3}{4\,\,(x-a)^2}\Bigg)\, y \end{aligned} \tag{84}$$

The parameters $\{A, B, D, E, F\}$ in (83) are related to $\{a, \sigma, \tau, \Delta\}$ by

$$\begin{aligned} A &= -2\,a\,(a-1)\,\Delta^2 + 2\,(2\,a-1)\,\sigma\,\Delta - 2\,\tau^2 - \frac{\tau + 1/2}{a} + \frac{1}{2}, \\ B &= 2\,a\,(a-1)\,\Delta^2 - 2\,(2\,a-1)\,\sigma\,\Delta + 2\,\tau^2 + \frac{\tau - a/2}{a-1}, \\ D &= -a^2\Delta^2 + 2\,a\,\sigma\,\Delta - \tau^2 + 1/4, \\ E &= -\,(a-1)^2\,\Delta^2 + 2\,(a-1)\,\sigma\,\Delta - \tau^2 + 1/4, \\ F &= -3/4 \end{aligned} \tag{85}$$

The parameters $\{\alpha, \beta, \gamma, \delta, \eta, h\}$ in (82) are related to $\{a, A, B, D, E, F\}$ by

$$\begin{aligned} \alpha^2 &= -4\,D + 2\,\alpha \\ \beta^2 &= -4\,E + 2\,\beta \\ \gamma^2 &= -4\,F + 2\,\gamma, \\ \delta^2 &= (-1 + \alpha + \beta + \gamma)\,\delta + 1/2\,\,(-\beta - \alpha)\,\gamma - 1/2\,\alpha\,\beta + B\,(-1 + a) + Aa \\ \eta &= \alpha + \beta + \gamma - \delta - 1 \\ h &= \frac{(\gamma + \beta\,a)\,\alpha + 2\,a\,A}{(\gamma + \beta)\,\alpha + (2\,A + 2\,B)\,a - 2\,B - \beta\,\gamma}, \end{aligned} \tag{86}$$

So the parameters $\{\alpha, \beta, \gamma, \delta, \eta, h\}$ are related to $\{a, \sigma, \tau, \Delta\}$ by

$$\begin{aligned} \alpha^2 &= 2\,(2\,a^2\Delta^2 - 4\,a\,\sigma\,\Delta + 2\,\tau^2 + \alpha) - 1 \\ \beta^2 &= 2\,(2\,(a-1)^2\,\Delta^2 + 4\,\sigma\,(1-a)\,\Delta + \beta + 2\,\tau^2) - 1 \end{aligned}$$

$$
\begin{aligned}
\gamma^2 &= 2\,\gamma+3 \\
\delta^2 &= 2\,a\,(1-a)\,\Delta^2+2\,(2\,a-1)\,\sigma\,\Delta+(\alpha+\beta+\gamma-1)\,\delta-2\,\tau^2-(\alpha\,\beta+1+(\alpha+\beta)\,\gamma)/2 \\
\eta &= \alpha+\beta+\gamma-\delta-1 \\
h &= \frac{4\,a^3\,(a-1)\,\Delta^2+4\,\sigma\,a\,(1-2\,a)\,\Delta+\alpha\,\gamma+1+\left(\alpha\,\beta+4\,\tau^2-1\right)a+2\,\tau}{4\,a\,(a-1)\,\Delta^2+\sigma\,4\,(1-2\,a)\,\Delta+(\alpha+\beta)\,\gamma+\alpha\,\beta+4\,\tau^2+1}
\end{aligned}
\tag{87}
$$

At these values of $\{\alpha,\beta,\gamma,\delta,\eta,h\}$, for $\sigma=\pm\tau$ the GHE (82) admits Liouvillian solutions, and for $\sigma^2\neq\tau^2$ the solution is obtained from (59).

Acknowledgments

This work was developed in the framework of the MITACS project and supported by both the Centre of Experimental and Constructive Mathematics, Simon Fraser University and the Maplesoft division of Waterloo Maple Inc. The author would like to thank K. von Bülow for a careful reading of this paper.

References

[1] "Heun's Differential Equations", Edited by A. Ronveaux, Oxford University Press (1995).

[2] A. Decarreau, M.Cl. Dumont-Lepage, P.Maroni, A.Robert, A.Ronveaux, "Formes Canoniques de Équations confluentes de l'équation de Heun", Annales de la Société Scientifique de Bruxelles, T. 92, I-II, pp. 53-78 (1978).

[3] A.O. Starinets, "Quasenormal modes of near extremal black branes", Physical Review D **66**, 124013 (2002).

[4] O.I. Tolstikhin, M. Matsuzawa, "Hyperspherical elliptic harmonics and their relation to the Heun equation", Physical Review A **63**, 032510 (2001).

[5] L. Anguelova, P. Langfelder, "Massive gravitino propagator in maximmaly symmetric spaces and fermions in dS/CFT", Journal of High Energy Physics 057 (2003).

[6] F. Finkel, A. González-López, A. L. Maroto, and Miguel Á. Rodríguez, "The Lamé equation in parametric resonance after inflation", Physical Review D **62** 103515 (1-7) (2000).

[7] S.Y. Slavyanov and W.Lay. "Special Functions, A Unified Theory Based on Singularities", Oxford Mathematical Monographs (2000).

[8] P.A. Clarkson, P.J. Olver, "Symmetry and Chazy Equation", Journal of Differential Equations 124, 225-246 (1996).

[9] A. Ronveaux, "Factorization of the Heun's differential operator", Applied Mathematics and Computation 141, 177-184 (2003).

[10] R.S. Maier, "Transforming the Heun equation to the Hypergeometric equation: 1. Polynomial Transformations", Foundations of Computational Mathematics '02, (2002).

[11] A.O. Smirnov, "Elliptic solitons and Heun's equation", CRM Proceedings and Lecture Notes v.32, 287-305 (2002).

[12] H. Exton, "A new solution of the biconfluent Heun equation", Rendiconti di Mathematica Serie VII, Volume 18, 615-622 (1998).

[13] E.S. Cheb-Terrab, A.D. Roche, "An Abel ODE class generalizing known integrable classes", European Journal of Applied Mathematics, Vol. 14, No. 2, 217-229 (2003).

[14] E.S. Cheb-Terrab, “A connection between Abel and ${}_pF_q$ hypergeometric differential equations”, submitted to the European Journal of Applied Mathematics (http://arxiv.org/math-ph/0402040).

[15] L. Gagnon, P. Winterniz, “Non-Painlevé reductions of nonlinear Schrödinger equations”, Physical Review A **42** 8 (1990).

[16] E. Kamke, “Differentialgleichungen”, N.Y. Chelsea Publ. Co. (1947).

[17] A.D. Polyanin, V.F. Zaitsev, “Handbook of Exact Solutions for Ordinary Differential Equations”. CRC Press, Boca Raton (1995).

[18] P.L. Sachdev, “A Compendium of Nonlinear Ordinary Differential Equations”. John Wiley & Sons (1997).

[19] E.S. Cheb-Terrab, A.D. Roche, “Abel Equations: Equivalence and Integrable Classes”, Computer Physics Communications 130 (2000).

[20] N.H. Abel, “Oeuvres Complétes II”. S.Lie and L.Sylow, Eds., Christiana, 1881.

[21] R. Liouville, “Sur une équation différentielle du premier ordre”, Acta Mathematica **26**, 55-78 (1902). See also R. Liouville, Comptes Rendus **103**, 476-479 (1886).

[22] P. Appell, “Sur les invariants de quelques équations différentielles”, Journal de Mathématique **5**, 361-423 (1889).

[23] A. Duval, “The Kovacic algorithm with applications to special functions”, Differential Equations and Computer Algebra, edited by M.Singer, Academic Press, London, 113-130 (1991).

[24] M. Abramowitz and I. A. Stegun, “Handbook of mathematical functions”, Dover (1964).