Theory and Algorithms for Trigonometric Polynomials.

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We describe some results for simplifying ratios of trigonometric polynomials. This Maple worksheet highlights the application of some results in a paper of Monagan and Mulholland which is in preparation. Included in the paper are other results on trigonometric polynomials, including results and algorithms for factoring trigonometric polynomials.

Consider the problem of simplifying

```
> f := 2*s*c^2/(s+c+1)/(s+c-1);
```

$$f := 2\,\frac{s\,c^2}{(s+c+1)\,(s+c-1)}$$

where $s = \sin(x)$ and $c = \cos(x)$. Our code for simplifying such expressions simplifies $\mathrm{f}(s, c)$ to c .

```
> read `trigpoly.mpl`;
> trig_simplify(f,s,c);
```

$$c$$

Therefore it must be that $2\,s\,c = (s+c+1)\,(s+c-1)$ where, remember, $s^2 + c^2 = 1$ because $\sin(x)^2 + \cos(x)^2 = 1$. To understand how to this in general we must take into account the relation $s^2 + c^2 = 1$ and hence we are led to study the following ring.

Let $\mathrm{R} = \mathrm{Q}[s, c]/\langle s^2 + c^2 - 1\rangle$ be the ring of polynomials in s, c modulo $s^2 + c^2 - 1$.
R consists of a set of equivalence classes where $\mathrm{a}(s, c)$ and $\mathrm{b}(s, c)$ are in the same equivalence class iff $s^2 + c^2 - 1$ divides their difference. We choose a canonical representative for each equivalence class of the form $\mathrm{p}(c)\,s + \mathrm{q}(c)$, that is, polynomials of degree at most one in $s = \sin(x)$. We obtain the unique representative by simply replacing s^2 by $1 - c^2$, for example:

```
> g := s^3+c^3+s^4-c^4;
```

$$g := s^3 + c^3 + s^4 - c^4$$

```
> algsubs(s^2=1-c^2,g);
```

$$(1 - c^2)\,s + c^3 + 1 - 2\,c^2$$

We define **tdeg**($\mathrm{a}(s, c)$), the trigonometric degree of a trigonometric polynomial $\mathrm{a}(s, c)$, to be the total degree of its canonical representative. Thus the trigonometric degree of our example polynomial is 3 not 4.

```
> n := trig_degree(g,s,c);
```

$$n := 3$$

One can show that $\mathrm{tdeg}(f\,g) = \mathrm{tdeg}(f) + \mathrm{tdeg}(g)$ holds in R from which it follows that R is an integral domain. It is obvious that R is not a unique factorization domain because, as is implicit in the simplification of $\mathrm{f}(s, c)$ above, $2\,s\,c = (c+s+1)\,(c+s-1)$ in R. To determine factorizations and greatest common divisors in R we make use the tan half angle map which we denote by $\Psi_t(\mathrm{f}(s, c))$. The complex exponential map could also be used but it leads to unecessary complex arithmetic. Let $t = \tan(x/2)$. Recall that $\Psi_t(\mathrm{f}(s, c))$, takes $(s, c) \rightarrow \left(\frac{2\,t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)$. One can show that $\Psi_t(\mathrm{a}(s, c))$ is a bijection between the equivalence classes of R and a class of rational functions in t. More precisely, we have proven the following lemma. Applying this lemma enables us to find all factorizations of a trigonometric polynomial.

Lemma 1. Let $\mathrm{a}(t)$ be any element of $Q[t]$ of degree $0 \le d$ satisfying (i) $1 + t^2$ does not divide $\mathrm{a}(t)$ and (ii) $d \le 2\,n$. Then there exists a unique equivalence class $[\mathrm{f}(s, c)]$ in $Q[s,c]/\langle\, s^2 + c^2 - 1 \,\rangle$ such that
$\Psi_t([\mathrm{f}(s, c)]) = \dfrac{\mathrm{a}(t)}{(1 + t^2)^n}$.

We have worked out how to find the trigonometric polynomial $\mathrm{f}(s, c)$ given $\mathrm{a}(t)$ and n satisfying these conditions, that is, how to invert Ψ_t. For example, the trigonometric polynomial of trigonometric degree 2 whose image is $4\,t^3 + 3$ is

```
> convert_to_sc( 4*t^3+4,t,s,c,2);
```

$$c^2 - s\,c + 2\,c + s + 1$$

The next result is an application of the Lemma. It tells us when a trigonometric polynomial is irreducible.

Theorem 1. Let $\mathrm{f}(s, c)$ be an element of $R = Q[s, c]/\langle\, s^2 + c^2 - 1 \,\rangle \setminus Q$ with trigonometric degree d. Then $\mathrm{f}(s, c)$ is irreducible in R if and only if
(i) if $1 < d$ then $1 + c$ does not divide $\mathrm{f}(s, c)$ in $Q[s, c]$ and
(ii) either $(1 + t^2)^n\, \Psi_t(\mathrm{f}(s, c))$ is irreducible in $Q[t]$ or $(1 + t^2)^n\, \Psi_t(\mathrm{f}(s, c)) = \mathrm{a}(t)\,\mathrm{b}(t)$ where $\mathrm{a}(t)$ and $\mathrm{b}(t)$ are both irreducible in $Q[t]$ and both of odd degree.

Now consider the following problem where we want to simplify the ratio of *a/b* .

```
> a :=
  (50*c^5+70*c^6+120*c^3+40*c^4+662*c^2+418*c+20*c^7+156)*s+35*c^8+325
  *c^7+168+1981*c^2+1255*c^6+2865*c^5+4305*c^4+3791*c^3+635*c;
```

$$a := (50\,c^5 + 70\,c^6 + 120\,c^3 + 40\,c^4 + 662\,c^2 + 418\,c + 20\,c^7 + 156)\,s + 35\,c^8 + 325\,c^7 + 168 + 1981\,c^2 + 1255\,c^6 + 2865\,c^5 + 4305\,c^4 + 3791\,c^3 + 635\,c$$

```
> b :=
  (-5*c^5-95*c^4-25*c^3+805*c^2+75*c^6+549*c+25*c^7+207)*s+50*c^8+455*
  c^7+225+1745*c^6+6025*c^4+3995*c^5+2771*c^2+5365*c^3+873*c;
```

$$b := (-5\,c^5 - 95\,c^4 - 25\,c^3 + 805\,c^2 + 75\,c^6 + 549\,c + 25\,c^7 + 207)\,s + 50\,c^8 + 455\,c^7 + 225 + 1745\,c^6 + 6025\,c^4 + 3995\,c^5 + 2771\,c^2 + 5365\,c^3 + 873\,c$$

How can we simplify *a/b* ? An obvious approach is to compute *g* = GCD(*a*,*b*) in R and output (*a/g*)/(*b/g*). We need to define and compute the trigonometric greatest common divisor (GCD). Our code computes the following polynomial for the GCD of $\mathrm{a}(s, c)$ and $\mathrm{b}(s, c)$.

```
> g := trig_gcd(a,b,s,c,'abar','bbar');
```

$$g := 5\,c^6 + 5\,s\,c^5 + 50\,c^5 + 155\,c^4 + 15\,s\,c^4 + 220\,c^3 - 10\,s\,c^3 - 30\,s\,c^2 + 235\,c^2 + 82\,c + 5\,s\,c + 21 + 15\,s$$

Dividing out by this GCD

```
> trig_divide(a,g,s,c,'abar');
  trig_divide(b,g,s,c,'bbar');
```

true

true

we have that *a/b* may be simplified to

```
> normal(abar/bbar);
```

$$\frac{11\,c^2 - 3\,s\,c + 18\,c + 11 + 7\,s}{15\,c^2 + 26\,c - 5\,s\,c + 15 + 9\,s}$$

We explain briefly how we obtained this result. The tan half angle map of $\mathrm{a}(s, c)$ is

```
> at := factor(convert_to_t(a,s,c,t));
```

$$at := 128\,\frac{(t+5)\,(t^2+2)\,(t^3+2)\,(t^5+2)\,(t^5+3)}{(1+t^2)^8}$$

How could $a(s, c)$ factor? Applying Lemma 1, if we take the first factor $t+5$ of odd degree, and put one $1+t^2$ factor under it to reconstruct a trigonometric polynomial, we will not have enough $1+t^2$ factors to reconstruct a trigonometric polynomial from the remaining factors in the numerator. It follows that the odd degree factor $t+5$ must be combined with another odd degree factor. There are three distinct ways the factors of odd degree may be combined, namely

$$(t+5)\,(t^3+2),\ (t^5+2)\,(t^5+3)$$

$$(t+5)\,(t^5+2),\ (t^3+2)\,(t^5+3)$$

$$(t+5)\,(t^5+3),\ (t^3+2)\,(t^5+2)$$

This yields three distinct factorizations each of which will have three factors, one of which is the factor $c+3$ coming from the t^2+2 image factor of even degree. The first factorization is

```
> normal(128*
  convert_to_sc((t+5)*(t^3+2),t,s,c,2) *
  convert_to_sc((t^5+2)*(t^5+3),t,s,c,5) *
  convert_to_sc(t^2+2,t,s,c,1));
```

$$\frac{1}{2}(11\,c^2+18\,c-3\,s\,c+7\,s+11)\,(5\,c^5+35\,c^4+5\,s\,c^4+50\,c^3+70\,c^2-10\,s\,c^2+25\,c+7+5\,s)\,(c+3)$$

Now consider the GCD problem where we have

```
> 'Psi[t](a)' = factor( convert_to_t(a,s,c,t) );
```

$$\Psi_t(a) = 128\,\frac{(t+5)\,(t^2+2)\,(t^3+2)\,(t^5+2)\,(t^5+3)}{(1+t^2)^8}$$

```
> 'Psi[t](b)' = factor( convert_to_t(b,s,c,t) );
```

$$\Psi_t(b) = 128\,\frac{(t+7)\,(t^2+2)\,(t^3+2)\,(t^5+2)\,(t^5+3)}{(1+t^2)^8}$$

One would like to choose the image of the GCD to be $(t^2+2)\,(t^3+2)\,(t^5+2)\,(t^5+3)\,/\,(1+t^2)^8$. But since we are forced to pair up odd degree factors to obtain images of trigonometric polynomials, we have to omit one of the odd degree image factors from the GCD. Therefore we have three choices for greatest common divisors, namely

```
> g1 := 32*convert_to_sc( (t^2+2)*(t^3+2)*(t^5+2), t,s,c,5 );
```

$$g1 := 5\,c^5+27\,c^4+66\,c^3-4\,s\,c^3+102\,c^2-12\,s\,c^2+41\,c+4\,s\,c+15+12\,s$$

```
> g2 := 32*convert_to_sc( (t^2+2)*(t^3+2)*(t^5+3), t,s,c,5 );;
```

$$g2 := 7\,c^5-s\,c^4+41\,c^4+102\,c^3-8\,s\,c^3-14\,s\,c^2+146\,c^2+67\,c+8\,s\,c+15\,s+21$$

```
> g3 := 64*convert_to_sc( (t^2+2)*(t^5+2)*(t^5+3), t,s,c,6 );
```

$$g3 := 5\,c^6+5\,s\,c^5+50\,c^5+155\,c^4+15\,s\,c^4+220\,c^3-10\,s\,c^3-30\,s\,c^2+235\,c^2+82\,c+5\,s\,c+21+15\,s$$

In the context of simplifying *a/b* we would want to cancel out a GCD of greatest trigonometric degree, hence we choose g3. This is the GCD that our program computed. It is not unique in general. But Ψ_t

(a/b) is also equal to

```
> 'Psi[t](a/b)' = simplify(
  convert_to_t(a,s,c,t)/convert_to_t(b,s,c,t));
```

$$\Psi_t\left(\frac{a}{b}\right) = \frac{t+5}{t+7}$$

Multiplying the numerator and denominator by an artificial $1 + t^2$ factor we can write *a/b*

```
> 'a/b' =
  normal(convert_to_sc(t+5,t,s,c,1)/convert_to_sc(t+7,t,s,c,1));
```

$$\frac{a}{b} = \frac{5\,c + s + 5}{7\,c + s + 7}$$

This is the result we want when simplifying *a/b* . It is the result that our simplification program computes.

```
> trig_simplify(a/b,s,c);
```

$$\frac{5\,c + s + 5}{7\,c + s + 7}$$

Note, it does not correspond to cancelling out a common divisor which is a trigonometric polynomial. This was a surprise to us. Thus the GCD problem is not always the same as simplifying a ratio of trigonometric polynomials.