River Flow Computation Using Composite Numerical Integration

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Introduction

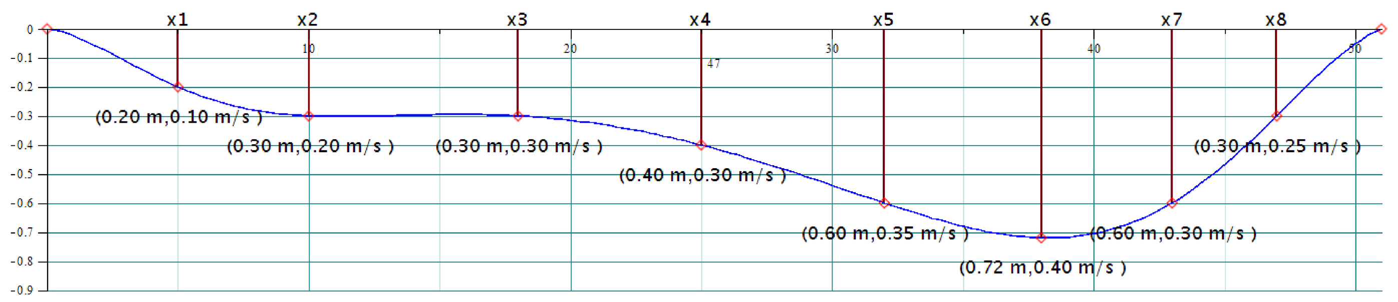

Figure 1: cross section of a river. measured data point: (x, d, v)

River flow (discharge) is the volume of water which flows through a cross section of a river per unit of time. It is commonly expressed in cubic meters per second. The data values (x, d, v) in Figure 1 mean at position x, the river is d meters deep, and the average velocity at position x is $v\ m/s$. The velocity is measured at a depth of 40% from the river bed. In deeper rivers(> 0.75 meters), we take the average of the two velocities at depths 20% and 80%.

In general, river flow is computed by multiplying the area of the cross section by the average velocity of the water in that cross section. The main goal of this project is to find accurate methods to calculate the river discharge from data measurements. We have researched and studied the existing methods, and developed new approximation methods.

Methods Used in Practice

A Trapezoidal Rule Estimate

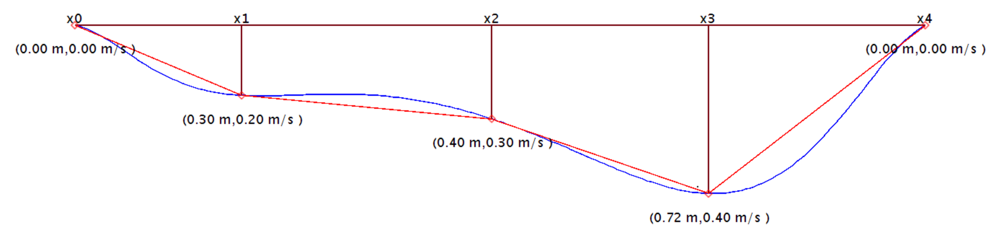

Figure 2: trapezoidal rule. measured data point: (x, d, v)

$$\textit{flow in a section} \approx \textit{trapezoid area} \times \textit{average velocity in that section}$$
$$= [(x_{i+1} - x_i)\frac{d_{i+1} + d_i}{2}] \times \frac{v_{i+1} + v_i}{2}$$

$$\textit{Total Flow} \approx T_n = \sum_{i=0}^{n-1}\left([(x_{i+1} - x_i)\frac{d_{i+1} + d_i}{2}] \times \frac{v_{i+1} + v_i}{2}\right)$$

A Midpoint Rule Estimate

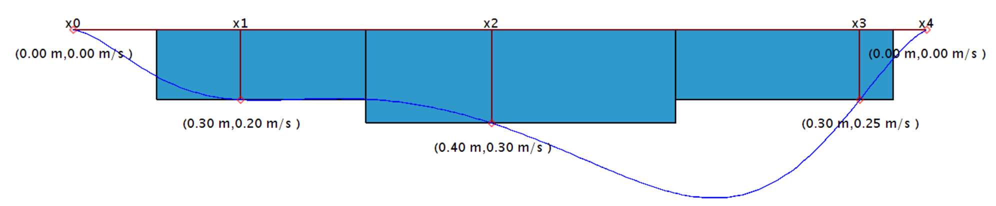

Figure 3: midpoint rule. measured data point: (x, d, v)

$$\textit{flow in a section} \approx \textit{rectangle area} \times \textit{average velocity in that section}$$
$$= [(\frac{x_i - x_{i-1}}{2} + \frac{x_{i+1} - x_i}{2}) \times d_i] \times v_i$$

$$\textit{Total Flow} \approx M_n = \sum_{i=1}^{n-1}\left([(\frac{x_{i+1} - x_{i-1}}{2}) \times d_i] \times v_i\right)$$

Exact River Flow

If we had analytical functions $d(x)$ for the depth at position x, and $v(x)$ for the velocity at position x, then the total flow from Calculus is given by the integral on the interval $[x_0, x_n]$.

$$\textit{exact total flow} = \int_{x_0}^{x_n} d(x) \times v(x)\, dx$$

However, in practice we have a finite number of data measurements so we use approximation methods.

Piecewise Approximation Method

Our approach used piecewise approximation. It involved estimating river flow by first generating the approximate $d_i(x)$ any $v_i(x)$ separately on a sequence of intervals. Then it applied the composite numerical integration to find the approximate flow.

$$\textit{estimated total flow} = \sum \int_{xL}^{xR} d_i(x) \times v_i(x)\, dx$$

The following three methods are the approximate methods to find the $d_i(x)$ and $v_i(x)$.

New Trapezoidal Rule

We noticed that the Trapezoidal rule is biased; it underestimates the flow on each section when the depth and velocity on one side is lower than the other side of the section. This is because there is more flow on the deeper faster side. To correct this bias we compute for a section (xL, dL, vL) to (xR, dR, vR) a formula for the following integral

$$\textit{flow in section} = \int_{xL}^{xR} D(x)V(x)dx = \frac{(xR - xL)}{6} \times (2dLvL + dLvR + dRvL + 2dRvR)$$

where $D(x) = dL + (dR - dL)/(xR - xL)x$ is the linear interpolant for the depth on the section and $V(x) = vL + (vR - vL)/(xR - xL)x$ is the linear interpolant for the velocity.

Simpson's Rule

Instead of approximating the depth and the velocities by straight lines, Simpson's rule approximates the depth and velocity on two sections, so three consecutive points, with a quadratic polynomial. Those three points are not necessarily equally spaced. The formula is complicated but easily computed with Maple.

Quartic Rule

An even more accurate rule is to interpolate five consecutive points with a quartic polynomial for the depth $d(x)$ and velocity $v(x)$. Again the formulas, although complicated, are easily computed using Maple's interp command as follows:

$$\text{interp}([x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}], [d_i, d_{i+1}, d_{i+2}, d_{i+3}, d_{i+4}]).$$

Experimental Data

To measure the accuracy of the methods we chose functions for the depth $d(x)$ and velocity $v(x)$ for which we can compute the exact flow using $\textit{Flow} = \int_{x_0}^{x_n} d(x)v(x)\, dx$.

Sample Data 1: Equally Spaced Intervals

$$d(x) = \frac{1}{400}(x-20)(x+20) \quad v(x) = \frac{1}{2.304 \times 10^9}(x^2-400)(x^2-1600)(x^2-3600) \quad \text{for } -20 \le x \le 20$$

	Trapezoid		Midpoint		New Trap		Simpson's		Quartic	
n	error	ratio	error	ratio	error	ratio	error	ratio	error	ratio
4	2.9661011	3.46	0.0071167	36.7	1.9797733	3.50	0.1672730	6.41	0.0070547	NA
8	0.7676737	3.87	0.0003265	21.8	0.5112671	3.87	0.0122843	13.6	0.0010058	7.01
16	0.1932586	3.97	0.0000187	17.5	0.1288453	3.97	0.0007892	15.6	0.0000201	50.1
32	0.0484130	3.99	0.0000011	16.4	0.0322757	3.99	0.0000496	15.9	0.0000003	60.7
64	0.0121094	4.00	0.0000001	16.1	0.0080730	4.00	0.0000031	16.0	0.0000000	63.2
128	0.0030277	4.00	0.0000000	16.0	0.0020185	4.00	0.0000002	16.0	0.0000000	63.8
256	0.0075696	4.00	0.0000000	16.0	0.0005046	4.00	0.0000000	16.0	0.0000000	64.0
512	0.0001892	4.00	0.0000000	16.0	0.0001262	4.00	0.0000000	16.0	0.0000000	64.0
1024	0.0000473	4.00	0.0000000	16.0	0.0000315	4.00	0.0000000	16.0	0.0000000	64.0

Table 1: Exact Flow = $\int_{-20}^{20} d(x)v(x)\, dx = 20.26102293\ m^3/s$

Sample Data 2: Equally Spaced Intervals(Sine Functions)

$$d(x) = \sin\left(\frac{\pi(x+20)}{40}\right) \quad v(x) = \sin\left(\frac{\pi(x+20)}{40}\right) \quad \text{for } -20 \le x \le 20$$

	Trapezoid		Midpoint		New Trap		Simpson's		Quartic	
n	error	ratio	error	ratio	error	ratio	error	ratio	error	ratio
4	2.0928932	3.41	0.0000000	NA	1.9526214	3.41	0.2287638	5.83	0.0078731	NA
8	0.7612047	3.84	0.0000000	NA	0.5074698	3.85	0.0154515	14.8	0.0014386	5.47
16	0.1921472	3.96	0.0000000	NA	0.1280981	3.97	0.0009845	15.7	0.0000237	60.4
32	0.0481527	3.99	0.0000000	NA	0.0321018	3.99	0.0000618	15.9	0.0000004	63.1
64	0.0120454	4.00	0.0000000	NA	0.0080302	4.00	0.0000039	15.9	0.0000000	63.8
128	0.0030118	4.00	0.0000000	NA	0.0020079	4.00	0.0000002	16.0	0.0000000	63.9
256	0.0007530	4.00	0.0000000	NA	0.0005020	4.00	0.0000000	16.0	0.0000000	64.0
512	0.0001882	4.00	0.0000000	NA	0.0001255	4.00	0.0000000	16.0	0.0000000	64.0
1024	0.0000471	4.00	0.0000000	NA	0.0000314	4.00	0.0000000	16.0	0.0000000	64.0

Table 2: Exact Flow = $\int_{-20}^{20} d(x)v(x)\, dx = 20\ m^3/s$

Sample Data 3: Randomly Spaced Intervals

Data points x_i are taken at random from the middle half of each interval to simulate realistic data.

For this experiment $-20 \le x \le 20$ and

$$d(x) = \sin\left(\frac{1}{40}\pi(x+20)\right) \quad v(x) = \frac{1}{2.0 \times 10^8}(-x^4 + 15x^3 + 100x^2 - 6000x + 1.2 \times 10^5)(x^2 + 1000)$$

	Trapezoid		Midpoint		New Trap		Simpson's		Quartic	
n	error	ratio	error	ratio	error	ratio	error	ratio	error	ratio
4	3.1622248	2.87	0.3892531	8.74	2.2178991	3.11	0.3774289	7.29	0.1155416	NA
8	0.6857692	4.61	0.1007413	3.86	0.4733846	4.66	0.1084517	3.48	0.0150185	7.70
16	0.1723907	3.98	0.0090092	11.2	0.1166641	4.06	0.0078219	13.9	0.0005089	29.5
32	0.0432944	3.98	0.0011178	7.73	0.0291405	4.00	0.0004769	16.4	0.0000080	63.7
64	0.0104214	4.15	0.0002130	5.52	0.0070291	4.15	0.0000157	30.4	0.0000002	51.3
128	0.0026366	3.95	0.0000581	3.73	0.0017595	4.00	0.0000033	4.80	0.0000000	51.4
256	0.0006617	3.98	0.0000065	8.97	0.0004421	3.98	0.0000003	11.4	0.0000000	92.1
512	0.0001663	3.98	0.0000008	7.97	0.0001110	3.98	0.0000000	13.6	0.0000000	40.8
1024	0.0000412	4.04	0.0000002	3.94	0.0000274	4.05	0.0000000	17.7	0.0000000	27.1

Table 3: Exact Flow = $\int_{-20}^{20} d(x)v(x)\, dx = 15.60866260\ m^3/s$.

Analysis and Remarks

(1) Experimental Error: As the table[1] indicated, we conclude that:

$$\begin{aligned} \textit{Trapezoidal}: &\quad O(h^2) \\ \textit{New Trapezoidal}: &\quad O(h^2) \\ \textit{Midpoint}: &\quad O(h^4) \\ \textit{Simpson}: &\quad O(h^4) \\ \textit{Quartic}: &\quad O(h^6) \end{aligned}$$

where h represents the width of the interval. We were surprised by how good the simple midpoint rule is.

(2) We noticed that if we use the sine function for both the depth and velocity, that is, $d(x) = \sin\left(\frac{\pi x}{w}\right)$ and $v(x) = \sin\left(\frac{\pi x}{w}\right)$ where w is the width of the river, and the data points $x_0 = 0, x_1, x_2, ..., x_n = w$ are equally spaced, then the error of the Midpoint rule is always zero! This is because of the following identity

$$M_n = \sum_{i=1}^{n-1}\frac{w}{n}\left(\sin\left(\frac{\pi\, i}{n}\right)\right)^2 = \int_0^w \sin^2\left(\frac{\pi x}{w}\right) dx = \frac{w}{2} \text{ (exact flow)}$$

References

[1] Michael Monagan. Measuring Water Flow of Rivers http://www.cecm.sfu.ca/CAG/papers/ FlowdataReporter.pdf

[2] U.S.Geological Survey. How streamflow is measured http://water.usgs.gov/edu/measureflow.html