Parallel Sparse Polynomial Multiplication Using Heaps

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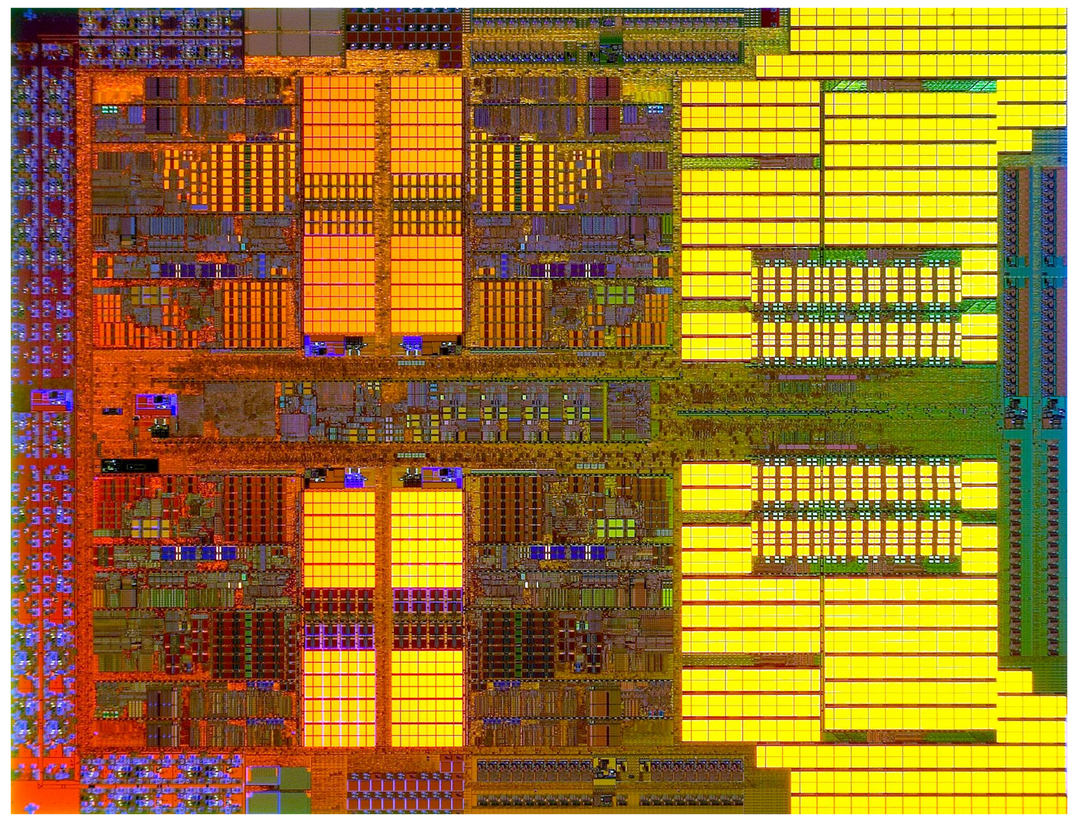
Multicore Processors



The AMD Phenom II is an entry level CPU in 2009. It has four cores (left), each with 512KB L2 cache, and a 6 MB L3 cache (right) which can be used for communication between the cores.

Most programs do not benefit from additional cores.

Sparse Polynomials

Computer algebra systems like Maple spend a lot of time multiplying and dividing polynomials. E.g.:

$$f = 9xy^3z - 4y^3z^2 - 6xy^2z - 8x^3 - 5$$

$$g = 5x^3y^2 + 2xyz^3 - 3y^2z^2 + 7xy + 1$$

The polynomials are often stored in a *sparse format* which represents only non-zero terms.

To compute $f \times g$ we multiply each term of f by all the terms of g, sort the products, and add like terms.

Doubling the size of f and g quadruples the time or worse. A lot of time is spent doing large problems.

Even seemingly unrelated tasks like integration use sparse polynomial routines. Their speed is critical.

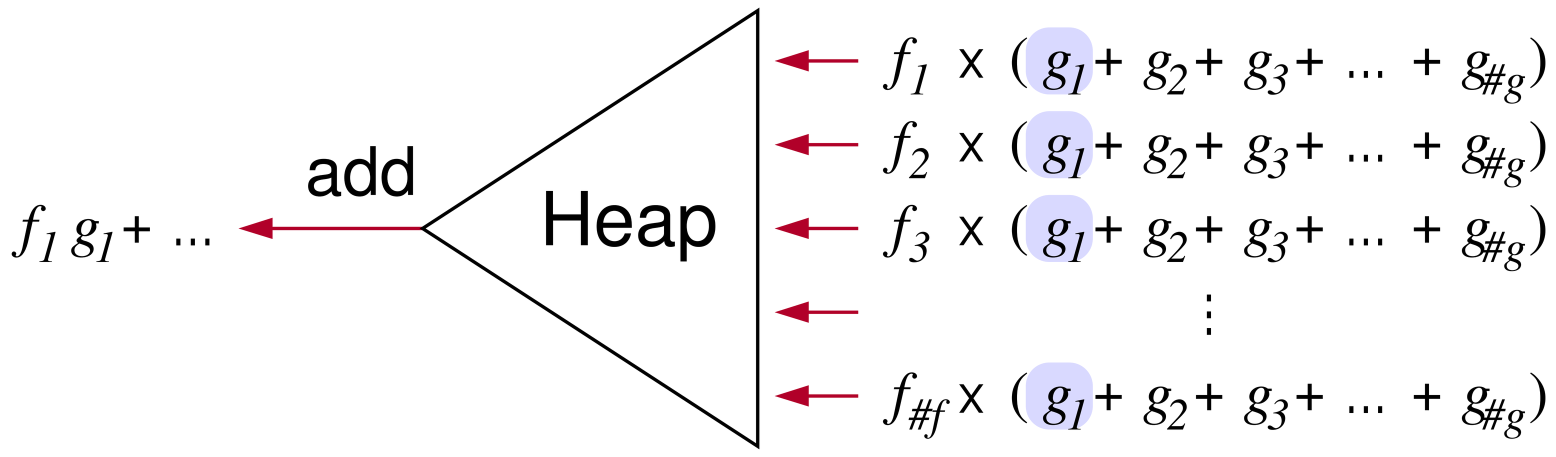
So how fast can we multiply sparse polynomials ?

Multiplication Using a Heap

The key to high performance is the CPU cache. Main memory is slow.

Johnson's algorithm uses a heap to simultaneously merge each $f_i \times g$.

The first term of each $f_i \times g$ is put into a heap, from which we extract terms in descending order. When $f_i \times g_j$ is extracted from the heap it is added to the end of the result and we insert $f_i \times g_{j+1}$ if it exists.



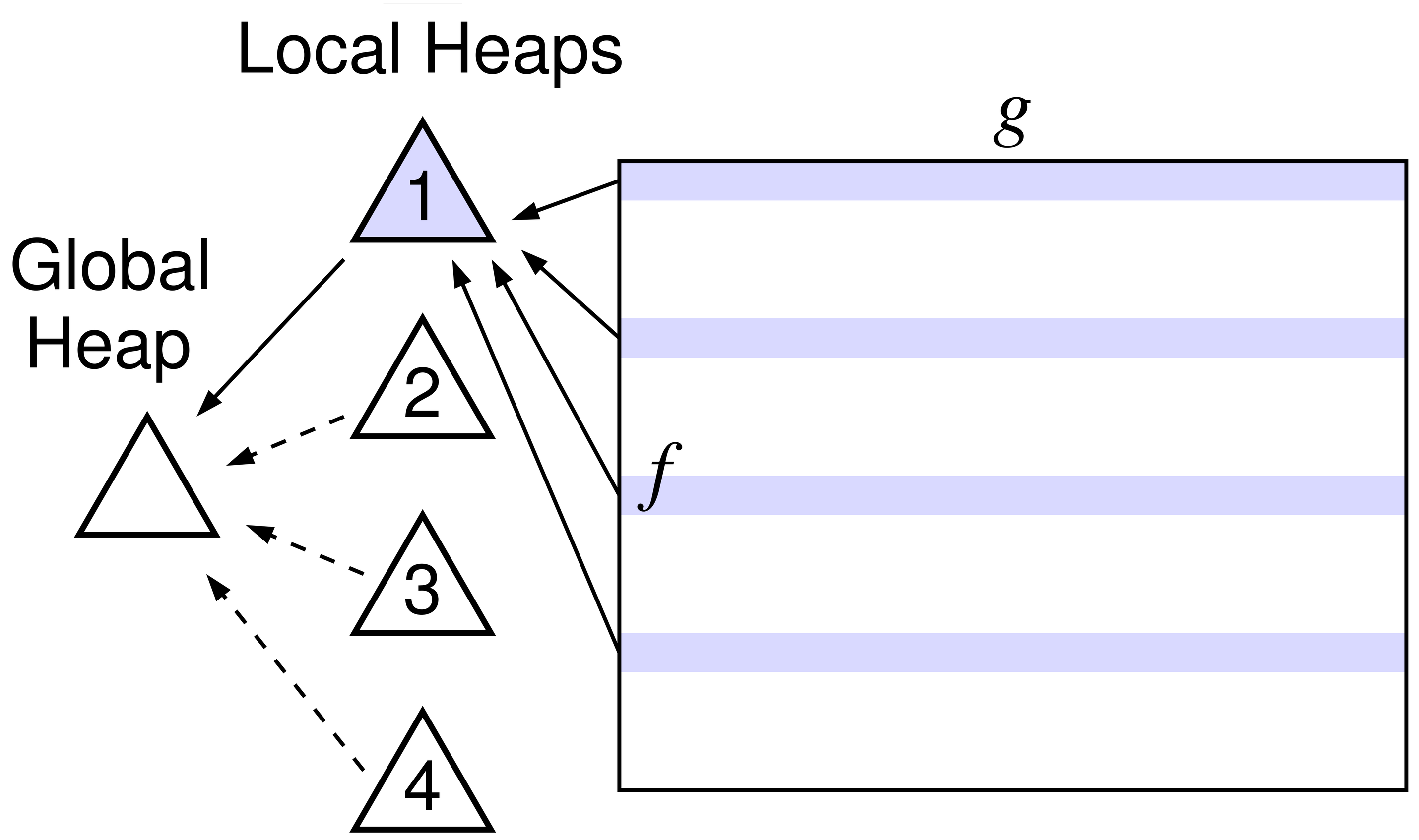
The heap is $O(\#f)$ so it fits in the CPU cache.

Inserting and extracting terms is $O(\log \#f)$ monomial comparisons.

We do at most $O(\#f \#g \log \#f)$ comparisons in total.

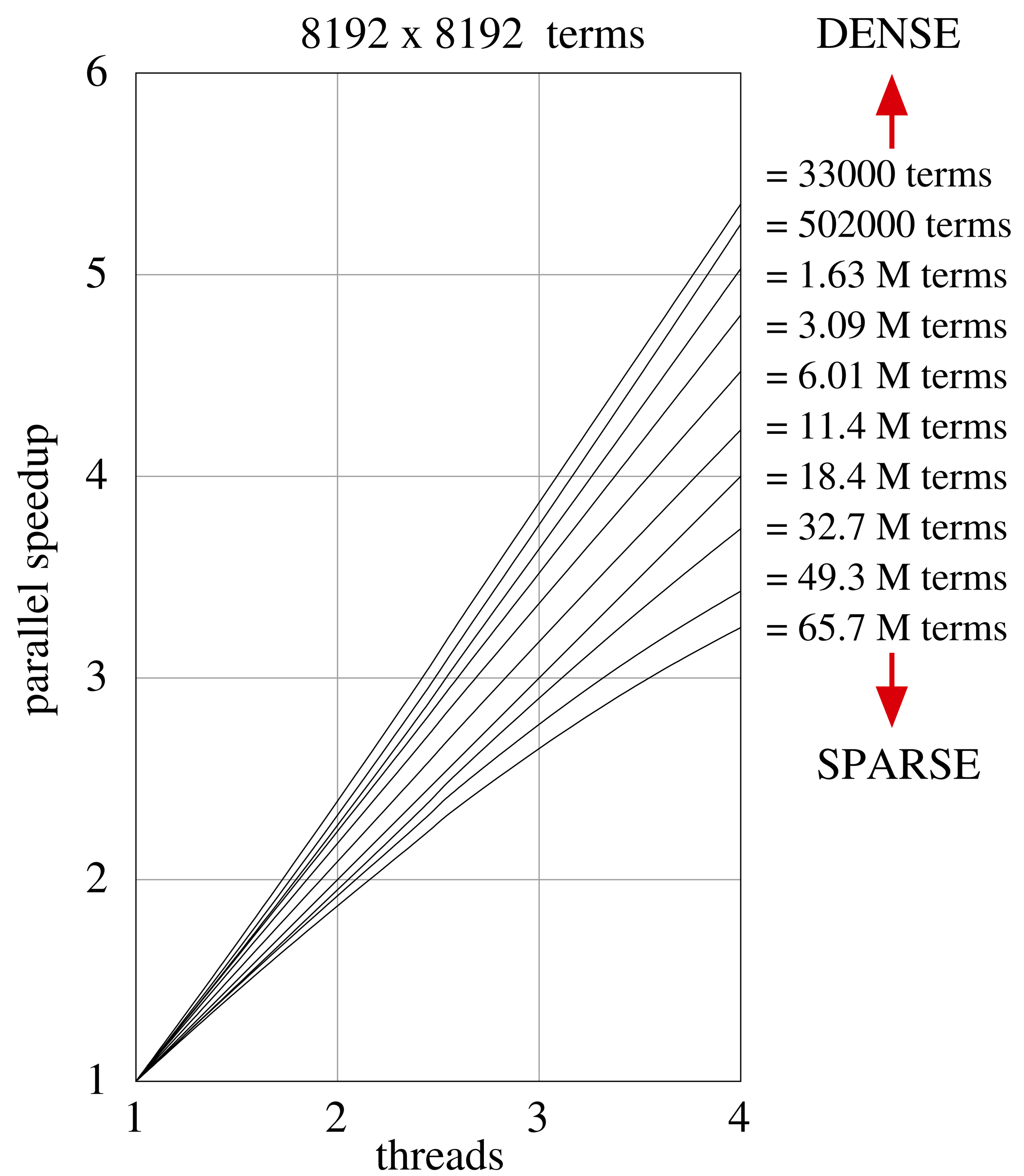
Parallel Algorithm

- Each thread uses a local heap to multiply some of the $f_i \times g$.
- Intermediate results are written to buffers in shared L3 cache.
- The threads take turns combining the buffers to form the result.



Parallel Performance

We multiplied random univariate polynomials with 8192 terms. Typical problems run 5x faster with 4 cores on a Core i7 CPU.



The speedup is relative to the fastest sequential code available. Our code is 50x faster than other computer algebra systems.

$$f = (1 + x + y + z + t)^{30} \qquad g = f + 1$$

$46376 \times 46376 = 635376$ terms $W(f,g) = 3332$

	threads	Core i7		Core 2 Quad	
our software (sdmp)	4	11.48 s	6.15x	14.15 s	4.25x
	3	16.63 s	4.24x	19.43 s	3.10x
	2	28.26 s	2.50x	28.29 s	2.13x
	1	70.59 s		60.25 s	
Magma 2.15-8	1	526.12 s			
Pari/GP 2.3.3	1	642.74 s		707.61 s	
Singular 3-1-0	1	744.00 s		1048.00 s	
Maple 13	1	5849.48 s		9343.68 s	

[1] Stephen C. Johnson. Sparse Polynomial Arithmetic. ACM SIGSAM Bulletin, Volume 8, Issue 3 (1974) 63–71.

[2] Michael Monagan, Roman Pearce. Parallel Sparse Polynomial Multiplication Using Heaps. *Proceedings of ISSAC 2009.*

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