# Triangular Decompositions for Solving Parametric Polynomial Systems

Changbo Chen[1]

joint work with
Marc Moreno Maza[1], Bican Xia[2] and Lu Yang[3]

[1]University of Western Ontario, Ontario, Canada
[2]Peking University, Beijing, China
[3]East China Normal University, Shanghai, China

May 9, 2008

# Parametric Polynomial Systems

- A polynomial system of $\mathbb{K}[U, X]$ consists of equations $(=)$.

$$P(s, x, y) := \begin{cases} x(1+y) - s & = & 0 \\ y(1+x) - s & = & 0 \end{cases}, \tag{1}$$

the solution set of (1) in $\overline{\mathbb{K}}$ is called an algebraic variety.

- Add inequations $(\neq)$ to system (1).

$$\begin{cases} P(s, x, y) \\ x + y - 1 \neq 0 \end{cases}, \tag{2}$$

the solution set of (2) in $\overline{\mathbb{K}}$ is called a constructible set.

- Add inequalities $(>, \geq, <, \leq)$ to system (2).

$$\begin{cases} P(s, x, y) \\ x + y - 1 > 0 \end{cases}, \tag{3}$$

the solution set of (3) in $\mathbb{R}$ is called a semi-algebraic set.

# Objectives

For a parametric polynomial system $F \subset \mathbb{K}[U][X]$, the following problems are of interest:

1. compute the values $u$ of the parameters for which $F(u)$ has solutions, or has finitely many solutions.
2. compute the solutions of $F$ as continuous functions of the parameters.
3. provide an automatic case analysis for the number (dimension) of solutions depending on the parameter values.

# Related work ($C^3$)

- (Comprehensive) Gröbner bases (CGB): (V. Weispfenning, 92, 02), (D. Kapur 93), (A. Montes, 02), (M. Manubens & A. Montes, 02), (A. Suzuki & Y. Sato, 03, 06), (D. Lazard & F. Rouillier, 07) and others.

- (Comprehensive) triangular decompositions (CTD): (S.C. Chou & X.S. Gao 92), (X.S. Gao & D.K. Wang 03), (D. Kapur 93), (D.M. Wang 05), (L. Yang, X.R. Hou & B.C. Xia, 01), (C. Chen, O. Golubitsky, F. Lemaire, M. Moreno Maza & W. Pan, 07) and others.

- Cylindrical algebraic decompositions (CAD): (G.E. Collins 75), (G.E. Collins, H. Hong 91), (H. Hong 92), (S. McCallum 98), (A. Strzeboński 00), (C.W. Brown 01) and others.

# Main Results

We extend comprehensive triangular decomposition of an algebraic variety (CGLMP, CASC 2007) naturally to:

- comprehensive triangular decomposition of a parametric constructible sets
- and apply it to complex roots counting
- comprehensive triangular decomposition of a parametric semi-algebraic sets
- and apply it to real roots counting

# Constructible Set

Roughly speaking, a constructible set is the solution set of a system of polynomial equations and inequations or any finite union of these solution sets.

**Example**

$$\left\{ \begin{array}{rcl} x(1+y)-s & = & 0 \\ y(1+x)-s & = & 0 \\ x+y-1 & \neq & 0 \end{array} \right. \tag{4}$$

**Defi nition**

A constructible set of $\overline{\mathbb{K}}^m$ is any finite union

$$(A_1 \setminus B_1) \cup \cdots \cup (A_e \setminus B_e)$$

where $A_1, \ldots, A_e, B_1, \ldots, B_e$ are algebraic varieties over $\mathbb{K}$.

# How to Represent a Constructible Set?

## Defi nition

A pair $R = [T, h]$ is called a regular system if $T$ is a regular chain, and $h$ is a polynomial which is regular w.r.t $\mathrm{sat}(T)$. The zero set of $R$ is defined as $Z(R) := V(T) \setminus V(hh_T)$.

## Proposition

*The zero set of any regular system is unmixed and nonempty. Every constructible set can be written as a finite union of the zero sets of regular systems.*

## Example

The constructible set (4) can be represented by two regular systems

$$R_1 : \left| \begin{array}{l} T_1 = \begin{cases} (y+1)x - s \\ y^2 + y - s \end{cases} \\ h_1 = y - 2s + 1 \end{array} \right. \qquad R_2 : \left| \begin{array}{l} T_2 = \begin{cases} x + 1 \\ y + 1 \\ s \end{cases} \\ h_2 = 1 \end{array} \right.$$

# Specialization

## Definition

A regular system $R := [T, h]$ specializes well at $u \in \overline{\mathbb{K}}^d$ if $[T(u), h(u)]$ is a regular system of $\overline{\mathbb{K}}[X]$ after specialization and no initials of polynomials in $T$ vanish during the specialization.

## Example

$$R_1 : \left| \begin{array}{l} T_1 = \left\{ \begin{array}{l} (y+1)x - s \\ y^2 + y - s \end{array} \right. \\ h_1 = y - 2s + 1 \end{array} \right.$$

does not specializes well at $s = 0$ or at $s = \frac{3}{4}$

$$R_1(0) : \left| \begin{array}{l} T_1(0) = \left\{ \begin{array}{l} (y+1)x \\ (y+1)y \end{array} \right. \\ h_1(0) = y + 1 \end{array} \right. \qquad R_1(\frac{3}{4}) : \left| \begin{array}{l} T_1(\frac{3}{4}) = \left\{ \begin{array}{l} (y+1)x - \frac{3}{4} \\ (y - \frac{1}{2})(y + \frac{3}{2}) \end{array} \right. \\ h_1(\frac{3}{4}) = y - \frac{1}{2} \end{array} \right.$$

# Pre-comprehensive Triangular Decomposition (1/2)

## Defi nition

The set of parameter values $u \in \overline{\mathbb{K}}^d$ where a regular system $R$ specializes well is called the defining set of $R$, denoted by $D(R)$. Define $Z_C(R) = Z(R) \cap \pi_U^{-1}(D(R))$.

## Example

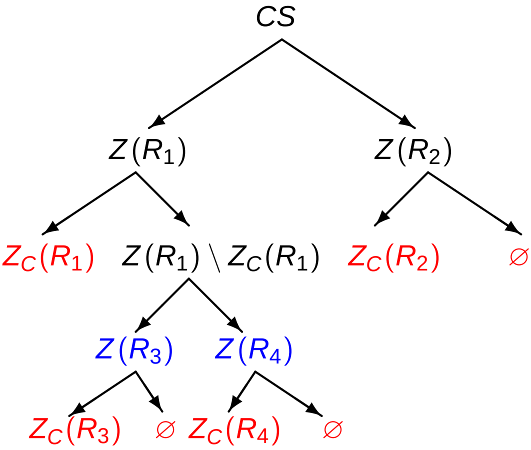
$$R_1 : \left| \begin{array}{l} T_1 = \begin{cases} (y+1)x - s \\ y^2 + y - s \end{cases} \\ h_1 = y - 2s + 1 \end{array} \right. \qquad D(R_1) := \{s \in \overline{\mathbb{K}} \mid s(s - \tfrac{3}{4}) \neq 0\}$$

## Defi nition

A triangular decomposition $\mathcal{R}$ of a constructible set $CS$ is called a pre-comprehensive triangular decomposition of $CS$ if we have

$$CS = \bigcup_{R \in \mathcal{R}} Z(R) = \bigcup_{R \in \mathcal{R}} Z_C(R).$$

# PCTD: Algorithm




$$R_1 : \left| \begin{array}{l} T_1 = \begin{cases} (y+1)x - s \\ y^2 + y - s \end{cases} \\ h_1 = y - 2s + 1 \end{array} \right. \quad R_2 : \left| \begin{array}{l} T_2 = \begin{cases} x+1 \\ y+1 \\ s \end{cases} \\ h_2 = 1 \end{array} \right. \quad R_3 : \left| \begin{array}{l} T_3 = \begin{cases} x \\ y \\ s \end{cases} \\ h_3 = 1 \end{array} \right. \quad R_4 : \left| \begin{array}{l} T_4 = \begin{cases} 2x+3 \\ 2y+3 \\ 4s-3 \end{cases} \\ h_4 = 1 \end{array} \right.$$

# Comprehensive Triangular Decomposition (CTD)

## Definition

Let $CS$ be a constructible set of $\mathbb{K}[U, X]$. A *comprehensive triangular decomposition* of $CS$ is given by :

1. a finite partition $\mathcal{C}$ of the parameter space $\overline{\mathbb{K}}^d$,
2. for each $C \in \mathcal{C}$ a set of regular systems $\mathcal{R}_C$ s.t. for $u \in C$
   - 2.1 each of the regular systems $R \in \mathcal{R}_C$ specializes well at $u$
   - 2.2 and we have

$$\pi_U^{-1}(u) \cap CS = \bigcup_{R \in \mathcal{R}_C} Z(R(u)).$$

## Example

A CTD of (4) is as follows:

1. $s \neq 0$ and $s \neq \frac{3}{4} \longrightarrow \{R_1\}$
2. $s = 0 \longrightarrow \{R_2, R_3\}$
3. $s = \frac{3}{4} \longrightarrow \{R_4\}$

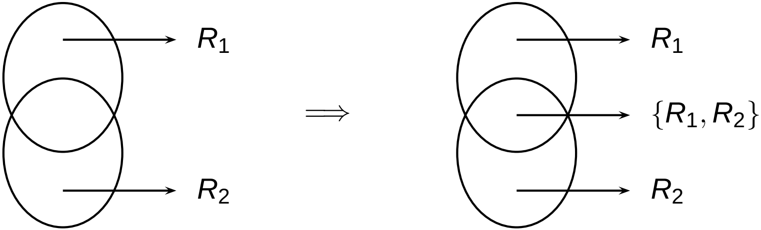
$$R_1 : \left| \begin{array}{l} T_1 = \left\{ \begin{array}{l} (y+1)x - s \\ y^2 + y - s \end{array} \right. \\ h_1 = y - 2s + 1 \end{array} \right. \qquad R_2 : \left| \begin{array}{l} T_2 = \left\{ \begin{array}{l} x+1 \\ y+1 \\ s \end{array} \right. \\ h_2 = 1 \end{array} \right.$$

$$R_3 : \left| \begin{array}{l} T_3 = \left\{ \begin{array}{l} x \\ y \\ s \end{array} \right. \\ h_3 = 1 \end{array} \right. \qquad R_4 : \left| \begin{array}{l} T_4 = \left\{ \begin{array}{l} 2x+3 \\ 2y+3 \\ 4s-3 \end{array} \right. \\ h_4 = 1 \end{array} \right.$$

# Algorithm of CTD

There are two main steps for computing a CTD of a constructible set $CS$.

- Compute a PCTD $\mathcal{R}$ of $CS$.
- Compute the intersection-free basis of the defining sets of regular systems in $\mathcal{R}$.



# Separation

## Definition

A squarefree regular system $R := [T, h]$ separates well at $u \in \overline{\mathbb{K}}^d$ if: $R$ specializes well at $u$ and $R(u)$ is a squarefree regular system of $\overline{\mathbb{K}}[X]$.

$$R_1 : \left| \begin{array}{l} T_1 = \begin{cases} (y+1)x - s \\ y^2 + y - s \end{cases} \\ h_1 = y - 2s + 1 \end{array} \right.$$

specializes well but not separates well at $s = -\frac{1}{4}$:

$$R_1(-\frac{1}{4}) : \left| \begin{array}{l} T_1(-\frac{1}{4}) = \begin{cases} (y+1)x + \frac{1}{4} \\ (y + \frac{1}{2})^2 \end{cases} \\ h_1(-\frac{1}{4}) = y + \frac{3}{2} \end{array} \right.$$

where the second polynomial of $T_1(-\frac{1}{4})$ is not squarefree.

# Disjoint Squarefree Comprehensive Triangular Decomposition (DSCTD)

## Defi nition

A disjoint squarefree comprehensive triangular decomposition of a constructible set $CS$ is given by:

1. a finite partition $\mathcal{C}$ of the parameter space $\overline{\mathbb{K}}^d$,
2. for each $C \in \mathcal{C}$ a set of squarefree regular systems $\mathcal{R}_C$ such that for each $u \in C$:
   - 2.1 each $R \in \mathcal{R}_C$ separates well at $u$,
   - 2.2 the zero sets $Z(R(u))$, for $R \in \mathcal{R}_C$, are pairwise disjoint and

$$\pi_U^{-1}(u) \cap CS = \bigcup_{R \in \mathcal{R}_C} Z(R(u)).$$

# DSCTD and Complex Root Counting

## Example

$$R_1 : \left| \begin{array}{l} T_1 = \begin{cases} (y+1)x - s \\ y^2 + y - s \end{cases} \\ h_1 = y - 2s + 1 \end{array} \right. \quad R_2 : \left| \begin{array}{l} T_2 = \begin{cases} x+1 \\ y+1 \\ s \end{cases} \\ h_2 = 1 \end{array} \right. \quad R_3 : \left| \begin{array}{l} T_3 = \begin{cases} x \\ y \\ s \end{cases} \\ h_3 = 1 \end{array} \right.$$

$$R_4 : \left| \begin{array}{l} T_4 = \begin{cases} 2x+3 \\ 2y+3 \\ 4s-3 \end{cases} \\ h_4 = 1 \end{array} \right. \quad R_5 : \left| \begin{array}{l} T_5 = \begin{cases} 2x+1 \\ 2y+1 \\ 4s+1 \end{cases} \\ h_5 = 1 \end{array} \right.$$

A DSCTD of system (4) is as follows:

1. $s \neq 0, s \neq -\frac{1}{4}, s \neq \frac{3}{4} \longrightarrow \{R_1\}$
2. $s = 0 \longrightarrow \{R_2, R_3\}$
3. $s = \frac{3}{4} \longrightarrow \{R_4\}$
4. $s = -\frac{1}{4} \longrightarrow \{R_5\}$

Therefore, we conclude that: if $(s+\frac{1}{4})(s-\frac{3}{4}) = 0$, system (4) has 1 complex root; otherwise system (4) has 2 complex roots.

# Semi-algebraic Set

Roughly speaking, a semi-algebraic set is the set of real solutions of a system of polynomial equations, inequations and inequalities or any finite union of these solution sets.

### Example

$$\left\{ \begin{array}{rcl} x(1+y)-s & = & 0 \\ y(1+x)-s & = & 0 \\ x+y-1 & > & 0 \end{array} \right. \tag{5}$$

### Defi nition

A semi-algebraic set of $\mathbb{R}^n$ is a finite union of the form:

$$\{x \in \mathbb{R}^n \mid \forall f \in F, g \in G, f(x) = 0 \text{ and } g(x) > 0\},$$

where $F$ and $G$ are any finite polynomial sets over $\mathbb{R}$.

# Regular Semi-algebraic System

### Definition

A pair $A := [T, G+]$ is called a regular semi-algebraic system if

1. $G+$ is a finite set of inequalities $\{g > 0 \mid g \in G\}$, where $G$ is a finite polynomial set over $\mathbb{R}$.
2. $[T, \prod_{g\in G} g]$ is a squarefree regular system over $\mathbb{R}$.

The set $Z(A) := Z([T, \prod_{g\in G} g]) \cap \{x \in \mathbb{R}^n \mid \forall g \in G, g > 0\}$ is called the zero set of $A$. We say $A$ separates well at $u \in \mathbb{R}^d$ if the squarefree regular system $[T, \prod_{g\in G} g]$ separates well at $u \in \mathbb{R}^d$.

### Example

$$A : \left| \begin{array}{l} T = \left\{ \begin{array}{l} (y+1)x - s \\ y^2 + y - s \end{array} \right. \\ G+ = \{x + y - 1 > 0\} \end{array} \right.$$

# CTD of a Semi-algebraic Set

## Definition

A CTD of a semi-algebraic set $\mathcal{S}$ of $\mathbb{R}[U, X]$ is given by:

1. a finite partition $\mathcal{C}$ of the parameter space $\mathbb{R}^d$ into connected semi-algebraic sets,
2. for each $C \in \mathcal{C}$, an associated sample point $s \in C$,
3. for each $C \in \mathcal{C}$ a set of regular semi-algebraic systems $\mathcal{A}_C$ of $\mathbb{R}[U, X]$ such that for each $u \in C$
   - 3.1 each $A \in \mathcal{A}_C$ separates well at $u$,
   - 3.2 the zero sets $Z(A(u))$, for $A \in \mathcal{A}_C$, are pairwise disjoint and

$$\pi_U^{-1}(u) \cap \mathcal{S} = \bigcup_{A \in \mathcal{A}_C} Z(A(u)).$$

**Algorithm:**

1. Compute the DSCTD of $CS(\mathcal{S})$.
2. Apply CAD to refine each cell obtained into connected semi-algebraic sets.

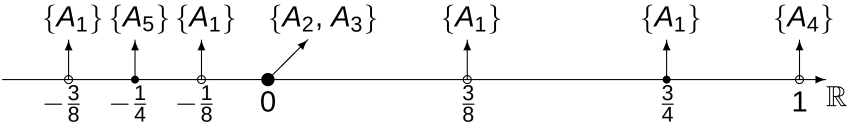
# CTD of a Semi-algebraic Set

## Example

$$A_1 : \left| \begin{array}{l} T_1 = \begin{cases} (y+1)x - s \\ y^2 + y - s \end{cases} \\ G+ = \{x + y - 1 > 0\} \end{array} \right. \quad A_2 : \left| \begin{array}{l} T_2 = \begin{cases} x+1 \\ y+1 \\ s \end{cases} \\ G+ = \{x + y - 1 > 0\} \end{array} \right. \quad A_3 : \left| \begin{array}{l} T_3 = \begin{cases} x \\ y \\ s \end{cases} \\ G+ = \{x + y - 1 > 0\} \end{array} \right.$$

$$A_4 : \left| \begin{array}{l} T_4 = \begin{cases} 2x+3 \\ 2y+3 \\ 4s-3 \end{cases} \\ G+ = \{x + y - 1 > 0\} \end{array} \right. \quad A_5 : \left| \begin{array}{l} T_5 = \begin{cases} 2x+1 \\ 2y+1 \\ 4s+1 \end{cases} \\ G+ = \{x + y - 1 > 0\} \end{array} \right.$$

A CTD of system (5) is as follows:



# Real Root Counting

## Proposition

*Let $A = [T, G+]$, where $X \subseteq \mathrm{mvar}(T)$, be a regular semi-algebraic system of $\mathbb{R}[U, X]$ and $C$ be a connected semi-algebraic set of $\mathbb{R}^d$. If $A$ separates well at any $u \in C$, then the number of solutions of $A$ is constant over $C$ and each solution is a continuous function of the parameters in $C$.*

## Example

$$A_1 : \left| \begin{array}{l} T_1 = \begin{cases} (y+1)x - s \\ y^2 + y - s \end{cases} \\ G+ = \{x + y - 1 > 0\} \end{array} \right.$$

The regular semi-algebraic system $A_1$ separates well at $s > \frac{3}{4}$ and the number of its solutions is 1.

Moreover, if $s > \frac{3}{4}$, the number of real solutions of system (5) is 1; otherwise system (5) has no real solutions.

# Mad Cow Disease

Mad cow disease causes cattle to behave strangely.

# Laurent Model

- Mad cow disease is a transmissible disease of the central nervous system, thought to be caused prion proteins.
- Prion proteins exist in a normal form $\mathrm{PrP^C}$ and a pathogenic form $\mathrm{PrP^{Sc}}$.
- An excess of $\mathrm{PrP^{Sc}}$ causes prion diseases. Can a small amount of $\mathrm{PrP^{Sc}}$ lead finally to excess of $\mathrm{PrP^{Sc}}$?

The model of Laurent reduces this question to studying the equilibria of the dynamical system below, where $x$ and $y$ are the concentrations of $\mathrm{PrP^C}$ and $\mathrm{PrP^{Sc}}$.

$$\left\{\begin{array}{lcl}\frac{\mathrm{d}x}{\mathrm{d}t} & = & f_1\\ \frac{\mathrm{d}y}{\mathrm{d}t} & = & f_2\end{array}\right. \quad \text{with} \quad \left\{\begin{array}{lcl} f_1 & = & \frac{16000+800y^4-20k_2x-k_2xy^4-2x-4xy^4}{20+y^4}\\ f_2 & = & \frac{2(x+2xy^4-500y-25y^5)}{20+y^4}\end{array}\right. \tag{6}$$

# From Dynamical System to Semi-algebraic Set

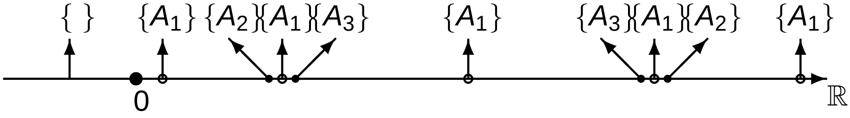
By Routh-Hurwitz criterion, the parametric semi-algebraic set

$$\mathcal{S} : \{f_1 = f_2 = 0, k_2 > 0, \Delta_1 > 0, \Delta_2 > 0\}$$

encodes exactly the asymptotically hyperbolic equilibria of system (6), where

$$\begin{aligned} \Delta_1 &:= -\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right) > 0 \\ \Delta_2 &:= \frac{\partial f_1}{\partial x} \cdot \frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x} > 0 \end{aligned}$$

A comprehensive triangular decomposition of $\mathcal{S}$ is illustrated as follows:

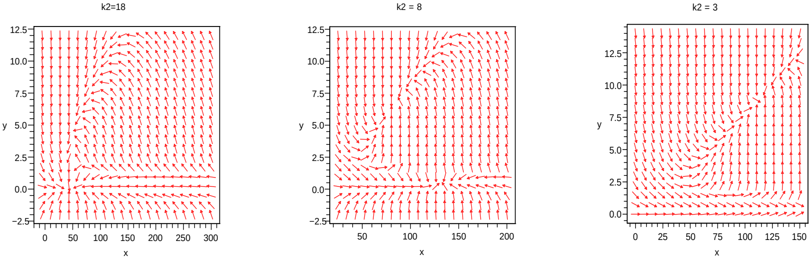


# Stability Analysis

Let

$$R_1 = 100000k_2^8 + 1250000k_2^7 + 5410000k_2^6 + 8921000k_2^5 - 9161219950k_2^4 - 5038824999k_2^3 - 1665203348k_2^2 - 882897744k_2 + 1099528405056.$$

If $R_1 > 0$, then the system has 1 equilibrium, which is asymptotically stable. If $R_1 < 0$, then the system has 3 equilibria, two of which are asymptotically stable. If $R_1 = 0$, the system experiences a bifurcation.



# Biochemical explanation

- The turnover rate $k_2$ determines whether it is possible for a pathogenic state to occur.
- As an answer to our question, if $k_2$ is large, a small amount of $\mathrm{PrP}^{\mathrm{Sc}}$ will not lead to a pathogenic state.
- Compounds that inhibit addition of $\mathrm{PrP}^{\mathrm{Sc}}$ can be seen as a possible therapy against prion diseases.
- Compounds that increase the turnover rate $k_2$ would be the best therapeutic strategy against prion diseases.