Computing with Constructible Sets

Liyun Li & Yuzhen Xie

Joint work with
Changbo Chen, Marc Moreno Maza, Wei Pan

ORCCA, UWO

MOCAA M^3 Workshop, May 9, 2008

Outline

- Constructible set and removing redundancy
- Difference: $(A, B) \longmapsto A \setminus B$
- PairRefine: $(A, B) \longmapsto (A \setminus B,\ A \cap B, B \setminus A)$
- MPD: C (with redundancy) $\longmapsto C$ (without redundancy)
- SMPD:

$$\begin{array}{ccl} C = \bigcup C_i & \longmapsto & C = \bigcup D_j \\ \text{(with redundancy)} & & \text{(without redundancy} \\ & & \text{and the } D_j's \text{ refine the } C_i's) \end{array}$$

- Complexity Analysis
- Experimental comparison (different algorithms for SMPD)

What is a Constructible Set?

Definition (Constructible Set)

A *constructible set* of $\overline{\mathbb{K}}^n$ is a finite union

$$(A_1 \setminus B_1) \cup \cdots \cup (A_e \setminus B_e)$$

where A_i's and B_i's are algebraic varieties in $\overline{\mathbb{K}}^n$.

Definition (Regular System)

A pair $[T, h]$ is a *regular system* if T is a regular chain, and $h \in \mathbb{K}[X]$ is regular with respect to $\mathrm{sat}(T)$. The zero set $Z(T, h)$ given by $[T, h]$ is $W(T) \setminus V(h)$.

Example (Regular Systems)

$$(Yes) \begin{cases} ax^2 + bx + c = 0 \\ a(b^2 - 4ac) \neq 0 \end{cases} \quad (No) \begin{cases} x^2 - 2xy + t = 0 \\ y^2 - t = 0 \\ x - y \neq 0 \end{cases}$$

Representation of Constructible Sets

Example (Constructible Set)

For what value of a, b, c, does the equation

$$ax^2 + bx + c = 0$$

have solutions over $\mathbb{C}$?

- when a is not zero; $rs_1 = [a \neq 0]$
- when a is zero but b is not; $rs_2 = [a = 0, b \neq 0]$
- when a, b, c are all zero. $rs_3 = [a = 0, b = 0, c = 0]$

$cs = \{rs_1, rs_2, rs_3\}$ describes the answer.

Another Example

- **Example**: given two elliptic curves in the complex plane of coordinates (x, y): $g_1(x, y) = 0$ and $g_2(x, y) = 0$, where

$$\begin{aligned} g_1(x,y) &= x^3 + a_1 x - y^2 + 1, \\ g_2(x,y) &= x^3 + a_2 x - y^2 + 1 \end{aligned}$$

In invariant theory, a classical question is whether there exists a
linear fractional map from the first curve to the second one:

$$f : (x, y) \mapsto \left(\frac{Ax + By + C}{Gx + Hy + K}, \frac{Dx + Ey + F}{Gx + Hy + K} \right)$$

Another Example

- This problem can be turned into a parametric system:

$$g_1(x,y) - (G\,x + H\,y + K)^3 g_2(f(x,y)) = 0.$$

$$\begin{cases} 1 - K^3 & = 0 \\ -a_2\,AK^2 + a_1 - 3\,GK^2 & = 0 \\ -3\,HK^2 - a_2\,BK^2 & = 0 \\ GD^2 - a_2\,G^2A - A^3 - G^3 + 1 & = 0 \\ -3\,H^2K + E^2K - 1 - 2\,a_2\,BHK & = 0 \\ -3\,G^2K - 2a_2\,GAK + D^2K & = 0 \\ GE^2 - 2a_2\,GBH - a_2\,AH^2 - 3\,AB^2 - 3\,GH^2 + 2\,DEH & = 0 \\ E^2H - H^3 - a_2\,BH^2 - B^3 & = 0 \\ D^2H - 3\,G^2H + 2\,GDE - 2a_2\,GAH - 3\,A^2B - a_2\,G^2B & = 0 \\ -3\,GHK - a_2\,AHK - a_2\,GBK + DEK & = 0 \end{cases}$$

- For which parameter values of a_1, a_2 does this system have solutions?

Another Example

- The output produced by the command `ComprehensiveTriangularize` of the module `ParametricSystemTools` consists of 11 regular chains $[T_1, \ldots, T_{11}]$ and 3 constructible sets C_1, C_2 and C_3.

$$\begin{array}{lll} C_1 & : & a_1^3 = a_2^3 = 9 \\ C_2 & : & a_1 = a_2 = 0 \\ C_3 & : & a_1^3 = a_2^3,\ a_2 \neq 0,\ a_2^3 \neq 9\ . \end{array}$$

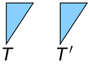
- The union of C_1, C_2, C_3 is the answer to our question: for which parameter values does the input system have solutions?

Redundancy in Computing with Constructible Sets

- Redundancy in a single constructible set
  - Two regular systems have a common part.
  - Remove redundancy: make regular systems pairwise disjoint (MPD)
- Redundancy in a list of constructible sets
  - Some zeroes appear in more than one constructible sets.
- Building block: compute the difference of two regular systems.

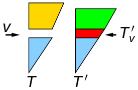
Sketch of Difference Algorithm to compute $V(T) \setminus V(T')$ by exploiting the triangular structure level by level.

Case 1:



$\langle T \rangle = \langle T' \rangle$, Easy!

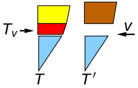
Case 2:



Output $[T, T'_v]$ and

$\texttt{Difference}(V(T_v) \cap V(T'_v), V(T'))$;

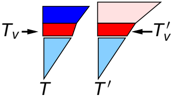
Case 3:



Output $\texttt{Difference}$

$(T, V(T_v) \cap V(T'))$;

Case 4:



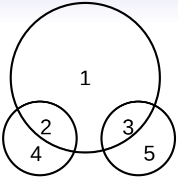
- $g = \mathrm{GCD}(T_v, T'_v, T_{<v})$; • $g \in \mathbb{K} \Rightarrow$ Output $[T, 1]$;
- $\mathrm{mvar}(g) < v \Rightarrow$ Output $[T, g]$,
  $\texttt{Difference}(V(g) \cap V(T), T')$;
- Output $\texttt{Difference}(T_{<v} \cup \{g\} \cup T_{>v}, T')$;
- Output $\texttt{Difference}(T_{<v} \cup \{T_v/g\} \cup T_{>v}, T')$;

Algorithm 2 MPD

Input: a list L of regular systems

Output: a pairwise disjoint representation of L

$n \leftarrow |L|$
if $n < 2$ **then**
    return L
else
    $d \leftarrow L[n]$
    $L^* \leftarrow$ MPD$(L[1, \ldots, n-1])$
    for $\ell' \in L^*$ **do**
        $d \leftarrow$ Difference(d, ℓ')
    end for
    return $d \cup L^*$
end if



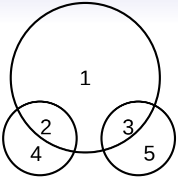
$$A = \{1, 2, 3\},$$

$$B = \boxed{\{2,4\}}, C = \boxed{\{3,5\}}.$$

Algorithm 2 MPD

Input: a list L of regular systems
Output: a pairwise disjoint representation of L

```
n ← |L|
if n < 2 then
    return L
else
    d ← L[n]
    L* ← MPD(L[1, ..., n − 1])
    for ℓ′ ∈ L* do
        d ← Difference(d, ℓ′)
    end for
    return d ∪ L*
end if
```



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$$A = A \setminus B = \{1, 3\}$$

Algorithm 2 MPD

Input: a list L of monic square-free zero-dimensional regular chains

Output: a pairwise disjoint representation of L

$n \leftarrow |L|$
if $n < 2$ **then**
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    $d \leftarrow L[n]$
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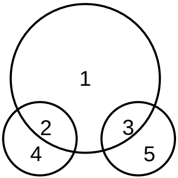

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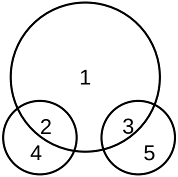
$A = A \setminus C = \boxed{\{1\}}$

Algorithm 2 MPD

Input: a list L of monic square-free zero-dimensional regular chains

Output: a pairwise disjoint representation of L

$n \leftarrow |L|$
if $n < 2$ **then**
  return L
else
  $d \leftarrow L[n]$
  $L^* \leftarrow \mathrm{MPD}(L[1, \ldots, n-1])$
  for $\ell' \in L^*$ **do**
    $d \leftarrow \mathsf{Difference}(d, \ell')$
  end for
  return $d \cup L^*$
end if



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$A = A \setminus B = \{1, 3\}$

$A = A \setminus C = \boxed{\{1\}}$

$\mathrm{MPD}(A, B, C) = \{1\}, \{2, 4\}, \{3, 5\}$

RCPairRefine

In dimension 0, we can do better.

- RCPairRefine performs Difference in a simple case.
- Compute two-side differences and intersection **in one pass**.

Example

$$\mathrm{RCPairRefine}([x = 0, y(y+1) = 0], [x = 0, y(y+2) = 0])$$

outputs

$$\underbrace{[x = 0, y+1 = 0]}_{\textit{difference}}, \quad \underbrace{[x = 0, y = 0]}_{\textit{intersection}}, \quad \underbrace{[x = 0, y+2 = 0]}_{\textit{difference}}$$

RCPairRefine Algorithm

Algorithm 1 RCPairRefine

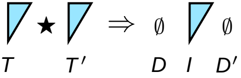
Input: two monic squarefree zero-dimensional regular chains T and T'

Output: three constructible sets D, I and D', such that $V(T) \setminus V(T') = Z(D)$, $V(T) \cap V(T') = Z(I)$ and $V(T') \setminus V(T) = Z(D')$

if $T = T'$ **then**
  return $\emptyset$, $[T]$, $\emptyset$
else
  $D \leftarrow \emptyset$; $I \leftarrow \emptyset$; $D' \leftarrow \emptyset$
  Let v be the largest variable s.t. $T_{<v} = T'_{<v}$
  for $(g, G) \in$ GCD$(T_v, T'_v, T_{<v})$ **do**
    if $g \in \mathbb{K}$ **or** $\mathrm{mvar}(g) < v$ **then**
      $T_q \leftarrow G \cup \{T_v\} \cup T_{>v}$;
      $T'_q \leftarrow G \cup \{T'_v\} \cup T'_{>v}$;
      $D \leftarrow D \cup T_q$;
      $D' \leftarrow D' \cup T'_q$
    else
      $q \leftarrow \mathrm{pquo}(T_v, g, G)$;
      $q' \leftarrow \mathrm{pquo}(T'_v, g, G)$;
      $T_g \leftarrow G \cup \{g\} \cup T_{>v}$;
      $T'_g \leftarrow G \cup \{g\} \cup T'_{>v}$
      if $\mathrm{mvar}(q) = v$ **then**
        $T_q \leftarrow G \cup \{q\} \cup T_{>v}$;
        $D \leftarrow D \cup T_q$
      end if
      if $\mathrm{mvar}(q') = v$ **then**
        $T'_q \leftarrow G \cup \{q'\} \cup T'_{>v}$;
        $D' \leftarrow D' \cup T'_q$
      end if
      $W, J, W' \leftarrow$ RCPairRefine(T_g, T'_g);
      $D \leftarrow D \cup W$;
      $I \leftarrow I \cup J$;
      $D' \leftarrow D' \cup W'$
    end if
  end for
  return D, I, D'
end if

RCPairRefine Algorithm

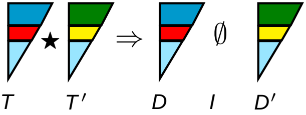
Case 1: $T = T'$



easy!

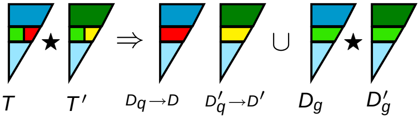
Case 2: $g = \color{blue}{GCD}(T_v, T'_v, T_{<v})$

$g \in \mathbb{K}$ or $\mathrm{mvar}(g) < v$



Case 3:

$g = \color{blue}{GCD}(T_v, T'_v, T_{<v}); \quad \mathrm{mvar}(g) = v$



$$
\begin{aligned}
D &= T \setminus T' \\
I &= T \cap T' \\
D' &= T' \setminus T
\end{aligned}
$$

Irredundant Representation of a Family of Constructible Sets by *Symmetrically Make Pairwise Disjoint* (SMPD)

- Let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be a set of constructible sets.
- An intersection-free basis of $\mathcal{C}$: $\mathcal{D} = \{D_1, \ldots, D_n\}$ satisfies
  (1) $D_i \cap D_j = \emptyset$ for $1 \le i \ne j \le n$,
  (2) each C_i can be uniquely written as a union of some D_j's.

- CSPairRefine $= \mathrm{SMPD}(C_1, C_2) = C_1 \backslash C_2,\ C_1 \cap C_2,\ C_2 \backslash C_1$

- Based on RCPairRefine, CSPairRefine can be deduced naturally.

Example

$\mathrm{SMPD}(\{[x = 0]\},\ \{[y = 0]\})$ outputs:

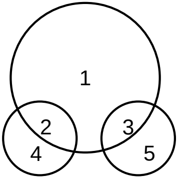
$\{[x = 0,\ y \ne 0]\},\ \{[x = 0, y = 0]\},\ \{[x \ne 0,\ y = 0]\}$

Algorithm 3 BachSMPD

Input: a list L of constructible sets with each consisting of a family of monic squarefree zero-dimensional regular chains

Output: an intersection-free basis of L

$n \leftarrow |L|$
if $n < 2$ **then**
  return L
else
  $I \leftarrow \emptyset$; $D' \leftarrow \emptyset$; $d \leftarrow L[n]$
  $L^* \leftarrow \text{BachSMPD}(L[1, \ldots, n-1])$
  for $l' \in L^*$ **do**
    $d, i, d' \leftarrow \text{CSPairRefine}(d, l')$
    $I \leftarrow I \cup i$; $D' \leftarrow D' \cup d'$
  end for
  return $\{d\} \cup I \cup D'$
end if



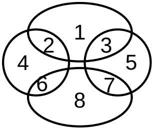
$$
\begin{aligned}
A &= \{1, 2, 3\}, \\
B &= \{2, 4\}, \\
C &= \{3, 5\}. \\
A \bigstar B &= \{1, 3\}, \{2\}, \{4\} \\
d \bigstar C &= \{1\}, \{3\}, \{5\}
\end{aligned}
$$

$\text{SMPD}(A, B, C) =$
$\{1\}, \{2\}, \{4\}, \{3\}, \{5\}$

Algorithm 4 DCSMPD

1: $n \leftarrow |L|$
2: **if** $n < 2$ **then**
3:     **return** L
4: **else**
5:     $z \leftarrow \lfloor n/2 \rfloor$
6:     $L_1 \leftarrow$ DCSMPD$(L[1, \ldots, z])$
7:     $L_2 \leftarrow$ DCSMPD$(L[z+1, \ldots, n])$
8:     **for** j in $1..|L_1|$ **do**
9:         **for** k in $1..|L_2|$ **do**
10:             $L_1[j], i, L_2[k] \leftarrow$ CSPairRefine$(L_1[j], L_2[k])$
11:             $I \leftarrow I \cup i$
12:         **end for**
13:     **end for**
14:     **return** $L_1 \cup I \cup L_2$
15: **end if**

Merge: Line 8-line 13



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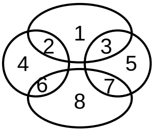

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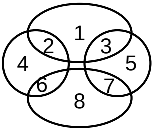

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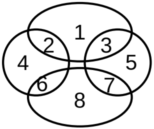

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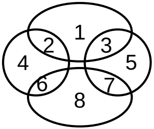

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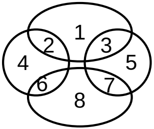
15: **end if**

Merge: Line 8-line 13



$A = \{2,4,6\}, B = \{3,5,7\}$,
$C = \{1,2,3\}, D = \{6,7,8\}$.

$A \star C = \{4,6\}, \boxed{\{2\}}, \{1,3\}$

$A \star D = \boxed{\{4\}}, \boxed{\{6\}}, \{7,8\}$

$B \star C = \{5,7\}, \boxed{\{3\}}, \boxed{\{1\}}$

$B \star D = \boxed{\{5\}}, \boxed{\{7\}}, \boxed{\{8\}}$

$\mathrm{SMPD}(A,B,C) = \{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{7\},\{8\}$

Complexity Analysis

- The main cost comes from GCD computation modulo regular chains.
- By means of evaluation/interpolation techniques, these GCDs can be performed modulo 0-dim regular chains.
- Let $T = [T_1, \ldots, T_n]$ be a squarefree regular chain in $\mathbb{K}[X_1 < \cdots < X_n]$.
- $\mathbb{K}(T) := \mathbb{K}[X_1, \ldots, X_n]/\langle T \rangle$ is a direct product of fields (DPF).
- Let $\deg_i T := \deg(T_i, Xi)$, for $1 \leq i \leq n$, and $\deg_T$ be their product.

Complexity Analysis

- For practical purpose, we rely on classical i.e. quadratic algorithms.
- We adapt the extended Euclidean algorithm for polynomials with coefficients in $\mathbb{K}$ to $\mathbb{K}(T)$.
- Recall an extended GCD of f_1 and f_2 in $\mathbb{K}[x]$ with degrees $d_1 \geq d_2$, can be computed in $O(d_1 d_2)$ operations in $\mathbb{K}$.

Complexity Analysis

- What is different in $\mathbb{K}(T)$ for EEA?

  In a Euclidean division step i, to do

  $$r_{i-1} \; rem \; r_i \; mod \; T,$$

  we need first to get the inverse u_i of $lc(r_i)$ s.t.

  $$u_i \times lc(r_i) = 1 \; mod \; T,$$

  which is called a quasi-inverse.

  Let X_k be the main variable of $lc(r_i)$. We need to compute

  $$GCD(lc(r_i), T_{x_k}) \; mod \; T_{<x_k}$$

Complexity Analysis

- In case of a zero-divisor, T splits into a family of $T^1, \cdots, T^m$. Then computation splits into m branches.

- All the elements then need to project to each branch. To perform this projection efficiently, each family of

$$T^1_{x_j}, \cdots, T^m_{x_j} \text{ for } 1 \leq j \leq m$$

  should be pair-wise coprime, which is called a *non-critical* triangular decomposition.

- Thus GCD-free basis computation modulo regular chains is needed. We extend Bach's augment refinement method to $\mathbb{K}(T)$ for this.

Complexity Analysis

- Following the inductive process applied in “On the complexity of D5 principle”, we obtain an arithmetic time for regular chains based on classical algorithms.
- Recall that an arithmetic time $T \mapsto \mathsf{A}_n(\deg_1 T, \ldots, \deg_n T)$ is an asymptotic upper bound for the cost of basic polynomial arithmetic operations in $\mathbb{K}(T)$ (counted in $\mathbb{K}$), i.e. $+, *$, quasi-inverse, projection and computation of non-critical decompositions.

An Arithmetic Time for Regular Chain based on Classical Algorithms

Theorem (An arithmetic time for regular chain)

There exists a constant C such that, writing

$$\mathsf{A}_n(d_1, \ldots, d_n) = C^n (d_1 \times \cdots \times d_n)^2,$$

the function $T \mapsto \mathsf{A}_n(\deg_1 T, \ldots, \deg_n T)$ is an arithmetic time for regular chain T in n variables, for all n.

Basic Complexity Result

- An extended GCD of f_1 and f_2 in $\mathbb{K}(T)[y]$ with degrees $d_1 \geq d_2$ can be computed in $O(d_1 d_2 \mathsf{A}_n(T))$ operations in $\mathbb{K}$.
- For a family of monic squarefree polynomials $F = \{f_1, \ldots, f_m\}$ in $\mathbb{K}(T)[y]$ with degrees $d_1, \ldots, d_m$, we can extend the augment refinement method to compute a GCD-free basis of F modulo T in $O(\sum_{1 \leq i < j \leq m}(d_i d_j)\mathsf{A}_n(T))$ operations in $\mathbb{K}$.
- With $\deg(T) = d$ and $\deg(T') = d'$, RCPairRefine or $\texttt{Difference}(T, T')$ costs $O(C^{n-1} d d')$ operations in $\mathbb{K}$.

Main Complexity Result

Assume all regular chains are monic and squarefree in dimension zero.

Theorem (Complexity of MPD)

Let $L = \{U_1, \ldots, U_m\}$ be a set of regular chains and the degree of U_i be d_i for $1 \le i \le m$. Then a pairwise disjoint representation of L can be computed in $O(C^{n-1} \sum_{1 \le i < j \le m} d_i d_j)$ operations in $\mathbb{K}$.

Remark

Consider the case when $d_i = d_j = d$, then MPD can be computed in $O(C^{n-1}(md)^2)$ operations.

Main Complexity Result

Theorem (Complexity of SMPD)

Given a set $L = \{C_1, \ldots, C_m\}$ of constructible sets, each of which is given by some pairwise disjoint regular chains. Let D_i be the number of points in C_i for $1 \leq i \leq m$. An intersection-free basis of L can be computed in $O(C^{n-1} \sum_{1 \leq i < j \leq m} D_i D_j)$ operations in $\mathbb{K}$.

Remark

A special case is when $D_i = D_j = D$, then the number of operations needed to compute an SMPD is bounded by $O(C^{n-1}(mD)^2)$.

An Experimental Comparison

- OldSMPD:
  - the defining regular systems of each constructible set are made (symmetrically) pairwise disjoint;
  - the set theoretical differences and the intersections are computed separately.
- BachSMPD
- DCSMPD: combining a divide-and-conquer approach with the augment refinement method for the operation SMPD.

Timing(s) of 15 Examples Computed by 3 SMPD Algorithms

Sys	OldSMPD	BachSMPD	DCSMPD
1	12.302	3.494	0.786
2	0.303	0.103	0.062
3	1.123	0.259	0.271
4	2.407	1.184	0.703
5	0.574	0.091	0.159
6	0.548	0.293	0.300
7	0.733	0.444	0.211
8	3.430	0.584	0.633
9	25.413	8.292	9.530
10	1097.291	82.468	122.575
11	11.828	0.930	0.985
12	54.197	1.934	1.778
13	0.530	0.047	0.064
14	27.180	13.705	4.626
15	–	1838.927	592.554

An Experimental Comparison

- Improved OldSMPD: extending the RCPairRefine algorithm to positive dimension and manages to compute the difference and the intersection in one pass
- BachSMPD + MPD and DCSMPD + MPD: cleaning each constructible set after SMPD.

Timing(s) of 15 Examples Computed by 3 SMPD Algorithms

Sys	OldSMPD(improved)	BachSMPD(+MPD)	DCSMPD+MPD
1	3.949	3.766	0.914
2	0.118	0.103	0.062
3	0.271	0.259	0.271
4	1.442	1.449	0.927
5	0.116	0.100	0.173
6	0.257	0.300	0.290
7	0.460	0.444	0.211
8	0.607	0.584	0.633
9	9.291	8.347	9.592
10	95.690	82.795	125.286
11	0.912	0.930	1.784
12	12.330	1.934	2.900
13	0.065	0.047	0.065
14	16.792	16.280	6.323
15	2272.550	1876.061	624.679

Conclusions

- We provide different algorithms for removing the redundancies and distinguished two cases:

MPD: *removal* of redundant components within a constructible set

SMPD: *removal with refinement* among a family of constructible sets.

- We rely on quadratic arithmetic motivated by practical concerns.

- We give a complexity analysis of these algorithms in dimension zero:
    - following the work of Bach, Driscoll and Shallit, we have obtained *essentially quadratic time complexity* based on classical (=quadratic) polynomial arithmetic.
    - the GCD of polynomials modulo a zero-dimensional regular chain can be done in *essentially quadratic time complexity* based on quadratic arithmetic.

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