π related Developments since 1988

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1 A COMPUTATIONAL UPDATE

Since the original publication of *Scientific American* article ('Ramanujan and Pi', February,1988) our predictions have been matched and more. All the subsequent calculations have been performed using methods described within the paper. We touch upon the most interesting.

In *1989*, The Chudnovsky Brothers computed 1,011,196,961 digits of π using a series of Ramanujan type (but missed by Ramanujan perhaps because it relies on analytic continuation). It has the attractive feature that it can effectively be used for a pure integer calculation and so restarted at will. It is

$$\frac{1}{\pi} = 12\sum_{n=0}^{\infty}(-1)^n \frac{(6n)!}{(n!)^3(3n)!}\frac{13591409+n545140134}{(640320^3)^{n+1/2}}.$$

The calculation required an IBM3090/VF for 120 hours and a CRAY2 for 28 hours. In short order Kanada then computed 1,073,741,799 digits (2^{30}-25) with a verification run. These together took under 161 hours. Kanada used the quartic algorithm described herein and the related original Gauss-Legendre-Salamin-Brent algorithm (see "Pi and the AGM", John Wiley 1987). In *1991* the Chudnovskys then computed in excess of 2.16 billion digits. Many details of their work on a largely home-built computer are given in the delightful profile "The Mountains of Pi" in the *New Yorker* magazine of March 2 , 1992.)

In *June 1995*, Kanada computed in excess of 3 billion digits after which the Chudnovskys reported (personal communication) that in *May 1994* they had passed the 4 billion digit mark (4,044,000,000 digits).

The two most recent record computations were performed by Yasumasa Kanada and his colleagues during *1995*. They were run using two CPUs on the HITAC S-3800/480. It is worth emphasizing that each of his computations effectively performs the equivalent of a few hundred full precision multiplications. A single such 6.4 billion digit multiplication performed on the same machine but without using FFT ideas *would take 4 months to a year* and thus the whole computation would take centuries. A few highlights follow. Much more is available at Kanada's web site – including detailed analysis of the digits which seem very normal (in the technical sense).

July-August 1995: 2^{32} digits. Independent calculations based on two different algorithms generated 4,294,967,296 (2^{32}) decimal digits of π and the two sequences matched to 4,294,967,286 decimal digits, e.g., a 10 decimal digit difference. The digits of π ending at the 4 billionth (after the decimal point) are 94375 34306 22684 47216.
September-October 1995: $3 \cdot 2^{31}$ digits. Independent calculations based on two different algorithms generated 6,442,450,944 ($3 \cdot 2^{31}$) decimal digits of π and the two sequences matched to 6,442,450,938 decimal digits, e.g., a 6 decimal digit difference. The exact times of this computation were:

Main program run:
Job start : 19th September 1995 20:54
Job end : 24th September 1995 17:32
Elapsed time: 116:38:12
Vector CPU: 112:36:06
Main memory : 1856.75 MB
ES memory : 32764 MB
Algorithm : Borweins' 4-th order convergent algorithm

Verification program run:
Job start: 06th October 1995 09:58
Job end: 11th October 1995 21:38
Elapsed time: 131:40:24
Vector CP: 135:45:52
Main memory: 1792.75 MB
ES memory : 32328 MB
Algorithm: Gauss-Legendre algorithm

2 SOME THEORETICAL DEVELOPMENTS

Various parts of the theory have been expanded over the last decade. Papers 1], 2], and 3] listed below highlight such developments. In particular, some striking identities and algorithms arre given in 1] and 2], while very rapidly converging series for $1/\pi$ that correspond to Ramanujan's theory for class number three are exhibited in 2]. The last paper below gives a very efficient algorithm for computing binary digits of π individually.
Finally, most of Ramanujan's notebooks, as edited and expanded upon by Bruce Berndt, have now appeared and so Ramanujan's works are thus much more accessible.

3 REFERENCES

1] Borwein, J. M., and Borwein, P. B.,"A cubic counterpart of Jacobi's identity and the AGM," *Trans. Amer. Math. Soc.*, **323** (1991), 691–701.

2] Borwein, J. M., and Borwein, P. B., "Class number three Ramanujan type series for $1/\pi$. Computational complex analysis," *J. Comput. Appl. Math.* **46** (1993), 281–290.

3] Borwein, J. M., Borwein, P. B., and Garvan, F. G., "Some cubic modular identities of Ramanujan," *Trans. Amer. Math. Soc.***343** (1994), 35–47.

4] Bailey, D., Borwein, P. B., and Plouffe S., "On the rapid computation of various polylogarithmic constants," (To appear.)