$3x + 1$ Problem Annotated Bibliography

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Abstract

The $3x + 1$ problem concerns iteration of the map $T : \mathbb{Z} \to \mathbb{Z}$ given by

$$T(x) = \begin{cases} \dfrac{3x+1}{2} & \text{if } x \equiv 1 \pmod{2} \ . \\[2ex] \dfrac{x}{2} & \text{if } x \equiv 0 \pmod{2} \ . \end{cases}$$

The $3x + 1$ Conjecture asserts that each $m \geq 1$ has some iterate $T^{(k)}(m) = 1$. This is an annotated bibliography of work done on the $3x + 1$ problem and related problems subsequent to the survey of Lagarias (1985).

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1. Terminology

The $3x + 1$ *function* is

$$T(x) = \begin{cases} \dfrac{3x+1}{2} & \text{if } x \equiv 1 \pmod 2) \\[2ex] \dfrac{x}{2} & \text{if } x \equiv 0 \pmod 2) \ . \end{cases}$$

The $3x + 1$ *Conjecture* states that every $m \geq 1$ has some iterate $T^{(k)}(m) = 1$.

The *stopping time* of m is the least k such that $T^{(k)}(m) < m$, and the *total stopping time* $\sigma_\infty(m)$ is the least k such that $T^{(k)}(m) = 1$. The *height* $h(m) = \sigma_\infty(m) + d(m)$ where $d(m)$ counts the number of iterates $T^{(k)}(m) \equiv 1 \pmod 2$ for $0 \leq k < \sigma_\infty(m)$. The height $h(m)$ is also the least k for which the *Collatz function*

$$C(x) = \begin{cases} 3x+1 & \text{if } x \equiv 1 \pmod 2) \ , \\[2ex] \dfrac{x}{2} & \text{if } x \equiv 0 \pmod 2) \ \ , \end{cases}$$

has $C^{(k)}(m) = 1$. Finally, the function $\pi_a(x)$ counts the number of n with $|n| \leq x$ that eventually iterate to a under T.

2. Bibliography

This bibliography covers articles on the $3x+1$ problem appearing since 1984. For references to earlier work see the survey article of Lagarias (1985).

1. E. Akin (1993^+), Why is the $3x + 1$ Problem So Hard?, preprint.

This paper analyzes the $3x+1$ problem by viewing the map T as acting on the domain

$\mathbb{Z}_2$ of 2-adic integers. The map T is topologically conjugate over $\mathbb{Z}_2$ to the 2-adic shift map

$$S(x) = \begin{cases} \dfrac{x-1}{2} & \text{if } x \equiv 1 \pmod{2}\ , \\ \\ \dfrac{x}{2} & \text{if } x \equiv 0 \pmod{2}\ , \end{cases}$$

by a conjugacy map $h : \mathbb{Z}_2 \to \mathbb{Z}_2$, i.e. $h \circ T = S \circ h$. The $3x+1$ Conjecture can be reformulated in terms of the behavior of h, namely that h maps $\mathbb{Z}^+$ into $\frac{1}{3}\mathbb{Z}$. More generally the paper considers the Rationality Conjecture, which asserts that h maps the rationals with odd denominators into themselves. Akin describes a heuristic argument suggesting that the Rationality Conjecture is true for the $3x+1$ map but is not true for the $mx+1$ map where m is any odd integer greater than 5.

2. S. Albeverto, D. Merlini and R. Tartini (1989), Una breve introduzione a diffusioni su insiemi frattali e ad alcuni essempi di sistemi dinamici semplici, Note di matematica e fisica, Edizioni Cerfim Locarno **3**, 1–39.

   Gives a short description of the $3x+1$ problem, along with other iteration problems.

3. S. Anderson (1987), Struggling with the $3x+1$ problem, Math. Gazette **71** (1987) 271–274.

   Studies simple analogues of $3x+1$ function such as

$$g(x) = \begin{cases} x+k & \text{if } x \equiv 1 \pmod{2}\ , \\ \\ \dfrac{x}{2} & \text{if } x \equiv 0 \pmod{2}\ . \end{cases}$$

   For $k = 1$ when iterated this map gives the binary expansion of x. The paper also reformulates the $3x+1$ Conjecture using the function:

$$f(x) = \begin{cases} \dfrac{x}{3} & \text{if } x \equiv 0 \pmod{3}\ , \\ \\ \dfrac{x}{2} & \text{if } x \equiv 2 \text{ or } 4 \pmod{6}\ , \\ \\ 3x+1 & \text{if } x \equiv 1 \text{ or } 5 \pmod{6}\ . \end{cases}$$

4. D. Applegate and J. C. Lagarias (1995a), Density Bounds for the $3x+1$ Problem I. Tree-Search Method, Math. Comp., **64**, 411–426. (MR95c:11024)

Let $n_k(a)$ count the number of integers n having $T^{(k)}(n) = a$. Then for any $a \not\equiv 0 \pmod 3$ and sufficiently large k, $(1.299)^k \le n_k(a) \le (1.361)^k$. Let $\pi_k(a)$ count the number of $|n| \le x$ which eventually reach a under iteration by T. If $a \not\equiv 0 \pmod 3$ then $\pi_a(x) > x^{.643}$ for all sufficiently large x. The extremal distribution of number of leaves in $3x+1$ trees with root a and depth k (under iteration of T^{-1}) as a varies are computed for $k \le 30$. The proofs are computer-intensive.

5. D. Applegate and J. C. Lagarias (1995b), Density Bounds for the $3x+1$ Problem II. Krasikov Inequalities, Math. Comp., **64**, 427–438. (MR95c:11025)

   Let $\pi_a(x)$ count the number of $|n| \le x$ which eventually reach a under iteration by T. If $a \not\equiv 0 \pmod 3$, then $\pi_a(x) > x^{.809}$ for all sufficiently large x. It is shown that the inequalities of Krasikov (1989) can be used to construct nonlinear programming problems which yield lower bounds for the exponent γ in $\pi_a(x) > x^{\gamma}$. The exponent above was derived by computer for such a nonlinear program having about 20000 variables.

6. D. Applegate and J. C. Lagarias (1995c), On the distribution of $3x+1$ trees, Experimental Mathematics **4**, 101–117.

   The extremal distribution of the number of leaves in $3x+1$ trees with root a and depth k (under iteration of T^{-1}) as a varies were computed for $k \le 30$ in Applegate and Lagarias (1995a). Here these are shown to have a much narrower spread around the mean value $(\frac{4}{3})^k$ of leaves in a $3x+1$ tree of depth k than is predicted by (repeated drawings from) the branching process models of Lagarias and Weiss (1992). Rigorous results are proved for the branching process models.

   The paper also derives formulas for the expected number of leaves in a $3x+1$ tree of depth k whose root node is $a \pmod{3^\ell}$. A 3-adic limit is proved to exist almost everywhere as $k \to \infty$, the expected number of leaves being $W_\infty(a)\left(\frac{4}{3}\right)^k$ where the function $W_\infty : \mathbb{Z}_3^\times \to \mathbb{R}$ almost everywhere satisies the 3-adic functional equation

$$W(\alpha) = \frac{3}{4}\left(W(2\alpha) + \psi(\alpha \bmod 9) W\left(\frac{2\alpha - 1}{3}\right)\right) , \qquad (*)$$

   in which $\psi(\alpha) = 1$ if $\alpha \equiv 2$ or $8 \pmod 9$ and is 0 otherwise. (Here $\mathbb{Z}_3^* = \{\alpha \in \mathbb{Z}_3 : \alpha \not\equiv 0 \pmod 3\}$). It is conjectured that W_∞ is continuous and everywhere nonzero. It is an open problem to characterize solutions of the functional equation $(*)$.

7. J. Arsac (1986), Algorithmes pour vérifier la conjecture de Syracuse, C. R. Acad. Sci. Paris **303**, Serie I, no. 4, 155–159. [Also: RAIRO, Inf. Théor. Appl. **21** (1987), 3–9.] (MR87m:11128)

Studies the computational complexity of algorithms to compute stopping times of $3x+1$ function on all integers below a given bound x.

8. C. Ashbacher (1992), Further Investigations of the Wondrous Numbers, J. Recreational Math. **24**, 1–15.

Numerically studies the "MU" functions $F_D(x)$ of Wiggin (1988) on $x \in \mathbb{Z}^+$ for $2 \le D \le 12$. He finds no exceptions for Wiggin's conjecture that all cycles of F_D on $\mathbb{Z}^+$ contain an integer smaller than D, for $x < 1.4 \times 10^7$. He tabulates integers in this range that have a large stopping time, and observes various patterns. These are easily explained by observing that, for $n \not\equiv 0 \pmod{D}$, $F_D^{(2)}(n) = \frac{n(D+1)-R}{D}$ if $n \equiv R \pmod{D}$, $-1 \le R \le D-2$, hence, for most n, $F_D^{(2)}(n) > n$, although F_D decreases iterates on the average.

9. S. Beltraminelli, D. Merlini and L. Rusconi (1994), Orbite inverse nel problema del $3n+1$, Note di matematica e fisica, Edizioni Cerfim Locarno **7**, 325–357.

[I have not seen this paper.]

10. L. Berg and G. Meinardus (1994), Functional equations connected with the Collatz problem, Results in Math. **25**, 1–12. (MR95d:11025).

This paper reformulates the $3x+1$ problem in terms of generating functions for the $3x+1$ mapping. Let $F(z,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} T^{(m)}(n) z^m y^n$, which converges for $|z| < 1$ and $|y| < \frac{2}{3}$. If $\omega = \exp\left(\frac{2\pi i}{3}\right)$, then

$$F(z^3, y) = \frac{z^3}{(1-z^3)^2} + y\ F(z^6, y) + \frac{y}{3z} \sum_{j=0}^{2} \omega^j F(\omega^j z^2, y)\ .$$

Consider the functional equation $h(z^3) = h(z^6) + \frac{1}{3z} \sum_{j=0}^{2} \omega^j h(\omega^j z^2)$. The $3x+1$ Conjecture is equivalent to the assertion that the only solutions of this equation holomorphic in $|z| < 1$ are $c_0 + \frac{c_1 z}{1-z}$ for complex constants c_0, c_1. If $f_m(z) = \sum_{n=0}^{\infty} T^{(m)}(n) z^n$ then $f_m(z) = p_m(z)(1-z^{2m})^{-2}$ for a polynomial $p_m(z) \in \mathbb{Z}[z]$, and various recurrence formulae for $p_m(z)$ are derived.

11. D. J. Bernstein (1994), A Non-Iterative 2-adic Statement of the $3x+1$ Conjecture, Proc. Amer. Math. Soc., **121**, 405–408. (MR94h:11108).

Let $\mathbb{Z}_2$ denote the 2-adic integers, and for $x \in \mathbb{Z}_2$ write $x = \sum_{i=0}^{\infty} 2^{d_i}$ with $0 \le d_0 < d_1 < d_2 < \cdots$. Set $\Phi(x) = -\sum_{j=0}^{\infty} \frac{1}{3^{j+1}} 2^j$. The map Φ is shown to be a homeomorphism of the 2-adic integers to itself, which is the inverse of the map Q_∞ defined in Lagarias (1985). The author proves that $\Phi \circ T \circ \Phi^{-1} = S$, where T is the $(3x+1)$-function on $\mathbb{Z}_2$, and S is the shift map

$$S(x) = \begin{cases} \dfrac{x-1}{2} & \text{if } x \equiv 1 \pmod{2}\ , \\[2ex] \dfrac{x}{2} & \text{if } x \equiv 0 \pmod{2}\ . \end{cases}$$

He shows that the $3x+1$ Conjecture is equivalent to the conjecture that $\mathbb{Z}^+ \subseteq \Phi(\frac{1}{3}\mathbb{Z})$. He rederives the known results that $\mathbb{Q} \cap \mathbb{Z}_2 \subseteq Q_\infty(\mathbb{Q} \cap \mathbb{Z}_2)$, and that Q_∞ is nowhere differentiable, cf. Müller (1991).

12. D. J. Bernstein and J. C. Lagarias (1995^+), The $3x+1$ Conjugacy Map, Canadian J. Math., to appear.

Studies the map $\Phi : \mathbb{Z}_2 \to \mathbb{Z}_2$ of Bernstein (1994) that conjugates the 2-adic shift map to the $3x+1$ function. This is the inverse of the map Q_∞ in Lagarias (1985); see also Akin (1993^+). The map $\bar{\Phi}_n \equiv \Phi \pmod{2^n}$ is a permutation of $\mathbb{Z}/2^n\mathbb{Z}$. This permutation is shown to have order 2^{n-4} for $n \ge 6$. Let $\hat{\Phi}_n$ denote the restriction of this permutation to $(\mathbb{Z}/2^n\mathbb{Z})^* = \{x : x \equiv 1 \pmod{2}\}$. The function Φ has two odd fixed points $x = -1$ and $x = 1/3$ and the 2-cycle $\{1, -1/3\}$, hence each $\hat{\Phi}_n$ inherits two 1-cycles and a 2-cycle coming from these points. Empirical evidence indicates that $\hat{\Phi}_n$ has about $2n$ fixed points for $n \le 1000$. A heuristic argument based on this data suggests that -1 and $1/3$ are the only odd fixed points of Φ. The analogous conjugacy map $\Phi_{25,-3}$ for the '$25x-3$' problem is shown to have no nonzero fixed points.

13. K. Borovkov and D. Pfeifer (1993^+), Estimates for the Syracuse problem via a probabilistic model, preprint.

Studies a multiplicative random walk imitating the $3x+1$ iteration. Let X_0 be given and set $Y_j = X_0 X_1 \cdots X_j$ where each X_i for $i \ge 1$ are i.i.d. random variables assuming

the value $\frac{1}{2}$ or $\frac{3}{2}$ with probability $\frac{1}{2}$ each. Let $\sigma_\infty(X_0, \omega)$ be a random variable equal to the smallest J such that $Y_j < 1$. Then $E[\sigma_\infty(X_0, \omega)] = (\frac{1}{2}\log\frac{4}{3})^{-1}\log X_0$ and the normalized variable

$$\hat{\sigma}_\infty(X_0, \omega) = \frac{\sigma_\infty(X_0, \omega) - c_1 \log X_0}{c_2(\log X_0)^{1/2}}$$

with $c_1 = (\frac{1}{2}\log\frac{4}{3})^{-1}$ and $c_2 = c_1^{3/2}(\frac{1}{2}\log 3)$, has a distribution converging to a standard normal distribution as $X_0 \to \infty$. Various refinements of this result are given. Comparisons are made to empirical $3x + 1$ data which show good agreement.

14. D. Boyd (1985), Which rationals are ratios of Pisot sequences?, Canad. Math. Bull. 343–349. (MR86j:11078).

    The Pisot sequence $E(a_0, a_1)$ is defined by $a_{n+2} = \left[\frac{a_{n+1}^2}{a_n} + \frac{1}{2}\right]$, where a_0, a_1 are integer starting values. If $0 < a_0 < a_1$ then $\frac{a_n}{a_{n+1}}$ converges to a limit θ as $n \to \infty$. The paper asks: which rationals $\frac{p}{q}$ can occur as a limit? If $\frac{p}{q} > \frac{q}{2}$ then $\frac{p}{q}$ must be an integer. If $\frac{p}{q} < \frac{q}{2}$ the question is related to a stopping time problem resembling the $3x + 1$ problem.

15. S. Brocco (1995), A Note on Mignosi's Generalization of the $3x + 1$ Problem, J. Number Theory, **52**, 173–178.

    Mignosi (1992) studied the function $T_\beta : \mathbb{N} \to \mathbb{N}$ defined by

$$T_\beta(n) = \begin{cases} \lceil \beta n \rceil & \text{if } n \equiv 1 \pmod 2 \\ \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \end{cases},$$

    where $\lceil x \rceil$ denotes the smallest integer $n \ge x$. He also formulated Conjecture C_β asserting that T_β has finitely many cycles and that every $n \in \mathbb{N}$ eventually enters a cycle under T_β. This paper shows that Conjecture C_β is false whenever β is a Pisot number or a Salem number. The result applies further to functions $T_{\beta,\alpha}(n) = \lceil \beta n + \alpha \rceil$ if $n \equiv 1 \pmod 2$, $\frac{n}{2}$ if $n \equiv 0 \pmod 2$, for certain ranges of values of α.

16. S. Burckel (1994), Functional equations associated with congruential functions, Theoretical Computer Science **123**, 397–406. (MR94m.11147).

    The author proves undecidability results for periodically linear functions generalizing those of J. H. Conway [Proc. 1972 Number Theory Conference, Boulder, Colorado, 49–52]. A periodically linear function $f : \mathbb{Z} \to \mathbb{Z}$ is one which is a linear function on each congruence class (mod L) for some finite L. The author shows it is undecidable whether a given function has $f^{(k)}(1) = 0$ for some $k \geq 1$, and also whether a given function has the property: for each $n \geq 1$, some $f^{(k)}(n) = 0$. He also shows that the $3x + 1$ conjecture is equivalent to a certain functional equation having only the trivial solution over the set of all power-series $R(z) = \sum_{n=}^{\infty} a_n z^n$ with all $a_i = 0$ or 1. The functional equation is

$$3z^3 R(z^3) - 3z^9 R(z^6) - R(z^2) - R(\omega z^2) - R(\omega^2 z^2) = 0$$

where $\omega = \exp(\frac{2\pi i}{3})$.

17. R. N. Buttsworth and K. R. Matthews (1990), On some Markov matrices arising from the generalized Collatz mapping, Acta Arithmetica **55**, 43–57. (MR92a:11016).

    This paper studies maps $T(x) = \frac{m_i x - r_i}{d}$ for $x \equiv i (\text{mod } d)$, where $r_i \equiv i m_i (\text{mod } d)$. In the case where g.c.d. $(m_0, \ldots, m_{d-1}, d) = 1$ it gives information about the structure of T-ergodic sets (mod m) as m varies. A set $S \subseteq \mathbb{Z}$ is *T-ergodic (mod m)* if it is a union of k congruence classes (mod m), $S = C_1 \cup \ldots \cup C_k$, such that $T(S) \subseteq S$ and there is an n such that $C_j \cap T^{(n)}(C_i) \neq \phi$ holds for all i and j. It characterizes them in many cases. As an example, for

$$T(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \equiv 0 \pmod 2 , \\[2ex] \dfrac{5x-3}{2} & \text{if } x \equiv 1 \pmod 3 , \end{cases}$$

    the ergodic set (mod m) is unique and is $\{n : n \in \mathbb{Z} \text{ and } (n, m, 15) = 1\}$, i.e. it is one of $\mathbb{Z}, \mathbb{Z} - 3, \mathbb{Z} - 5\mathbb{Z}$ or $\mathbb{Z} - 3\mathbb{Z} - 5\mathbb{Z}$ as m varies. An example is given having infinitely many different ergodic sets (mod m) as m varies.

18. C. C. Cadogan (1984), A note on the $3x+1$ problem, Caribbean J. Math **3**, No. 2, 69–72. (MR87a:11013).

    Using an observation of S. Znam, this paper shows that to prove the $3x+1$ Conjecture it suffices to check it for all $n \equiv 1$ (mod 4). Contrast this with the obvious fact that to prove the $3x + 1$ Conjecture it suffices to check it for all $n \equiv 3$ (mod 4). See Korec and Znam (1987) for stronger results in this direction.

19. N. Chamberland (1995^+), A Continuous Extension of the $3x + 1$ Problem to the Real Line, preprint.

This paper studies the iterates on $\mathbb{R}$ of the function

$$\begin{aligned} f(x) &= \frac{\pi x}{2}\left(\cos\frac{\pi x}{2}\right)^2 + \frac{3x+1}{2}\left(\sin\frac{\pi x}{2}\right)^2 \\ &= x + \frac{1}{4} - \left(\frac{x}{2} + \frac{1}{4}\right)\cos \pi x \ . \end{aligned}$$

which interpolates the $3x+1$ function T. All iterates of f contract to a fixed point 0 on $[0, \mu_1)$ where $\mu_1 = 0.27773\ldots$. The interval $[\mu_1, \mu_3]$ is invariant under f, where $\mu_3 = 2.44570\ldots$ and this interval includes the trivial cycle $A_1 = \{1, 2\}$. On this interval almost every point is attracted to one of two attracting cycles, which are A_1 and $A_2 = \{1.19253\ldots,\ 2.13865\ldots\}$. There is also an uncountable set of measure 0 on which the dynamics is "chaotic." On the interval $[\mu_3, \infty)$ the set of x that do not eventually iterate to a point in $[\mu_1, \mu_3]$ is conjectured to be of measure zero. The point μ_3 is proved to be a "homoclinic point," in the sense that for any $\epsilon > 0$ the iterates of $[\mu_3, \mu_3 + \epsilon)$ cover the whole interval (μ_1, ∞). It is shown that any nontrivial cycle of the $3x+1$ function would be an attracting periodic orbit of f. A fact crucial to the analysis is that f has negative Schwartzian derivative $Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$ on $\mathbb{R}^+$.

20. Busiso P. Chisala (1994), Cycles in Collatz Sequences, Publ. Math. Debrecen **45**, 35–39.

    The author shows that for any m-cycle of the Collatz map on positive *rationals*, the least element is at least as large as $(2^{[m\theta]/m} - 3)^{-1}$, where $\theta = \log_2 3$. Using this result, he derives a lower bound for $3x+1$ cycle lengths based on the continued fraction of $\theta = [1, a_1, a_2, a_3, \ldots]$, in which the n-th convergent is $\frac{p_n}{q_n}$, and the intermediate convergent denominator q_n^i is $iq_{n+1} + q_n$ for $0 \le i < a_{n+1}$. If the $3x+1$ conjecture is true for $1 \le n \le N$, and $N \ge (2^{C(i,k)} - 3)^{-1}$, where $C(i,k) = \frac{[q_k^i \theta]}{q_k^i}$, then there are no nontrivial cycles of the $3x+1$ function on $\mathbb{Z}^+$ containing less than q_k^{i+1} *odd* terms. Using the known bound $N = 2^{40} \doteq 1.2 \times 10^{12}$, the author shows that there are at least $q_{15} = 10\ 787\ 915$ odd terms in any cycle of the $3x+1$ function on $\mathbb{Z}^+$.

    [Compare these results with those of Eliahou (1993).]

21. Dean Clark (1995), Second-Order Difference Equations Related to the Collatz $3n+1$ Conjecture, J. Difference Equations & Appl., **1**, 73–85.

    The paper studies the integer-valued recurrence $\frac{x_{n+1}+x_n}{2}$ if $x_{n+1} + x_n$ is even, and

$x_n = \frac{b|x_{n+1}-x_n|+1}{2}$ if $x_{n+1} + x_n$ is odd, for $b \geq 1$ an odd integer. For $b = 1, 3, 5$ all recurrence sequences stabilize at some fixed point depending on x_1 and x_2, provided that $x_1 = x_2 \equiv \frac{b+1}{2} \pmod{b}$. For $b \geq 7$ there exist unbounded trajectories, and periodic trajectories of period ≥ 2. In the "convergent" cases $b = 3$ or 5 the iterates exhibit an interesting phenomenon, which the author calls *digital convergence,* where the low order digits in base b of x_n successively stabilize before the high order bits stabilize.

22. D. Clark and J. T. Lewis (1995^+), A Collatz-Type Difference Equation, preprint.

    This paper studies the difference equation

$$x_n = \begin{cases} \dfrac{x_{n-1} + x_{n-2}}{2} & \text{if } x_{n-1} + x_{n-2} \text{ is even },\\[2ex] x_{n-1} - x_{n-2} & \text{if } x_{n-1} + x_{n-2} \text{ is odd }. \end{cases}$$

    Without loss of generality $(x_0, x_1) = 1$. For such initial conditions the recurrence always converges to one of the 1-cycles 1 or -1 or to the 6-cycle $\{3, 2, -1, -3, -2, -1\}$.

23. T. Cloney, E. C. Goles and G. Y. Vichniac (1987), The $3x + 1$ Problem: a Quasi-Cellular Automaton, Complex Systems **1**, 349–360. (MR88d:68080).

    Draws computer graphics pictures of binary expansions of $\{T^{(i)}(m) \ldots 1, 2, \ldots\}$ for "random" large m, using black and white pixels to represent 1 resp. 0 (mod 2). It discusses patterns seen in these pictures. No theorems.

24. L. Collatz (1986), On the Origin of the $(3n + 1)$-Problem, J. of Qufu Normal University, Natural Science Edition **12**, No. 3, 9–11 (Chinese, transcribed by Zhi-Ping Ren). [German translation: Über den Ursprung des $(3n + 1)$-Problems, by Zhang-Zheng Yu 1991, U. Hamburg]

    Lothar Collatz describes his interest since 1928 in iteration problems represented using associated graphs and hypergraphs. He describes the structure of such graphs for several different problems. He states that he invented the $3x + 1$ problem and publicized it in many talks. He says: "Because I couldn't solve it I never published anything. In 1952 when I came to Hamburg I told it to my colleague Prof. Dr. Helmut Hasse. He was very interested in it. He circulated the problem in seminars and in other countries."

25. P. Devienne, P. Lebègue, J.-C. Routier (1993), Halting Problem of One Binary Horn Clause is Undecidable, *Proceedings of STACS 1993,* Lecture Notes in Computer Science No. **665**, Springer–Verlag, pp. 48–57. (MR95e:03114).

The halting problem for derivations using a single binary Horn clause for reductions is shown to be undecidable, by encoding Conway's problem of iterating periodically linear functions having no constant terms [Proc. 1972 Number Theory Conf., Boulder, Univ. of Colorado, pp. 49–52]. In contrast, the problem of whether or not ground can be reached using reductions by a single binary Horn clause is decidable. [M. Schmidt-Schauss, Theor. Comp. Sci. **59** (1988), 287–296.]

26. J. M. Dolan, A. F. Gilman and S. Manickam (1987), A generalization of Everett's result on the Collatz $3x+1$ problem, Adv. Appl. Math. **8**, 405–409. (MR89a:11018)

Shows that for any $k \geq 1$, the set of $m \in \mathbb{Z}+$ having k distinct iterates $T^{(i)}(m) < m$ has density one.

27. P. Eisele and K. P. Hadeler (1990), Game of Cards, Dynamical Systems, and a Characterization of the Floor and Ceiling Functions, Amer. Math. Monthly **97**, 466–477. (MR91h:58086).

Studies iteration of the mappings $f(x) = a + \lceil \frac{x}{b} \rceil$ on $\mathbb{Z}$ where a, b are positive integers. These are periodical linear functions (mod b). For $b \geq 2$, every trajectory becomes eventually constant or reaches a cycle of order 2.

28. S. Eliahou (1993), The $3x+1$ problem: New Lower Bounds on Nontrivial Cycle Lengths, Discrete Math., **118**, 45–56. (MR94h:11017).

The author shows that any nontrivial cycle on $\mathbb{Z}+$ of the $3x+1$ function $T(x)$ has period $p = 301994A + 17087915B + 85137581C$ with A, B, C nonnegative integers where $B \geq 1$, and at least one of A or C is zero. Hence the minimal possible period length is at least 17087915. The method uses the continued fraction expansion of $\log_2 3$, and the truth of the $3x+1$ Conjecture for all $n < 2^{40}$.

29. P. D. T. A. Elliott (1985), *Arithmetic Functions and Integer Products,* Springer-Verlag, New York.

An *additive function* is a function with domain $\mathbb{Z}^+$, which satisfies $f(ab) = f(a) + f(b)$ if $(a,b) = 1$. In Chapters 1–3 Elliott studies additive functions having the property that $|f(an+b) - f(An+B)| \le c_0$ for all $n \ge n_0$, for fixed positive integers a, b, A, B with $\det \begin{bmatrix} a & b \\ A & B \end{bmatrix} \ne 0$, and deduces that $|f(n)| \le c_1(\log n)^3$. For the special case $\begin{bmatrix} a & b \\ A & B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ an earlier argument of Wirsching yields a bound $|f(n)| \le c_2(\log n)$. On page 19 Elliott indicates that the analogue of Wirsching's argument for $\begin{bmatrix} a & b \\ A & B \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$ leads to the $3x+1$ function, and implies that $|f(n)| \le c_3\sigma_\infty(n)$. A strong form of the $3x+1$ Conjecture claims that $\sigma_\infty(n) \le c_4 \log n$, see Lagarias and Weiss (1992). Elliott proves elsewhere by other arguments that in fact $|f(n)| \le c_4 \log n$ holds. [P. D. T. A. Elliott, J. Number Theory 16 (1983) 285–310]

30. M. R. Feix, A. Muriel, D. Merlini, and R. Tartini (1995), The $(3x+1)/2$ Problem: A Statistical Approach, in: *Stochastic Processes, Physics and Geometry II,* Locarno 1991. (Eds: S. Albeverio, U. Cattaneo, D. Merlini) World Scientific, 1995, pp. 289–300.

    Heuristic study of random walk models imitating "average" behavior of the $3x+1$ function.

31. M. R. Feix, M. Muriel and J. L. Rouet (1994), Statistical Properties of an Iterated Arithmetic Mapping, J. Stat. Phys., **76**, 725–741.

    This paper interprets iteration of the $3x+1$ map as exhibiting a "forgetting" mechanism concerning the iterates (mod 2^k), i.e. after k iterations starting from elements it draws from a fixed residue class (mod 2^k), the iterate $T^k(n)$ is uniformly distributed (mod 2^k). It proves that certain associated $2^k \times 2^k$ matrices M_k has $(M_k)^k = J_{2^k}$ where J_{2^k} is the doubly-stochastic $2^k \times 2^k$ matrix having all entries equal to 2^{-k}.

32. P. Filipponi (1991), On the $3n+1$ Problem: Something Old, Something New, Rendiconti di Mathematica, Serie VII, Roma **11**, 85–103. (MR92i:11031).

    Derives by elementary methods various facts about coalescences of trajectories and divergent trajectories. For example, the smallest counterexample n_0 to the $3x+1$ Conjecture, if one exists, must have $n_0 \equiv 7, 15, 27, 31$, 39, 43, 63, 75, 79, 91 (mod 96). [The final Theorem 16 has a gap in its proof, because formula (5.11) is not justified.]

33. Z. Franco (1990), Diophantine Approximation and the $qx + 1$ Problem, Ph.D. Thesis, Univ. of Calif. at Berkeley. (H. Helson, Advisor).

Considers iteration of the $qx + 1$ function defined by $C_q(x) = \frac{qx+1}{2^{\mathrm{ord}_2(qx+1)}}$, where $2^{\mathrm{ord}_2(y)} || y$, and both q and x are odd integers. The first part of the thesis studies a conjecture of Crandall (1978), and the results appear in Franco and Pomerance (1995). The second part of the thesis gives a method to determine for a fixed q whether there are any orbits of period 2, i.e. solutions of $C_q^{(2)}(x) = x$, and it shows that for $|q| < 10^{11}$, only $q = \pm 1, \pm 3, 5, -11, -91$, and 181 have such orbits. The method uses an inequality of F. Beukers, [Acta Arith. **38** (1981) 389–410].

34. Z. Franco and C. Pomerance (1995), On a Conjecture of Crandall Concerning the $qx + 1$ Problem, Math. Comp., **64**, 1333–1336.

Considers iterates of the $qx+1$ function $C_q(x) = \frac{qx+1}{2^{\mathrm{ord}_2(qx+1)}}$, where $2^{\mathrm{ord}_2(y)} || y$ and both q and x are odd integers. Crandall (1978) conjectured that for each odd $q \geq 5$ there is some $n > 1$ such that the orbit $\{C_q^{(k)}(n) : k \geq 0\}$ does not contain 1, and proved it for $q = 5, 181$ and 1093. This paper shows that $\{q :$ Crandall's conjecture is true for $q\}$ has asymptotic density 1, by showing the stronger result that the set $\{q : C^{(2)}(m) \neq 1$ for all $m \in \mathbb{Z}\}$ has asymptotic density one.

35. D. Gale (1991), Mathematical Entertainments: More Mysteries, Mathematical Intelligencer **13**, No. 3, 54–55.

Discusses the possible undecidability of the $3x + 1$ Conjecture, and also whether the orbit containing 8 of the original Collatz function

$$U(n) = \begin{cases} \frac{3}{2}n & \text{if } n \equiv 0 \pmod 2 \\ \frac{3}{4}n + \frac{1}{4} & \text{if } n \equiv 1 \pmod 4 \\ \frac{3}{4}n - \frac{1}{4} & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

is infinite.

36. G.-G. Gao (1993), On consecutive numbers of the same height in the Collatz problem, Discrete Math., **112**, 261–267. (MR94i:11018).

Proves that if there exists one k-tuple of consecutive integers all having the same height and same total stopping time, then there exists infinitely many such k-tuples. (He attributes this result to P. Penning.) There is a 35654-tuple starting from $2^{500}+1$. He conjectures that the set $\{n : C^{(k)}(n) = C^{(k)}(n+1) \text{ for some } k \leq \log_2 n\}$ has natural density one, and proves that it has a natural density which is at least 0.389.

37. L. E. Garner (1985), On heights in the Collatz $3n+1$ problem, Discrete Math. **55**, 57–64. (MR86j:11005)

Shows that infinitely many pairs of consecutive integers have equal (finite) heights and equal total stopping times. To do this he studies how trajectories of consecutive integers can coalesce. Given two consecutive integers m, $m+1$ having $T^{(i)}(m) \neq T^{(i)}(m+1)$ for $i < k$ and $T^{(k)}(m) = T^{(k)}(m+1)$, associate to them the pair $(\mathbf{v}, \mathbf{v}')$ of $0-1$ vectors of length k encoding the parity of $T^{(i)}(m)$ (resp. $T^{(i)}(m+1)$) for $0 \leq i \leq k-1$. Call the set $\mathcal{A}$ of pairs $(\mathbf{v}, \mathbf{v}')$ obtained this way *admissible pairs*. Garner exhibits collections $\mathcal{B}$ and $\mathcal{S}$ of pairs of equal-length $0-1$ vectors $(\mathbf{b}, \mathbf{b}')$ and $(\mathbf{s}, \mathbf{s}')$ called *blocks* and *strings*, respectively, which have the properties: If $(\mathbf{v}, \mathbf{v}') \in \mathcal{A}$ and $(\mathbf{b}, \mathbf{b}') \in \mathcal{B}$ then the concatenated pair $(\mathbf{bv}, \mathbf{b}'\mathbf{v}') \in \mathcal{A}$, and if $(\mathbf{s}, \mathbf{s}') \in \mathcal{S}$ then $(\mathbf{sv}', \mathbf{s}'\mathbf{v}) \in \mathcal{A}$. Since $(001, 100) \in \mathcal{A}$, $(10, 01) \in \mathcal{B}$ and $(000011, 101000) \in \mathcal{S}$, the set $\mathcal{A}$ is infinite. He conjectures that: (1) a majority of all positive integers have the same height as an adjacent integer (2) arbitrarily long runs of integers of the same height occur.

38. H. Glaser and H.-G. Wiegand, Das-ULAM Problem-Computergestütze Entdeckungen, DdM **2** (1989), 114–134.

[I have not seen this paper.]

39. G. Gonnet (1991), Computations on the $3n+1$ Conjecture, MAPLE Newsletter, to appear.

Describes how to write computer code to efficiently compute $3x+1$ function iterates for very large x using MAPLE. It displays a computer plot of the total stopping function for $n < 4000$, revealing an interesting structure of well-spaced clusters of points.

40. R. K. Guy (1986), John Isbell's Game of Beanstalk and John Conway's Game of Beans Don't Talk, Math. Magazine **59**, 259–269. (MR88c:90163)

John Isbell's game of Beanstalk has two players alternately make moves using the rule

$$n_{i+1} = \begin{cases} \dfrac{n_1}{2} & \text{if } n_i \equiv 0 \pmod 2 , \\ 3n_i \pm 1 & \text{if } n_i \equiv 1 \pmod 2 , \end{cases}$$

where they have a choice if n_i is odd. The winner is the player who moves to 1. In Conway's game the second rule becomes $\dfrac{3n \pm 1}{2^*}$, where 2^* is the highest power of 2 that divides the numerator. It is unknown whether or not there are positions from which neither player can force a win. If there are then the $3x+1$ problem must have a nontrivial cycle or a divergent trajectory.

41. G. Hasenjager (1990), Hasse's Syracuse-Problem und die Rolle der Basen, in: *Mathesis rationis. Festschrift für Heinrich Schepers,* Nodus Publications: Müllernster (ISBN 3-89323-229-X), 329–336.

This paper, written for a philosopher's anniversary, comments on the complexity of the $3x+1$ problem compared to its simple appearance. It suggests looking for patterns in the iterates written in base 3 or base 27, rather than in base 2. Contains heuristic speculations and no theorems.

42. B. Hayes (1984), Computer recreations: The ups and downs of hailstone numbers, Scientific American **250**, No. 1, 10–16.

Introduces the $3x+1$ problem to a general audience under yet another name — hailstone numbers.

43. F. Jarvis (1989), 13, 31 and the $3x+1$ problem, Eureka **49**, 22–25.

Studies the function $g(n) = h(n+1) - h(n)$, where $h(n)$ is the height of n, and observes empirically that $g(n)$ appears unusually often to be representable as $13x + 31y$ with small values of x and y. It offers a heuristic explanation of this observation in terms of Diophantine approximations to $\log_6 2$. Several open problems are proposed, mostly concerning $h(n)$.

44. I. Korec (1992), The $3x+1$ Problem, Generalized Pascal Triangles, and Cellular Automata, Math. Slovaca, **42**, 547–563. (MR94g:11019).

This paper shows that the iterates of the Collatz function $C(x)$ can actually be encoded in a simple way by a one-dimensional nearest-neighbor cellular automaton with 7 states. The automaton encodes the iterates of the function $C(x)$ written in base 6. The encoding is possible because the map $x \to 3x+1$ in base 6 does not have carries propagate. (Compare the $C(x)$-iterates of $x = 26$ and $x = 27$ in base 6.) The $3x+1$ Conjecture is reformulated in terms of special structural properties of the languages output by such cellular automata.

45. I. Korec (1994), A Density Estimate for the $3x+1$ Problem, Math. Slovaca **44**, 85–89. (MR95h:11022).

Let $S_\beta = \{n : \text{some } T^{(k)}(n) < n^\beta\}$. This paper shows that for any $\beta > \frac{\log\ 3}{\log\ 4} \doteq .7925$ the set S_β has density one. The proof follows the method of E. Heppner [Arch. Math. **31** (1978) 317-320].

46. I. Korec and S. Znam (1987), A Note on the $3x+1$ Problem, Amer. Math. Monthly **94**, 771–772. (MR90g:11023)

Shows that to prove the $3x+1$ Conjecture it suffices to verify it for the set of all numbers $m \equiv a \pmod{p^n}$, for any fixed $n \geq 1$, provided that 2 is a primitive root (mod p) and $(a,p) = 1$. This set has density p^{-n}.

47. I. Krasikov (1989), How many numbers satisfy the $3x+1$ Conjecture?, Internatl. J. Math. & Math. Sci. **12**, 791–796. (MR91c:11013).

Shows that the number of integers $\leq x$ for which the $3x+1$ function has an iterate that is 1 is at least $x^{3/7}$. More generally, if $\theta_a(x) = \{n \leq x : \text{ some } T^{(k)}(n) = a\}$, then he shows that, for $a \not\equiv 0 \pmod 3$, $\theta_a(x)$ contains at least $x^{3/7}$ elements, for large enough x. The paper derives a set of difference inequalities which can be used to obtain an exponent larger than 3/7, see Applegate and Lagarias (1995b).

48. J. R. Kuttler (1994), On the $3x+1$ Problem, Adv. Appl. Math. **15**, 183–185.

The author states the oft-discovered fact that $T^{(k)}(2^k n - 1) = 3^k n - 1$ and derives his main result that if r runs over all odd integers $1 \leq r \leq 2^k - 1$ then $T^{(k)}(2^k n + r) = 3^p n + s$, in which $p \in \{1, 2, \ldots, k\}$ and $1 \leq s \leq 3^p$, and each value of p occurs exactly $\binom{k-1}{p-1}$ times. Thus the density of integers with $T^{(k)}(n) > n$ is exactly $\frac{\alpha}{2^k}$, where $\alpha = \sum_{p > \theta k} \binom{k-1}{p-1}$, and $\theta = \log_2\ 3$. This counts the number of inflating vectors in Theorem C in Lagarias (1985).

49. J. C. Lagarias (1985), The $3x+1$ problem and its generalizations, Amer. Math. Monthly **92**, 3–23. (MR86i:11043)

A survey with extensive bibliography of known results on $3x+1$ problem and related problems up to 1984. It also contains improvements of previous results and some new results, including in particular Theorems D, E, F, L and M.

Theorem O is stated incorrectly. The method of Conway (1972) gives, for any partial recursive function f, a periodic piecewise linear function g, with the property: iterating g with the starting value 2^n will never be a power of 2 if $f(n)$ is undefined, and will eventually reach a power of 2 if $f(n)$ is defined, and the first such power of 2 will be $2^f(n)$. Thus, in parts (ii) and (iii) of Theorem O, occurrences of n must be replaced by 2^n.

50. J. C. Lagarias (1990), The set of rational cycles for the $3x+1$ problem, Acta Arithmetica **56**, 33–53. (MR91i:11024).

This paper studies the sets of those integer cycles of

$$T_k(x) = \begin{cases} \dfrac{3x+k}{2} & \text{if } x \equiv 1 \pmod 2, \\[2ex] \dfrac{x}{2} & \text{if } x \equiv 0 \pmod 2 \text{ ,} \end{cases}$$

for positive $k \equiv \pm 1 \pmod 6$ which have $(x,k)=1$. These correspond to rational cycles $\frac{x}{k}$ of the $3x+1$ function T. It conjectures that every T_k has such an integer cycle. It shows that infinitely many k have at least $k^{1-\epsilon}$ distinct such cycles of period at most $\log k$, and infinitely many k have no such cycles having period length less than $k^{1/3}$. Estimates are given for the counting function $C(k,y)$ counting the number of such cycles of T_k of period $\le y$, for all $k \le x$ with $y = \beta \log x$. In particular $C(k, 1.01k) \le 5k(\log k)^5$.

51. J. C. Lagarias, H. A. Porta and K. B. Stolarsky (1993), Asymmetric Tent Map Expansions I. Eventually Periodic Points, J. London Math. Soc., **47**, 542–556. (MR94h:58139).

This paper studies the set $\mathrm{Per}(T_\alpha)$ of the asymmetric tent map

$$T_\alpha(x) = \begin{cases} \alpha x & \text{if } 0 < x \le \dfrac{1}{\alpha} \\[2ex] \dfrac{\alpha}{\alpha-1}(1-x) & \text{if } \dfrac{1}{\alpha} \le x < 1 \text{ ,} \end{cases}$$

where $\alpha > 1$ is real. It shows that $\mathrm{Per}(T_\alpha) = \mathbb{Q}(\alpha) \cap [0,1]$ for those α such that both α and $\frac{\alpha}{\alpha-1}$ are Pisot numbers. It conjectures that the same property holds for certain other α, including the real root of $x^5 - x^3 - 1 = 0$. The problem of proving that $\mathrm{Per}(T_\alpha) = \mathbb{Q}(\alpha) \cap [0,1]$ in these cases appears analogous to the problem of proving that the $3x+1$ function has no divergent trajectories.

52. J. C. Lagarias and A. Weiss (1992), The $3x+1$ Problem: Two Stochastic Models, Annals of Applied Probability **2**, 229–261. (MR92k:60159).

    Studies two stochastic models that mimic the "pseudorandom" behavior of the $3x+1$ function. For both models it proves that there is a constant $c_0 = 41.677...$ such that with probability one for any $\epsilon > 0$ only finitely many m take $(c_0 + \epsilon)\log m$ iterations to take the value 1, while infinitely many m take at least $(c_0 - \epsilon)\log m$ iterations to do so. This prediction is shown to be consistent with empirical data for the $3x+1$ function.

53. G. T. Leavens and M. Vermeulen (1992), $3x+1$ Search Programs, Computers & Mathematics, with Applications **24**, No. 11, 79–99. (MR43k:68047).

    Describes methods for computing $3x+1$ function iterates, and gives results of extensive computations done on a distributed network of workstations, taking an estimated 10 CPU-years in total. The $3x+1$ Conjecture is verified for all $n < 5.6 \times 10^{13}$. Presents statistics on various types of extremal trajectories of $3x+1$ iterates in this range. Gives a detailed discussion of techniques used for program optimization. (The program is still running and has now verified the $3x+1$ Conjecture for all $n < 6 \times 10^{13}$.)

54. G. M. Leigh (1986), A Markov process underlying the generalized Syracuse algorithm, Acta Arith. **46**, 125–143. (MR87i:11099)

    This paper considers mappings $T(x) = \frac{m_i x - r_i}{d}$ if $x \equiv i \pmod{d}$, where all $r_i \equiv i m_i \pmod{d}$. It presents conjectural formulae for limiting densities that divergent trajectories (should they exist) spend in a residue class $j \pmod{m}$.

55. G. M. Leigh and K. R. Matthews (1987), A generalization of the Syracuse algorithm to $F_q[x]$, J. Number Theory **25**, 274–278. (MR88f:11116)

    Defines mappings analogous to the $3x+1$ function on polynomials over finite fields, e.g. $T(f) = \frac{f}{x}$ if $f \equiv 0 \pmod{x}$ and $\frac{(x+1)^3 f + 1}{x}$ if $f \equiv 1 \pmod{x}$, over GF(2). It proves that divergent trajectories exist for certain such maps. These divergent trajectories have a regular behavior.

56. D. Marcu (1991), The powers of two and three, Discrete Math. **89**, 211-212. (MR92h:11026).

Erdös raised the question: "Does there exist an integer $m \neq 0, 2, 8$ such that 2^m is a sum of distinct integral powers of 3?" This paper shows that if $N(T)$ denotes the number of such $m \leq T$ then $N(T) \leq 2.495 T^{\theta}$ where $\theta = \frac{\log 2}{\log 3}$.

57. K. R. Matthews (1992), Some Borel measures associated with the generalized Collatz mapping, Colloq. Math. **53**, 191–202. (MR93i:11090).

This paper studies maps of the form

$$T(x) = \frac{m_i x - r_i}{d} \quad \text{if} \quad x \equiv i (\bmod\ d)\ .$$

These extend first to maps on the d-adic integers $\mathbb{Z}_d$, and then further to maps on the polyadic integers $\hat{\mathbb{Z}}$. Here $\hat{\mathbb{Z}}$ is the projective limit of the set of homomorphisms $\phi_{n,m}$: $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ where $m|n$. The open sets $B(j,m) := \{x \in \hat{\mathbb{Z}} : x \equiv j (\bmod\ m)\}$ put a topology on $\hat{\mathbb{Z}}$, which has a Haar measure $\sigma(B(j,m)) = \frac{1}{m}$. The paper proves conjectures of Buttsworth and Matthews (1990) on the structure of all ergodic open sets (mod m). In particular the ergodic sets link together to give finitely many projective systems, each giving T-invariant measure on $\hat{\mathbb{Z}}$. The paper gives examples where there are infinitely many ergodic sets.

58. K. R. Matthews (1992^+), A survey of some recent work on the generalized Collatz mapping, preprint.

[I have not seen this paper.]

59. K. Matthews and A. M. Watts (1984) A generalization of Hasse's generalization of the Syracuse algorithm, Acta. Arith. **43**, 167–175. (MR85i:11068)

This paper studies functions $T(x) = \frac{m_i x - r_i}{d}$ for $x \equiv i$ (mod d), where all m_i are positive integers and $r_i \equiv i m_i$ (mod d). It is shown that if $\{T^{(k)}(m) : k \geq 0\}$ is unbounded and uniformly distributed (mod d) then $m_1 m_2 \cdots m_d > d^d$ and $\lim_{k\to\infty} |T^{(k)}(m)|^{1/k} = \frac{1}{d}(m_1 \cdots m_d)^{1/d}$. The function T is extended to a mapping on the d-adic integers and is shown to be strongly mixing, hence ergodic, on $\mathbb{Z}_d$. The trajectories $\{T^{(k)}(\omega : k \geq 0\}$ for almost all $\omega \in \mathbb{Z}_d$ are equidistributed (mod d^k) for all $k \geq 1$.

60. K. Matthews and A. M. Watts (1985), A Markov approach to the generalized Syracuse algorithm, Acta Arith. **45**, 29–42. (MR87c:11071).

    This paper studies the same functions T considered in Matthews and Watts (1984). Given a modulus m one associates to T a row-stochastic matrix $Q = [q_{jk}]$ in which j, k index residue classes (mod m) and q_{jk} equals $\frac{1}{md}$ times the number of residue classes (mod md) which are $\equiv k(\text{mod } m)$ and whose image under T is $\equiv j(\text{mod } m)$. It gives sufficient conditions for the entries of Q^l to be the analogous probabilities associated to the iterated mapping $T^{(l)}$. Matthews and Watts conjecture that if $\mathcal{S}$ is an ergodic set of residues (mod m) and $(\alpha_i;\ i \in \mathcal{S})$ is the corresponding stationary vector on $\mathcal{S}$, and if $A = \Pi_{i \in \mathcal{S}} \left(\frac{m_i}{d}\right) \alpha_i < 1$ then all trajectories of T starting in $\mathcal{S}$ will eventually be periodic, while if $A > 1$ almost all trajectories starting in $\mathcal{S}$ will diverge. Some numerical examples are given.

61. P. Michel (1993), Busy Beaver Competition and Collatz-like Problems, Archive Math. Logic **32**, 351–367. (MR94f:03048)

    The Busy Beaver problem is to find that Turing machine which, among all k-state Turing machines, when given the empty tape as input, eventually halts and produces the largest number $\Sigma(k)$ of ones on the output tape. The function $\Sigma(k)$ is well-known to be non-recursive. This paper shows that the current Busy Beaver record-holder for 5-state Turing machine computes a Collatz-like function. This machine M_5 of Marxen and Buntrock [Bulletin EATCS No. **40** (1990) 247–251] has $\Sigma(M_5) = 47\ 176\ 870$. Michel shows that M_5 halts on all inputs if and only if iterating the function $g(3m) = 5m + 6$, $g(3m + 1) = 5m + 9$, $g(3m + 2) = \uparrow$ eventually halts at $\uparrow$ for all inputs. Similar results are proved for several other 5-state Turing machines. The result of Mahler [J. Aust. Math. Soc. **8** (1968) 313-321] on Z-numbers is restated in this framework: If a Z-number exists then there is an input m_0 such that the iteration $g(2m) = 3m$, $g(4m + 1) = 6m + 2$, $g(4m + 3) = \uparrow$ never halts with $\uparrow$ when started from m_0.

62. F. Mignosi (1995), On a Generalization of the $3x+1$ Problem, J. Number Theory, to appear.

For real $\beta > 1$, define the function $T_\beta : \mathbb{N} \to \mathbb{N}$ by

$$T_\beta(n) = \begin{cases} \lceil \beta n \rceil & \text{if } n \equiv 1 \pmod 2 \\ \dfrac{n}{2} & \text{if } n \equiv 0 \pmod 2 \,. \end{cases}$$

Then $\beta = \frac{3}{2}$ gives the $(3x+1)$-function. Conjecture C_β asserts that T_β has finitely many cycles and every $n \in \mathbb{N}$ eventually enters a cycle under iteration of T_β. The author shows that, for any fixed $0 < \epsilon < 1$, and for β either transcendental or rational with an even denominator, if $1 < \beta < 2$, then the set $S(\epsilon, \beta) = \{n : \text{some } T_\beta^{(k)}(n) < \epsilon n\}$ has natural density one, while if $\beta > 2$ then $S^*(\epsilon, \beta) = \{n : \text{some } T_\beta^{(k)}(n) > \epsilon^{-1} n\}$ has natural density one. For certain algebraic β more complicated behavior occurs, and Conjecture C_β can sometimes be settled. In particular Conjecture C_β is true for $\beta = \sqrt{2}$ and false for $\beta = \frac{1+\sqrt{5}}{2}$.

63. Helmut A. Müller (1991), Das '$3n+1$' Problem, Mitteilungen der Math. Ges. Hamburg **12**, 231–251. (MR93c:11053).

A survey, with complete proofs, of basic results on $3x+1$ problem, much of which overlaps Lagarias (1985). Contains new observations on the $3x+1$ function T on the domain of 2-adic integers $\mathbb{Z}_2$. For $\alpha \in \mathbb{Z}_2$ the 2-adic valuation $|\alpha|_2 = 2^{-j}$ if $2^j || \alpha$. Müller observes that $T(\alpha) = \sum_{n=0}^{\infty} (-1)^{n-1}(n+1)2^{n-2}\binom{\alpha}{n}$ where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$. The function $T(\alpha)$ is locally constant but not analytic. Define the function $Q_\infty : \mathbb{Z}_2 \to \mathbb{Z}_2$ by $Q_\infty(\alpha) = \sum_{i=0}^{\infty} a_i 2^i$ where $a_i = 0$ if $|T^{(i)}(\alpha)|_2 < 1$ and $a_i = 1$ if $|T^{(i)}(\alpha)|_2 = 1$. Lagarias (1985) showed that this function is a measure-preserving homeomorphism of $\mathbb{Z}_2$ to itself. Müller proves that Q_∞ is nowhere differentiable.

64. Helmut A. Müller (1994), Über eine Klasse 2-adischer Funktionen im Zusammenhang mit dem "$3x+1$"-Problem, Abh. Math. Sem. Univ. Hamburg **64**, 293–302. (MR95e:11032).

Let $\alpha = \sum_{j=0}^{\infty} a_{0,j} 2^j$ be a 2-adic integer, and let $T^{(k)}(\alpha) = \sum_{j=0}^{\infty} a_{k,j} 2^j$ denote its k-th iterate. Müller studies the functions $Q_j : \mathbb{Z}_2 \to \mathbb{Z}_2$ defined by $Q_j(\alpha) = \sum_{k=0}^{\infty} a_{kj} 2^k$. He proves that each function Q_j is continuous and nowhere differentiable. He proves that the function $f : \mathbb{Z}_2 \to \mathbb{Q}_2$ given by $f = \sum_{j=1}^{N} A_j Q_j(\alpha)$ with constants $A_j \in \mathbb{Q}_2$ is differentiable at a point α with $T^{(k)}(\alpha) = 0$ for some $k \geq 0$ if and only if $2A_0 + A_1 = 0$ and $A_2 = A_3 = \ldots = A_N = 0$.

65. C. Pickover (1989), Hailstone $3n+1$ Number graphs, J. Recreational Math. **21**, 120–123.

Gives two-dimensional graphical plots of $3x + 1$ function iterates revealing "several patterns and a diffuse background of chaotically-positioned dots."

66. Susana Puddu (1986), The Syracuse problem (Spanish), 5[th] Latin American Colloq. on Algebra – Santiago 1985, Notas Soc. Math. Chile **5**, 199–200. (MR88c:11010).

This note considers iterates of the Collatz function $C(x)$. It shows every positive m has some iterate $C^k(m) \equiv 1 \pmod 4$. If $m \equiv 3 \pmod 4$ the smallest such k must have $C^k(m) \equiv 5 \pmod{12}$.

67. D. Rawsthorne (1985), Imitation of an iteration, Math. Magazine **58**, 172–176. (MR86i:40001)

Proposes a multiplicative random walk that models imitating the "average" behavior of the $3x + 1$ function and similar functions. Compares the mean and standard deviation that this model predicts with empirical $3x + 1$ function data, and with data for several similar mappings, and finds good agreement between them.

68. O. Rozier (1990), Demonstraton de l'absence de cycles d'une certain forme pour le probléme de Syracuse, Singularité **1**, no. 3, 9–12.

Proves that there are no cycles except the trivial cycle whose iterates (mod 2) repeat a pattern of the form $1^m 0^{m'}$. Such cycles are called *circuits* by R. P. Steiner, who obtained this result in 1978. Rozier's proof uses effective transcendence bounds similar to Steiner's, but is simpler since he can quote the recent bound

$$\left| \frac{\log 3}{\log 2} - \frac{p}{q} \right| > q^{-15},$$

when $(p, q) = 1$, which appears in M. Waldschmidt, Equationes diophantiennes et Nombres Transcendents, Revue Palais de la Découverte, **17**, No. 144 (1987) 10–24.

69. O. Rozier (1991), Probleme de Syracuse: "Majorations" Elementaires des Cycles, Singularité **2**, no. 5, 8–11.

Proves that if the integers in a cycle of the $3x + 1$ function are grouped into blocks of integers all having the same parity, for any k there are only a finite number of cycles having $\leq k$ blocks. The proof uses the Waldschmidt result $\left|\frac{\log 3}{\log 2} - \frac{p}{q}\right| > q^{-15}$.

70. Jurgen W. Sander (1990), On the $(3N+1)$-conjecture, Acta Arith. **55**, 241-248.

    The paper shows that the number of integers $\leq x$ for which the $3x+1$ Conjecture is true is at least $x^{3/10}$, by extending the approach of Crandall (1978). (Krasikov (1989) obtains a better lower bound $x^{3/7}$ by another method.) This paper also shows that if the $3x+1$ Conjecture is true for all $n \equiv \frac{1}{3}(2^{2k}-1) \pmod{2^{2k-1}}$, for any fixed k, then it is true in general.

71. B. G. Seifert (1988), On the arithmetic of cycles for the Collatz-Hasse ('Syracuse') conjectures, Discrete Math. **68**, 293–298. (MR89a:11031).

    Gives criteria for cycles of $3x+1$ function to exist, and bounds the smallest number in the cycle in terms of the length of the cycle. Shows that if the $3x+1$ Conjecture is true, then the only positive integral solution of $2^l - 3^r = 1$ is $l = 2, r = 1$. [All integer solutions of $2^l - 3^r = 1$ can be found unconditionally by a method of Størmer, Nyt. Tidsskr. Math. B **19** (1908) 1–7. This has been explicitly done as a special case of various more general results by S. Pillai, Bull. Calcutta Math. Soc. **37** (1945) 15–20 (MR# 7, 145i); W. LeVeque, Amer. J. Math. **74** (1952) 325–331 (MR 13, 822f); R. Hampel, Ann. Polon. Math. **3** (1956) 1–4 (MR 18, 561c); etc.]

72. J. O. Shallit (1991), The "$3x+1$" Problem and Finite Automata, Bulletin of the EATCS (European Association for Theoretical Computer Science), No. 46, 182-185.

    A set S of positive integers is said to be *2-automatic* if the binary representations of the integers in S form a regular language $L_S \subseteq \{0,1\}^*$. Let S_i denote the set of integers n which have some $3x+1$ function iterate $T^{(j)}(n) = 1$, and whose $3x+1$ function iterates include exactly i odd integers ≥ 3. The sets S_i are proved to be 2-automatic for all $i \geq 1$.

73. Gy. Targonski (1991), Open questions about KW-orbits and iterative roots, Aequationes Math. **41**, 277–278.

    Targonski suggests the possibility of applying a result of H. Engl [An analytic representation for self-maps of a countably infinite set and its cycles, Aequationes Math. **25** (1982), 90-96, MR85d.04001] to bound the number of cycles of the $3x+1$ problem. Engl's

result expresses the number of cycles as the geometric multiplicity of 1 as an eigenvalue of a map on the sequence space l^1.

74. B. Thwaites (1985), My conjecture, Bull. Inst. Math. Appl. **21**, 35-41. (MR86j:11022)

    The author reiterates his earlier assertion to have invented the $3x+1$ problem in 1952. He derives basic known results about iterates and makes conjectures on the "average" behavior of trajectories. [It appears, however, that L. Collatz was the first to propose the $(3x+1)$-problem, see Collatz (1986).]

75. G. Venturini (1982), Sul Comportamento delle Iterazioni di Alcune Funzioni Numeriche, Rend. Sci. Math. Institute Lombardo **A 116**, 115-130. (Zbl. Math. 583.10009; MR87i:11015).

    The author studies functions $g(n) = a_r n + b_r$ for $n \equiv r \pmod p$ where a_r $(0 \le r \le p)$ are positive rationals with denominator p. He mainly treats the case that the a_r take two distinct values. If $\tau = (a_0 a_1 \cdots a_{p-1})^{1/p}$ has $\tau < 1$ then for almost all n there is some k with $g^{(k)}(n) < n$, while if $\tau > 1$ then the iterates tend to increase. [The Zentralblatt reviewer says that proofs are incomplete but contain an interesting idea. Rigorous versions of these results have since been established, see Lagarias (1985), Sect. 3.2.]

76. G. Venturini (1989), On the $3x+1$ Problem, Adv. Appl. Math **10**, 344-347. (MR90i:11020).

    Shows for any fixed ρ with $0 < \rho < 1$ that the set of $m \in \mathbb{Z}^+$ which either have some $T^{(i)}(m) = 1$ or some $T^{(i)}(m) < \rho m$ has density one.

77. G. Venturini (1992), Iterates of Number Theoretic Functions with Periodic Rational Coefficients (Generalization of the $3x+1$ problem), Studies in Applied Math. **86**, 185–218. (MR93b:11102).

    This paper studies iteration of maps $g : \mathbb{Z} \to \mathbb{Z}$ of the form $g(m) = \frac{1}{d}(a_r m + b_r)$ if $m \equiv r \pmod d$ for $0 \le r \le d-1$, where all $a_r,\ b_r \in \mathbb{Z}$, and $d \ge 2$ is an arbitrary integer. These maps generalize the $(3x+1)$-function, and include a wider class of such functions than Venturini (1989). The author is concerned with classifying g-ergodic sets S of such g which are finite unions of congruence classes. For example, the mapping $g(3m) = 2m$, $g(3m+1) = 4m+3$ and $g(3m+2) = 4m+1$ is a permutation and has $S = \{m : m \equiv 0 \text{ or } 5 \pmod{10}\}$ as a g-ergodic set. He then associates a (generally finite)

Markov chain to such a g-ergodic set, whose stationary distribution is used to derive a conjecture for the distribution of iterates $\{g^{(k)}(n_0 : k \geq 1\}$ in these residue classes for a randomly chosen initial value n_0 in S. For the example above the stationary distribution is $p_0 = \frac{1}{3}$ and $p_5 = \frac{2}{3}$. One can obtain also conjectured growth rates $a(g|_S)$ for iterates of a randomly chosen initial value in a g-ergodic set S. For the example above one obtains $a(g|s) = (\frac{4}{3})^{2/3}(\frac{2}{3}) \cong 1.0583$. The author's methods are similar in spirit to Matthews and Watts [Acta Arith. **43** (1984) 167-175 and **45** (1985) 29-42]. The author classifies the maps g into classes $G_v(d)$, for $v = 0, 1, 2, \ldots$, and also a class $G_\infty(d)$, according to how the various a_r of iterates $g(v)$ have common factors with d, and obtains results for more general functions than Matthews and Watts studied. He cannot handle all maps in the exceptional class $G_\infty(d)$, which sometimes lead to infinite Markov chains. The class $G_\infty(d)$ presumably contains functions constructed by J. H. Conway [Proc. 1972 Number Theory Conf., U. of Colorado, Boulder 1972, 39-42] which have all $b_r = 0$ and which can encode computationally undecidable problems. The author's Corollary to Theorem 6 has insufficient proof: It remains an open problem whether a $\mathbb{Z}$-permutation having an ergodic set S with $a(g|_S) > 1$ contains any orbit that is infinite. Sections 6 and 7 of the paper contain many interesting examples of Markov chains associated to such functions g; these examples are worth looking at for motivation before reading the rest of the paper.

78. G. Venturini (1994^+), On a Generalization of the $3x + 1$ problem, Adv. Appl. Math., to appear.

    This paper considers mappings $T(x) = \frac{t_r x - u_r}{p}$ when $x \equiv r \pmod{p}$, having $t_r \in \mathbb{Z}^+$, and $u_r \equiv rt_r \pmod{p}$. It shows that if gcd $(t_0 t_1 \ldots t_{p-1}, p) = 1$ and $t_0 t_1 \ldots t_{p-1} < p^p$ then for any fixed ρ with $0 < \rho < 1$ almost all $m \in \mathbb{Z}$ have an iterate k with $|T^{(k)}(m)| < \rho |m|$. The paper also considers the question: when are such mappings T permutations of $\mathbb{Z}$? It proves they are if and only if $\sum_{r=0}^{p-1} \frac{1}{t_r} = 1$ and $T(r) \not\equiv T(s) \pmod{(t_r, t_s)}$ for $0 \leq r < s \leq p - 1$. The geometric-harmonic mean inequality implies that $t_0 \ldots t_{p-1} > p^p$ for such permutations, except in the trivial case that all $t_i = p$.

79. S. Wagon (1985), The Collatz problem, Math. Intelligencer **7**, 72-76. (MR86d:11103).

    Studies a random walk imitation of the "average" behavior of the $3x + 1$ function,

computes its expected value and compares it to data on $3x+1$ iterates.

80. B. C. Wiggin (1988), Wondrous Numbers – Conjecture about the $3n+1$ family, J. Recreational Math. **20**, No. 2, 52-56.

This paper calls Collatz function iterates "Wondrous Numbers" and attributes this name to D. Hofstadter, *Gödel, Escher, Bach.* He proposes studying iterates of the class of "MU" functions

$$F_D(x) = \begin{cases} \dfrac{x}{D} & \text{if } x \equiv 0 \ (\mathrm{mod}\ D)\ , \\ (D+1)x - j & \text{if } x \equiv j \ (\mathrm{mod}\ D)\ ,\ 1 \le j \le D-2\ , \\ (D+1)x + 1 & \text{if } x \equiv -1 \ (\mathrm{mod}\ D)\ , \end{cases}$$

F_D is the Collatz function for $D = 2$. Wiggin's analogue of the $3x+1$ conjecture for a given $D \ge 2$ is that all iterates of $F_D(n)$ for $n \ge 1$ reach some number smaller than D. Somewhat surprisingly, no D is known for which this is false. It could be shown false for a given D by exhibiting a cycle with all members $> D$; no such cycles exist for $2 \le D \le 12$, and $x < 3 \times 10^4$.

81. G. J. Wirsching (1993), An Improved Estimate Concerning $3N+1$ Predecessor Sets, Acta Arithmetica, **63**, 205–210. (MR94e:11018).

Shows that, for all $a \not\equiv 0 \pmod 3$, the set $\theta_a(x) := \{n \le x : \text{some } T^{(k)}(x) = a\}$ has cardinality at least $x^{.48}$, for sufficiently large x. This is achieved by exploiting the inequalities of Krasikov (1989).

82. G. J. Wirsching (1994), A Markov chain underlying the backward Syracuse algorithm, Rev. Roumaine Math. Pures Appl. **XXXIX**, 915–926.

The author constructs from the inverse iterates of the $3x+1$ function ('backward Syracuse algorithm') a Markov chain defined on the state space $[0,1] \times \mathbb{Z}_3^{\times}$, in which $\mathbb{Z}_3^{\times}$ is the set of invertible 3-adic integers. He lets $g_n(k,a)$ count the number of "small sequence" preimages of an element $a \in \mathbb{Z}_3^{\times}$ at depth $n+k$, which has n odd iterates among its preimages, and with symbol sequence $0^{\alpha_0}10^{\alpha_1}\cdots 10^{\alpha_{n-1}}1$, with $\sum_{i=0}^{n} \alpha_i = k$ satisfying the "small sequence" condition $0 \le \alpha_j < 2 \cdot 3^{j-1}$ for $0 \le j \le n-1$. These quantities

satisfy a functional equation

$$g_n(k,a) = \frac{1}{2\cdot 3^{n-1}} \sum_{j=0}^{2\cdot 3^{n-1}} g_{n-1}\left(k-j\ ,\ \frac{2^{j+1}a-1}{3}\right)\ .$$

He considers the "renormalized" quantities

$$\hat{g}\left(\frac{k}{n}, a\right) := \frac{1}{\Gamma_n} g_n(k,a)\ ,$$

with

$$\Gamma_n = 2^{1-n} 3^{-\frac{1}{2}(n-1)(n-2)} (3^n - n)\ ,$$

He obtains in a weak limiting sense, a Markov chain whose probability density of being at (x,a) is a limiting average of $\hat{g}(x,a)$ in a neighborhood of (x,a) as the neighborhood shrinks to (x,α). One step of the backward Syracuse algorithm induces (in some limiting average sense) a limiting Markovian transition measure, which has a density taking the form of a product measure $\frac{3}{2}\chi_{[\frac{x}{3},\frac{x+2}{3}]} \otimes \phi$, in which $\chi_{[\frac{x}{3},\ \frac{x+2}{3}]}$ is characteristic function of the interval $\left[\frac{x}{3},\ \frac{x+2}{3}\right]$ and ϕ is a nonnegative integrable function on $\mathbb{Z}_3^\times$ (which may take the value $+\infty$).

The results of this paper are included in Chapter IV of Wirsching (1995).

83. G. J. Wirsching (1995^+), On the Combinatorial Structure of $3N+1$ Predecessor Sets, Discrete Math., to appear.

This paper studies the set $P(a)$ of inverse images of an integer a under the $3x+1$ function. Encode iterates $T^{(k)}(n) = a$ by a set of nonnegative integers $(\alpha_0, \alpha_1, \ldots, \alpha_\mu)$ such that the 0-1 vector $\mathbf{v}$ encoding $\{T^{(j)}(n) \pmod 2\colon\ 0 \le j \le k-1\}$ has $\mathbf{v} = 0^{\alpha_0}1\ 0^{\alpha_1}\cdots 1\ 0^{\alpha_\mu}$. Wirsching studies iterates corresponding to "small sequences," which are ones with $0 \le \alpha_i < 2\cdot 3^{i-1}$. He lets $G_\mu(a)$ denote the set of "small sequence" preimages n of a having a fixed number μ of iterates $T^{(j)}(n) \equiv 1 \pmod 2$, and shows that $|G_\mu(a)| = 2^{\mu-1}3^{1/2(\mu-1\ldots\mu-2)}$. He lets $g_a(k,\mu)$ denote the number of such sequences with $k = \alpha_0 + \alpha_1 + \cdots + \alpha_\mu$, and introduces related combinatorial quantities $\psi(k,\mu)$ satisfying $\sum_{0\le l\le k}\psi(l,\mu-1) \ge g_a(k,\mu)$. The quantities $\psi(k,\mu)$ can be asymptotically estimated, and have a (normalized) limiting distribution

$$\psi(x) = \lim_{\mu\to\infty} \frac{3^\mu - \mu}{2^\mu 3^{1/2\mu(\mu-1)}} \psi([3^\mu - \mu)x], \mu).$$

$\psi(x)$ is supported on $[0,1]$ and is a C^∞-function. He suggests a heuristic argument to estimate the number of "small sequence" preimages of a smaller than $2^n a$, in terms of a double integral involving the function $\psi(x)$.

The results of this paper are included in Chapter IV of Wirsching (1995).

84. G. J. Wirsching (1995), The Dynamical System on the Natural Numbers Generated by the $3n+1$ Function, Habilitationsschrift, Katholische Universität Eichstätt.

This Habilitationsschrift studies the problem of showing that for each positive integer $a \not\equiv 0 \pmod 3$ a positive proportion of integers less than x iterate to a, as $x \to \infty$. It develops an interesting 3-adic approach to this problem.

Chapter I contains a history of work on the $3x+1$ problem, and a summary of known results.

Chapter II studies the graph of iterates of T on the positive integers, ("Collatz graph") and, in particular studies the graph $\Pi^a(\Gamma_T)$ connecting all the inverse iterates $P(a)$ of a given positive integer a. This graph is a tree for any noncyclic value a. Wirsching uses a special encoding of the symbolic dynamics of paths in such trees, which enumerates symbol sequences by keeping track of the successive blocks of 0's. He then characterizes which graphs $\Pi^a(\Gamma_T)$ contain a given symbol sequence reaching the root node a. He derives "counting functions" for such sequences, and uses them to obtain a formula giving a lower bound for the function

$$P_T^n(a) := \{m : 2^n a \le m < 2^{n+1} a \ \text{ and some iterate } \ T^{(j)}(m) = a\} ,$$

which states that

$$|P_T^n(a) := \{m : 2^n(a) := \sum_{\ell=0}^{\infty} e_\ell(n + \lfloor \log_2 \left(\frac{3}{2}\right) \ell \rfloor \ , \ \ a) \ , \tag{1}$$

where $e_\ell(k,a)$ is a "counting function." (Theorem II.4.9). The author's hope is to use (1) to prove that

$$|P_T^n(a)| > c(a) 2^n \ , \quad n = 1, 2, \ldots \ . \tag{2}$$

for some constant $c(a) > 0$.

Chapter III studies the counting functions appearing in the lower bound above, in the hope of proving (2). Wirsching observes that the "counting functions" $e_\ell(k,a)$, which are ostensibly defined for positive integer variables k, ℓ, a, actually are well-defined when the variable a is a 3-adic integer. He makes use of the fact that one can define a 'Collatz graph' $\Pi^a(\Gamma_T)$ in which a is a 3-adic integer, by taking a suitable limiting process. Now the right side of (1) makes sense for all 3-adic integers, and he proves that actually $s_n(a) = +\infty$ on a dense set of 3-adic integers a. (This is impossible for any integer a because $|P_T^n(a)| \le 2^n a$.) He then proves that $s_n(a)$ is a nonnegative integrable function of a 3-adic variable, and he proves that its expected value $\bar{s}_n$ is explicitly expressible using binomial coefficients. Standard methods of asymptotic analysis are used to estimate $\bar{s}_n$, and to show that

$$\liminf_{n\to\infty} \frac{\bar{s}_n}{2^n} > 0 \ , \tag{3}$$

which is Theorem III.5.2. In view of (1), this says that (2) ought to hold in some "average" sense. He then proposes that (2) holds due to the:

Heuristic Principle. *As $n \to \infty$, $s_n(a)$ becomes relatively close to $\bar{s}_n$, in the weak sense that there is an absolute constant $c_1 > 0$ such that*

$$s_n(a) > c_1 \bar{s}_n \quad \textit{for all } n > n_0(a) \ . \tag{4}$$

One expects this to be true for all positive integers a. A very interesting idea here is that it might conceivably be true for all 3-adic integers a. If so, there is some chance to rigorously prove it.

Chapter IV studies this heuristic principle further by expressing the counting function $e_\ell(k, a)$ in terms of simpler counting functions $g_\ell(k, a)$ via a recursion

$$e_\ell(k, a) = \sum_{j=0}^{k} p_\ell(k-j) g_\ell(j, a) \ ,$$

in which $p_\ell(m)$ counts partitions of m into parts of a special form. Wirsching proves that properly scaled versions of the functions g_ℓ "converge" (in a rather weak sense) to a limit function which is independent of a, namely

$$g_\ell(k, a) \approx 2^{\ell-1} 3^{\frac{1}{2}(\ell^2 - 5\ell + 2)} \psi\left(\frac{k}{3^\ell}\right) \ , \tag{5}$$

for "most" values of k and a. (The notion of convergence involves integration against test functions.) The limit function $\psi : [0, 1] \to \mathbb{R}$ satisfies the functional-differential equation

$$\psi'(t) = \frac{9}{2}(\psi(3t) - \psi(3t - 2)) \ . \tag{6}$$

He observes that ψ has the property of being C^∞ and yet is piecewise polynomial, with infinitely many pieces, off a set of measure zero. There is a fairly well developed theory for related functional-differential equations, cf. C. Ryavec, Russian Math. Surveys **45** No. 1 (1990), 87–120. Finally, Wirsching observes that if the sense of convergence in (5) can be strengthened, then the heuristic principle can be deduced, and the desired bound (2) would follow.

85. Jia Bang Wu (1992), The trend of changes of the proportion of consecutive numbers of the same height in the Collatz problem (Chinese), J. Huazhong (Central China) Univ. Sci. Technol. **20**, 171–174. (MR94b:11024).

    [From Math Reviews.] This paper studies the density of consecutive numbers having the same height. Set $d(2^N) := \frac{1}{2^N-1}\{n < 2^N : h(n) = h(n+1)\}$. It computes $d(2^N)$ for $N \le 24$ and asserts that $d(2^N)$ is increasing with N for $N \le 24$. The longest k-tuple of consecutive numbers of the same height for $n \le 2^{30}$ is a 176-tuple with initial value $n = 722\ 067\ 240$.

86. Jia Bang Wu (1993), On the consecutive positive integers of the same height in the Collatz problem, Math. Appl., suppl. **6** (1993), 150–153 (Chinese).

    [I have not seen this paper.]

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Remark. I would appreciate knowing of any other papers on the $3x+1$ problem. Please write or send email to: jcl@research.att.com