# Lec20B Handouts

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Using the Euclidean algorithm to compute the resultant

```
> A := randpoly(x,degree=5,dense);
  B := randpoly(x,degree=5,dense);
  r := resultant(A,B,x);
```

$$A := 68\,x^5 - 10\,x^4 + 31\,x^3 - 51\,x^2 + 77\,x + 95$$

$$B := x^5 + x^4 + 55\,x^3 - 28\,x^2 + 16\,x + 30$$

$$r := -956123557049826225$$

```
> res := proc(A,B,x) local m,n,l,R,bn,r;
      m,n := degree(A,x),degree(B,x);                          1
      if n=0 then return B^m; fi;
      if m<n then return (-1)^(m*n)*res(B,A,x); fi;            2
      R := rem(A,B,x);
      if R=0 then return 0; else l := degree(R,x); fi;          5
      bn := coeff(B,x,n);
      l := degree(R,x);
      print(m,n,R);
      r := (-1)^(n*(m-l))*bn^(m-l)*res(R,B,x);                  6
  end:
> res(A,B,x);
```

$$5,\,5,\,-78\,x^4 - 3709\,x^3 + 1853\,x^2 - 1011\,x - 1945$$

$$5,\,4,\,\frac{13946533}{6084}\,x^3 - \frac{6977453}{6084}\,x^2 + \frac{1205525}{2028}\,x + \frac{7244815}{6084}$$

$$4,\,3,\,-\frac{371517739073676}{194505782720089}\,x^2 + \frac{281140993477656}{194505782720089}\,x + \frac{386731315559160}{194505782720089}$$

$$3,\,2,\,\frac{777125244375610486522252279875}{226866256486548328209329 64}\,x + \frac{408973178868620892276039630 65}{22686625648654832820932964}$$

$$2,\,1,\,\frac{867648688505987197442056649828245542967236}{124196543053931056062400800982059507430 5625}$$

$$-956123557049826225$$

The fractions in the Euc. Alg. grow.

```
> Bnd := `resultant/bound`(A,B,x);
```

$$Bnd := 143161325120846158463$$

```
> R,M := 0,1:
  p := 1000:
  while M<2*Bnd do
      p := nextprime(p);
      if irem(lcoeff(A,x),p)=0 then next fi;
      r := Resultant(A,B,x) mod p;        ———— computes res(A,B) in Zp[x].
      R,M := chrem([r,R],[p,M]), p*M;
  od:
  mods(R,M);
```

$$-956123557049826225$$

The ring $R$ is closed under differentiation.

(i) Let $R$ be a ring (field). $R$ is a <u>differential ring (field)</u> if $\exists D: R \to R$ s.t. $\forall f, g \in R$

(i) $D(f+g) = D(f) + D(g)$ and

(ii) $D(f \cdot g) = D(f) \cdot g + f \cdot D(g)$.

(ii) Let $F$ and $G$ be differential fields with $D_F$ and $D_G$. $G$ is a <u>differential extension</u> of $F$ if

(i) $F \subset G$ and (ii) $\forall f \in F$ $D_F(f) = D_G(f)$.

$F = \mathbb{Q}(x)$ $G = F(\theta) = \mathbb{Q}(x)(\ln x)$.

(iii) Let $F(\theta)$ be a differential extension of $F$.

$\theta$ is <u>logarithmic</u> over $F$ if $\exists u \in F$ s.t. $\theta' = \frac{u'}{u}$,

$\theta$ is <u>exponential</u> over $F$ if $\exists u \in F$ s.t. $\theta' = u'\theta$,

$\theta$ is <u>algebraic</u> over $F$ if $\exists p \in F[z]$ s.t. $p(\theta) = 0$.

$\theta$ is <u>transcendental</u> over $F$ if $\theta$ is NOT algebraic.

$\theta = \ln u$ $\theta' = u'/u$ $\theta = e^u$ $\theta' = u' \cdot e^u = u' \cdot \theta$

(iv) $G = F(\theta_1, \theta_2, \ldots, \theta_n)$ is an elementary extension of $F$ if $\theta_i$ is logarithmic, exponential or algebraic over $F(\theta_1, \ldots, \theta_{i-1})$ for $i = 1, 2, \ldots, n$.

$\mathbb{Q}(x)(\ln x, \ln \ln x)$

(v) The set of <u>elementary functions</u> of $x$ is

$E = \{ f : f \in \mathbb{C}(x)(\theta_1, \ldots, \theta_n) = G \}$ where $G$ is an elementary extension of $\mathbb{C}(x)$.

E.g. $e^x \ln(1+x) \in \mathbb{Q}(x)(\theta_1 = e^x, \theta_2 = \ln(1+x)) \subset E$.

E.g. $\sin(x) = [e^{-ix} - ie^{-ix}]/2i \in \mathbb{Q}(i)(x)(e^{ix}) \subset E$

E.g. $e^{e^{\sqrt{x}}} \in \mathbb{Q}(x)(\theta_1 = \sqrt{x}, \theta_2 = e^{\sqrt{x}}, \theta_3 = e^{\theta_2}) \subset E$.

Examples of the ``Risch" integration algorithm.

```
> f := ln(x)^2;
```

$$f := \ln(x)^2$$

```
> int(f,x);
```

$$\ln(x)^2 x - 2\,x \ln(x) + 2\,x$$

```
> f := (1-x^2*ln(x)^3+(-x^2+1)*ln(x)^2+(3-x)*ln(x))*exp(-x)
     /(x*ln(x)+1)^2;
```

$$f := \frac{\left(1 - x^2 \ln(x)^3 + \left(-x^2 + 1\right) \ln(x)^2 + (3 - x) \ln(x)\right) e^{-x}}{(x \ln(x) + 1)^2}$$

```
> int(f,x);
```

$$e^{-x} \ln(x) + \frac{(x - 1)\, e^{-x}}{x} - \frac{e^{-x}\,(x - 1)}{(x \ln(x) + 1)\, x}$$

```
> int(int(f,x),x);
```

$$-e^{-x} \ln(x) - e^{-x} + \int \left(-\frac{e^{-x}\,(x - 1)}{(x \ln(x) + 1)\, x}\right) dx$$

> $\int \frac{e^x}{\ln(x)}\, dx$

$$\int \frac{e^x}{\ln(x)}\, dx$$

> $\int \frac{e^{-x}}{x}\, dx$

$$-\mathrm{Ei}(1, x)$$

```
> ?Ei
```

> $\int e^{-x^2}\, dx$

$$\frac{1}{2} \sqrt{\pi}\, \mathrm{erf}(x)$$

```
> ?erf
```