Lec25B Risch: exponential subcase: logarithmic part
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<u>Theorem 12.9 (Trager-Rothstein – Exponential case)</u>

$\int \frac{C(\theta)}{D(\theta)}\,dx$ $\quad C, D \in F[\theta],\ \theta = e^w,\ \theta \notin F,\ \theta' \neq 0.$

$0 \le \deg_\theta C < \deg_\theta D,\ \mathrm{lc}_\theta D = 1,\ \theta \nmid D$

$\gcd(C, D) = 1,\ \gcd(D, D') = 1$ in $F[\theta]$

Let $R(z) = \mathrm{res}_\theta(\underbrace{C - zD'}_{F[\theta]}, D) \in F[z]$.

(i) $\int \frac{C}{D}\,dx$ is elementary iff roots of $R(z)$ are constants

(ii) If $\int \frac{C}{D}$ is elementary then $\int \frac{C}{D} = \sum c_i \log v_i + g_i$

where c_i are the distinct roots of $R(z)$

$v_i = \gcd(C - c_i D', D) \in F[\theta]$

$g_i = -c_i \deg_\theta(v_i) \cdot w$

Example $\int \frac{1}{e^x + 1} = \int \frac{1}{\theta + 1}$ $\quad C = 1,\ D = \theta + 1,\ D' = \theta$

$F(\theta) = \mathbb{Q}(x)(e^x)$

$R(z) = \mathrm{res}(1 - z\theta, \theta + 1) = \det\left(\begin{bmatrix} -z & 1 \\ 1 & 1 \end{bmatrix}\right)$

$= -z - 1 \Rightarrow c_1 = -1.$

$v_1 = \gcd(1 + \theta, 1 + \theta) = \theta + 1$

$g_1 = -(-1)\cdot 1 \cdot x = x$

$\int \frac{1}{e^x + 1} = c_1 \log v_1 + g_1 = -\log(e^x + 1) + x.$

L.T.

$$\int \frac{1}{e^x+1}\,dx = \int \frac{1}{\theta+1} = \underset{\substack{\Uparrow \\ F(\theta)}}{v_0(\theta)} + c_1 \cdot \log(e^x+1) + \sum \underset{\substack{\Uparrow \\ C}}{c_i} \log \underset{\substack{\Uparrow \\ F[\theta]}}{v_i(\theta)}$$

$$F(\theta) = Q(x)(e^x)$$

$$\frac{1}{e^x+1} = v_0'(\theta) + c_1 \frac{e^x}{e^x+1} + \sum L'$$

$$\overset{?}{=}$$

$$c_1\left(1 + \frac{-1}{e^x+1}\right) \leftarrow$$

Take $c_1 = -1$.

We have $0 = v_0'(\theta) - 1 + \sum L'$

$v_0 = x + \text{constant}$ and $\sum L = 0$.

$$\int \frac{1}{e^x+1} = \underset{\substack{\Uparrow \\ g}}{x} - \log(e^x+1) + \text{constant}.$$