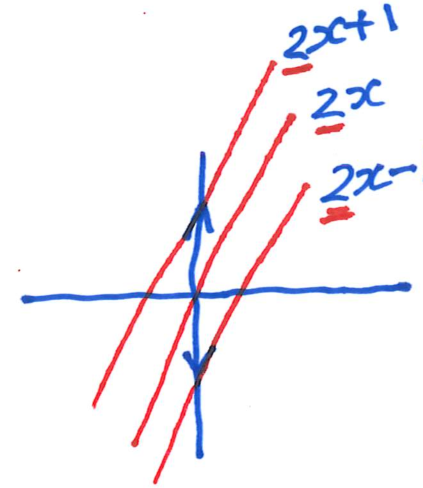
4.9 Antiderivatives

Math 152.

$f'(x)$	$f(x)$ ← antiderivative
2	? $2x$ $2x+3$ $2x+C$ ← general antiderivative.
$2x$	x^2+C
$\cos x$	$\sin x + C$
$1\cdot x+2$	$\frac{1}{2}x^2+2x+C$

particular antiderivatives



Theorem 1. If $f'(x)=g'(x)$ then $f(x)=g(x)+C$.

Def. If $f'(x)=g(x)$ then $f(x)$ is <u>an antiderivative of $g(x)$</u>.
and $f(x)+C$ is the <u>general antiderivative of $g(x)$</u>.

Ex 1. Find the general antiderivative of $2x+1$

$$f(x) = x^2 + x + C$$

What's C? Any value of $f(x)$, e.g. $f(2)=5$, determines C.

Ex 2. Given $f'(x) = 6x$ and $\boxed{f(0)=2}$, find $f(x)$.

$$f(x) = 3x^2 + C$$ ← general antiderivative

$$f(0) = \boxed{0 + C = 2} \Rightarrow C = 2.$$

$$f(x) = \underline{3x^2 + 2}$$ ← particular antiderivative.

Ex 3. Given $f''(x) = 6x$ and $\boxed{f(0)=3}$ and $\boxed{f'(0)=0}$ find $f(x)$.

$$f'(x) = 3x^2 + C$$

$$f'(0) = 0 + C = 0 \Rightarrow C = 0$$

$$f'(x) = 3x^2.$$

$$f(x) = x^3 + B$$ ← general antiderivative

$$f(0) = 0 + B = 3 \Rightarrow B = 3$$

$$f(x) = x^3 + 3.$$

Table of Antiderivatives

function	antiderivative
e^x $\xleftarrow{\prime}$	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$n \neq -1$ $\quad 1 \cdot x^n$	$\frac{1}{n+1} x^{n+1}$
$x^{-1} = \frac{1}{x}$	$\ln x$
a	$a \cdot x$
$c \cdot f'(x)$ $\xleftarrow{\prime}$	$c \cdot f(x)$
$f'(x) + g'(x)$	$f(x) + g(x)$
$1/\sqrt{1-x^2}$	$\sin^{-1} x$
$1/(1+x^2)$	$\tan^{-1} x$
$\sec^2 x$	$\tan x$
$\ln x$	?

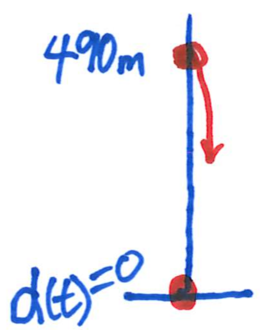
$1 \cdot x^2$ $\xleftarrow{\prime}$	$\frac{1}{3} x^3$
$\sqrt{x} = 1 \cdot x^{1/2}$	$\frac{1}{3/2} x^{3/2} = \frac{2}{3} \cdot x^{3/2}$
$2 \cdot \cos x$	$2 \cdot \sin x$

Linear Motion

Let $d(t)$, $v(t)$, $a(t)$ be the distance travelled, velocity, acceleration of an object at time t. Then $\boxed{v(t) = d'(t)}$ and $\boxed{a(t) = v'(t)}$.

Example The falling body problem.

If a stone is dropped from a height of 490m above ground, how long does it take before it hits the ground? How fast is it going when it hits the ground?



$d(0) = 490$

$\boxed{v(0) = 0}$

$a(t) = -9.8 \ m/s^2$

$$
\begin{aligned}
a(t) = v'(t) &= -9.8 \\
v(t) &= \boxed{-9.8t} + C \\
v(0) &= 0 + C = 0 \Rightarrow C = 0 \\
v(t) &= -9.8t \\
v(t) = d'(t) &= -9.8t \\
d(t) &= -4.9t^2 + B \\
d(0) &= 0 + B = 490 \Rightarrow B = 490
\end{aligned}
$$

$$d(t) = -4.9t^2 + 490.$$

The stone hits the ground when $d(t) = \boxed{0 = -4.9t^2 + 490} \Rightarrow \underline{t = 10s.}$

$v(10) = -9.8 \cdot 10 = \underline{-98\, m/s}.$

Sigma Notation

upper limit

summand.

Def the sum $\displaystyle\sum_{i=m}^{n} \underline{f(i)} = f(m) + f(m+1) + \cdots + f(n).$

index

lower limit

E.g. $\displaystyle\sum_{i=0}^{3} (\underbrace{2i+1}_{f(i)}) = \underset{i=0}{1} + \underset{i=1}{3} + \underset{i=2}{5} + \underset{i=3}{7} = 16.$

Properties ① $\displaystyle\sum_{i=m}^{n} c \cdot f(i) = c \sum_{i=m}^{n} f(i)$ ② $\displaystyle\sum_{i=m}^{n} (f(i) + g(i)) = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i).$

E.g. $\displaystyle\sum_{i=0}^{3} (2i+1) \overset{②}{=} \sum_{i=0}^{3} 2i + \sum_{i=0}^{3} 1 \overset{①}{=} 2 \sum_{i=0}^{3} i + \sum_{i=0}^{3} 1$

$= 2(0+1+2+3) + (1+1+1+1)$

$= 2 \cdot 6 + 4$

$= 16.$:)

Two Useful Formulas

① $\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

② $\sum_{i=1}^{n} i^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

E.g. $\sum_{i=1}^{n}(2i-1) = 1+3+5+\cdots+2n-1$ = sum of the first n odd numbers.

$$= \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} -1\cdot 1 = 2\left(\sum_{i=1}^{n} i\right) - \sum_{i=1}^{n} 1$$

$$= 2\left(\frac{n(n+1)}{2}\right) - n$$

$$= n(n+1) - n$$

$$= n^2 + n - n$$

$$= \underline{n^2}.$$

$\sum_{i=1}^{3}(2i-1) = 1+3+5 = \underline{9}$