5.5 Substitutions

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Antiderivatives $\int f(x)\,dx$

Definite Integrals $\int_a^b f(x)\,dx$

What is the dx for?

Consider $\int e^{2x+1}\,dx = \int e^u \cdot \frac{du}{2} = \int \frac{1}{2}e^u\,du = \frac{1}{2}e^u + C$

$= \frac{1}{2}e^{2x+1} + C.$

Let $\boxed{u = 2x+1}$

$\frac{du}{dx} = 2 \Rightarrow du = 2dx$

$\Rightarrow dx = \frac{du}{2}$

① $\int x^3\cos(x^4+2)\,dx$ $\quad u = x^4+2$

② $\int \frac{x}{1+x^2}\,dx$ $\quad u = 1+x^2$

③ $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$ $\quad u = \tan x$ ✓ $\quad u = \sin x$ ✓ $\quad u = \cos x$ ✓✓

④ $\int \frac{x^2}{\sqrt{1+x}}\,dx$ $\quad u = 1+x$

① $\int x^3\cos(x^4+2)\,dx = \int \frac{x^3\cos(u)\cdot du}{4x^3} = \int \frac{1}{4}\cos u\,du$

$= \frac{1}{4}\sin u + C$

$= \frac{1}{4}\sin(x^4+2) + C$

Let $u = x^4+2$

$\frac{du}{dx} = 4x^3 \Rightarrow dx = \frac{du}{4x^3}$

② $\int \frac{x}{1+x^2}\,dx = \int \frac{x}{u}\cdot\frac{du}{2x} = \int \frac{1}{2}\cdot\frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$

$= \frac{1}{2}\ln|\underbrace{1+x^2}_{\geq 0}| + C$

$= \frac{1}{2}\ln(1+x^2) + C.$

$u = 1+x^2$

$\frac{du}{dx} = 2x \quad dx = \frac{du}{2x}$

③ $\int \frac{\sin x}{\cos x}\,dx = \int \frac{\cancel{\sin x}}{u}\cdot\frac{(-du)}{\cancel{\sin x}} = \int -\frac{1}{u}\,du = -\ln|u| + C$

$= -\ln|\cos x| + C$

$u = \cos x$

$\frac{du}{dx} = -\sin x \Rightarrow dx = -\frac{du}{\sin x}$

$\int \frac{\sin x}{\cos x}\,dx = \int \frac{u}{\cos x}\cdot\frac{du}{\cos x} = \int \frac{u}{\cos^2 x}\,du = \int \frac{u}{1-\sin^2 x}\,du$

$= \int \frac{u}{1-u^2}\,du$

$u = \sin x$

$du/dx = \cos x$

$\sin^2 x + \cos^2 x = 1$

$\Rightarrow \cos^2 x = 1 - \sin^2 x$

$u = \sin x$ $\quad$ $\sin^2 x + \cos^2 x = 1$ $\quad$ $= \int \frac{u}{1-u^2}\,du$

$du/dx = \cos x$ $\quad$ $\Rightarrow \cos^2 x = 1 - \sin^2 x$ $\quad$ Let $v = 1-u^2$.

$dx = du/\cos x$

④ $$\int \frac{x^2}{\sqrt{1+x}}\,dx = \int \frac{x^2}{\sqrt{u}}\,du = \int \frac{(u-1)^2}{\sqrt{u}}\,du = \int \frac{u^2-2u+1}{\sqrt{u}=u^{1/2}}\,du$$

$u = 1+x \Rightarrow x = u-1.$

$du/dx = 1 \Rightarrow dx = du$

$$= \int (u^{3/2} - 2u^{1/2} + u^{-1/2})\,du$$

$$= \cdots$$

Calculating $\int_a^b f(x)\,dx$ using a substitution.

Ex. $$\int_0^1 \frac{x}{1+x^2}\,dx = \left[\tfrac{1}{2}\ln(1+x^2)\right]_0^1 = \tfrac{1}{2}\ln(2) - \tfrac{1}{2}\cancel{\ln(1)}^{0}$$
$$= \tfrac{1}{2}\ln 2.$$

Method ① $$\int \frac{x}{1+x^2}\,dx \overset{u=1+x^2}{=} \cdots = \underbrace{\tfrac{1}{2}\ln(1+x^2)}_{F(x)} + C$$

Method ② $$\int_0^1 \frac{x}{1+x^2}\,dx = \int_1^2 \frac{\cancel{x}}{u}\cdot\frac{du}{2\cancel{x}} = \int_1^2 \frac{1}{2}\cdot\frac{1}{u}\,du = \left[\tfrac{1}{2}\ln|u|\right]_1^2$$
$$= \tfrac{1}{2}\ln 2 - \tfrac{1}{2}\ln 1$$
$$= \tfrac{1}{2}\ln 2.$$

Let $\boxed{u = 1+x^2}$

$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$ $\quad$ $x=0$, $u = 1+0^2$ $\quad$ $x = 1$, $u = 1+1^2$

Warning! $$\int_0^1 \frac{x}{1+x^2}\,dx \neq \int_0^1 \frac{1}{2}\cdot\frac{1}{u}\,du = \left[\tfrac{1}{2}\ln|u|\right]_0^1 = \tfrac{1}{2}\ln 1 - \tfrac{1}{2}\ln 0.$$
X.

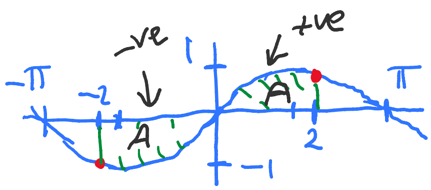
Property. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Proof: FTC (2): If $F'(x) = f(x)$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$
$$= -(F(a) - F(b))$$
$$= -\int_b^a f(x)\,dx \text{ by FTC (2).}$$

Application. Odd functions. $f(-x) = f(x)$.

E.g. $\sin x,\ x,\ x^3,\ x^5, \ldots$



$$\int_{-2}^{2} \sin x \, dx = A - A = 0.$$

$\sin 2 = -\sin(-2)$.

Suppose $f(x)$ is odd.

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$\| \, ?$

$$= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0.$$

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)(-du) = \int_{a}^{0} -f(u)(-du) = \int_{a}^{0} f(u)\,du$$

Let $\boxed{u = -x} \Rightarrow x = -u$

$\frac{du}{dx} = -1 \Rightarrow dx = -du$

$x = -a \qquad x = 0$

$u = +a \qquad u = 0$

Prop

$$= -\int_{0}^{a} f(u)\,du. = -\int_{0}^{a} f(x)\,dx$$

Let $\boxed{u = x}$

$\frac{du}{dx} = 1 \Rightarrow du = dx$