MATH 800 Course Project Fall 2023
The FGLM Gröbner Basis Conversion Algorithm

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The project is worth 20% of your final grade. Hand in by 11pm Friday December 15th, 2023.

Consider the following system in $\mathbb{R}[b, s, w, z, t, p]$ due to Trinks.

$$\begin{aligned} f_1 &= 45p + 35s - 165b - 36 \\ f_2 &= 35p + 40z + 25t - 27s \\ f_3 &= 15w + 25ps + 30z - 18t - 165b^2 \\ f_4 &= -9w + 15pt + 20zs \\ f_5 &= wp + 2zt - 11b^3 \\ f_6 &= 99w - 11sb + 3b^2 \end{aligned}$$

Let $F = \{f_1, f_2, \ldots, f_6\}$, $I = \langle f_1, f_2, \ldots, f_6 \rangle$ and $V = \mathbb{V}(f_1, f_2, \ldots, f_6)$ in $\mathbb{R}^6$. To solve the system, that is, to compute V we would like to apply the elimination theorem. So we want to compute a Gröbner basis for I in a lex monomial ordering e.g. $t > z > s > b > p > w$. It is known that this is much harder in general than computing a Grobner basis in a graded monomial ordering. The FGLM algorithm [1], discovered by Faugere, Gianni, Lazard, and Mora, will convert a Gröbner basis from any monomial ordering to any other monomial ordering, efficiently, using Linear Algebra, provided the ideal is zero dimensional (see Zero Dimensional Ideals below). Therefore to get our lex Gröbner basis efficiently we would first compute

```
> Groebner[Basis](F,grlex(t,z,s,b,p,w));
```

then use FGLM to get a Gröbner basis in the desired ordering plex(t, z, s, b, p, w). Maple does this automatically if you do

```
> Groebner[Basis](F,plex(t,z,s,b,p,w));
```

Study the notes on the FGLM algorithm. You are to implement the FGLM algorithm. You will need to solve systems of linear equations. You can do this using the `solve` command but that will not be the most efficient. Since you don't know the dimension of the system in advance (you get equations one at a time) you really should want to reduce a new equation wrt a linear system in reduced row Echelon form from the previous step. You can do this using the ReducedRowEchelonForm command in the LinearAlgebra package.

Zero Dimensional Ideals

Definition: Let $I = \langle f_1, f_2, \ldots, f_n \rangle$ be an ideal in $k[x_1, \ldots, x_n]$ for a field k. Let $R = k[x_1, \ldots, x_n]/I$. R is a quotient ring. We say I *is zero-dimensional* if the quotient ring R is a finite dimensional as a vector space over k.

There are several ways to characterize zero-dimensional ideals.

Theorem: Let I and R be as stated in the definition above.

Let $V = \mathbb{V}(f_1, f_2, \ldots, f_n)$ be the corresponding variety.
Let $L = \langle LT(I) \rangle = \{LT(f) : f \in I\}$ be the leading term ideal.
Let $C = \{monomials\ m : m \notin LT(I)\}$
Let $G = \{g_1, g_2, \ldots, g_t\}$ be any Gröbner basis for I.

The following statements are equivalent

1. I is zero-dimensional.
2. R is finite dimensional.
3. If k is algebraically closed then $|V|$ is finite.
4. $|C|$ is finite.
5. For $1 \leq i \leq n$ there exists a $g \in G$ s.t. $LM(g) = x_i^{m_i}$ for some $m_i \in \mathbb{N}$.

So if we want to solve a system of polynomial equations over $\mathbb{C}$ then zero dimensional ideals correspond to systems with finitely many solutions. Moreover there is a surprising connection between $|V|$ and $|C|$, namely $|V| \leq |C|$.

Do the following. For the Trinks system, for the two monomial orderings grlex(t, z, s, b, p, w) and plex(t, z, s, b, p, w) use Maple to compute Gröbner bases for F above, and for each ordering compute the set C described in the Theorem using any means. Use it to bound $|V|$. The LeadingMonomial command in the Gröbner basis package will be useful. If you print out the two Gröbner bases you will notice that the integer coefficients in the plex Gröbner basis are much bigger than those in the grlex Gröbner basis. Compute the size (in decimal digits) of the largest coefficient of both bases. Use the `maxnorm` and `length` commands.

Now program the FGLM algorithm in Maple and use it to convert the grlex Gröbner basis to the plex Gröbner basis. Before you do this for the Trinks problem, first get your code to work for the simpler system $[f_1 = x^2 + y^2 - 4, f_2 = x^2 - xy - 1]$ in $\mathbb{Q}[x, y]$. So first compute

```
> Groebner[Basis]([f1,f2],grlex(x,y));
```

$$[xy + y^2 - 3, x^2 + y^2 - 4, 2\,y^3 + 3\,x - 7\,y]$$

Thus the leading terms ideal $L = \langle xy, x^2, y^3 \rangle$ and $C = \{1, y, y^2, x\}$. Now use your FGLM code to get a plex(x, y) Gröbner basis. You could also test your code on the example in the handout which is $f_1 = xy + z - xz, f_2 = x^2 - z, f_3 = 2x^3 - x^2yz - 1$ in $\mathbb{Q}[x, y, z]$.

References

[1] J.C. Faugère; P. Gianni; D. Lazard; T. Mora (1993). Efficient Computation of Zero-dimensional Gröbner Bases by Change of Ordering. *J. Symbolic Computation* **16**(4):329-–344, 1993.