Linear Algebra I : Gaussian Elimination

September 21, 2023 9:22 AM

Computational Exact Linear Algebra.

Linear Algebra over fields e.g. $\mathbb{Q}$, $\mathbb{F}_q$, $\mathbb{Q}(\alpha)$ and integral domains e.g. $\mathbb{Z}$, $\mathbb{Z}[x_1, \ldots, x_n]$ and $F[x]$, F a field.

Let R be a comm. ring, $A \in R^{n \times n}$, $u \in R^n$, $B \in R^{n \times n}$.

$$A \cdot u = \begin{bmatrix} \times\times\times\times \\ \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} \cdot \begin{bmatrix} \times \\ \times \\ \times \\ \times \end{bmatrix} \qquad A \cdot B = \begin{bmatrix} \times\times\times\times \\ \text{---} \\ \text{---} \\ \text{---} \end{bmatrix} \begin{bmatrix} \times & | & | & | \\ \times & | & | & | \\ \times & | & | & | \\ \times & | & | & | \end{bmatrix} = \begin{bmatrix} \times & & \\ \times & & \\ \times & & \\ \times & & \end{bmatrix}$$

does n^2 mults in R Does $n^2 \cdot n = n^3$ mults in R.

$A \cdot B_1$

$$\underset{n^3 + n^2}{(A \cdot B) \cdot u} = \underset{n^2 + n^2}{A \cdot (B \cdot u)}$$

The order of operations matters costwise.

Today. Let F be a field, $A \in F^{n \times n}$, $b \in F^n$. Use Gaussian elimination to

Compute $\det(A)$ $\begin{bmatrix} \times\times\times \\ \times\times\times \\ \times\times\times \end{bmatrix} \sim \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}$

Solve $Ax = b$ $[A \mid b] \sim [I \mid x]$

Invert A $[A \mid I_n] \sim [I \mid A^{-1}]$.

Example. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\det(A) = 2 \cdot \frac{3}{2} = 3.$

$$[A \mid b] = \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 2 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - \frac{1}{2}R_1} \left[\begin{array}{ccc} 2 & 1 & 1 \\ 0 & 3/2 & 1/2 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{2}{3}R_2} \left[\begin{array}{ccc} 2 & 1 & 1 \\ 0 & 1 & 1/3 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc} 2 & 0 & 2/3 \\ 0 & 1 & 1/3 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{2}R_1} \left[\begin{array}{ccc} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \end{array}\right]$$

solution x.

A^{-1}

$$[A \mid I_2] = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - \frac{1}{2}R_1} \left[\begin{array}{cccc} 2 & 1 & 1 & 0 \\ 0 & 3/2 & -\frac{1}{2} & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{2}{3}R_2}$$

$$\left[\begin{array}{cccc} 2 & 1 & 1 & 0 \\ 0 & 1 & -1/3 & 2/3 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{cccc} 2 & 0 & 4/3 & -2/3 \\ 0 & 1 & -1/3 & 2/3 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{2}R_1} \left[\begin{array}{cccc} 1 & 0 & 2/3 & -1/3 \\ 0 & 1 & -1/3 & 2/3 \end{array}\right]$$

Algorithm Gaussian Elimination. Let F be a field.

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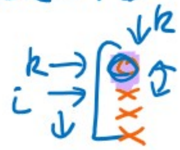
Input $A \in F^{n \times m}$ where $m \geq n$. Assume $\mathrm{rank}(A) = n$.

Output is $\det(A)$ and A in RREF.



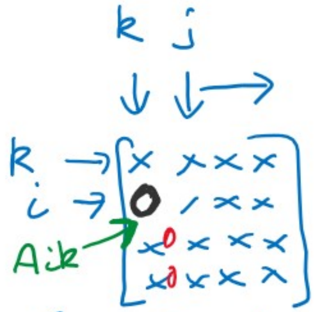
$det \leftarrow 1$.

for $k = 1, 2, \ldots, n$ do

Find $A_{ik} \neq 0$ for $k \leq i \leq n$.

If no such i exists then output 0 ($\det(A[n,n])$)

If $i \neq k$ then interchange row i and row k and set $det = -det$.



$R_i \leftarrow R_i - \frac{A_{ik}}{A_{kk}} \cdot R_k$

for $i = k+1, k+2, \ldots, n$

[if $A_{ik} = 0$ then next i else $\mu \leftarrow \frac{A_{ik}}{A_{kk}}$.

[for $j = k+1, k+2, \ldots, m$

$A_{ij} \leftarrow A_{ij} - \frac{A_{ik}}{A_{kk}} \cdot A_{kj} = A_{ij} - \mu \cdot A_{kj}$

$A_{ik} \leftarrow 0$.

Lets count #mults in F.

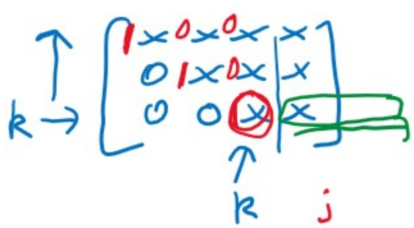
$$= \sum_{k=1}^{n} \sum_{i=k+1}^{n} \sum_{j=k+1}^{m} 1 = \frac{n^2 m}{2} - \frac{nm}{2} - \frac{n^3}{6} + \frac{n}{6}$$

Sum(sum(sum(1, j=k+1..m), i=k+1..n), k=1..n)

$m = n$ $\quad \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$

$m = n+1$ $\quad 1/3\, n^3 - 1/3\, n$ $\quad \in O(n^3)$.

$m = 2n$ $\quad 5/6\, n^3 - n^2 + n/6$



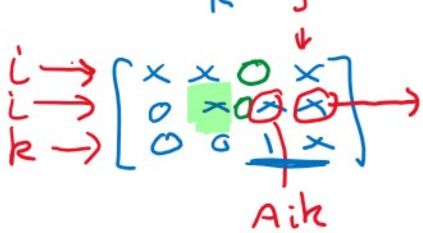
for $k = n, n-1, \ldots, 1$ do

$det = det * A_{kk}$.

$\mu = 1/A_{kk}$.

for $j = k+1, k+2, \ldots, m$ do

$A_{kj} \leftarrow A_{kj} \cdot \mu$.



for $i = 1, 2, \ldots, k-1$ do

$k \to [\ \cdots\ ]$

A_{ik}

for $i = 1, 2, \ldots, k-1$ do
[if $A_{ik} = 0$ then next i
for $j = n+1, n+2, \ldots, m$ do
$A_{ij} \leftarrow A_{ij} - A_{ik} \cdot A_{kj}$.

The # mults is $\displaystyle\sum_{k=n}^{1} \sum_{j=k+1}^{m} 1 + \sum_{k=n}^{1} \sum_{i=1}^{k-1} \sum_{j=n+1}^{m} 1$

$$= \tfrac{1}{2}n^2 m + \tfrac{1}{2}nm - \tfrac{1}{2}n^3 - \tfrac{1}{2}n^2$$

$\det(A)$	$m=n$	: 0
$Ax=b$	$m=n+1$	: $\frac{1}{2}n^2 + \frac{1}{2}n$.
A^{-1}	$m=2n$	: $\frac{1}{2}n^3 + \frac{1}{2}n^2$

Total			
	$\det(A)$	$m=n$	$\frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$
	$Ax=b$	$m=n+1$	$1/3\,n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$
	A^{-1}	$m=2n$	$4/3\,n^3 - \frac{1}{2}n^2 + \frac{1}{6}n$