Lecture 23 Euler Circuits
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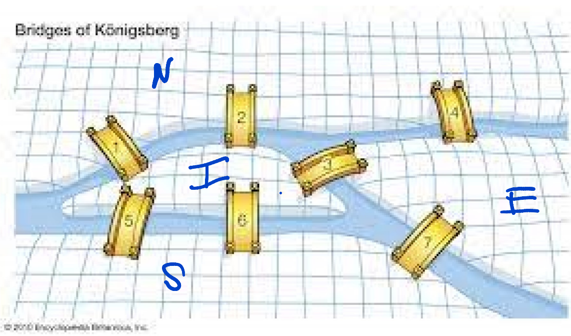
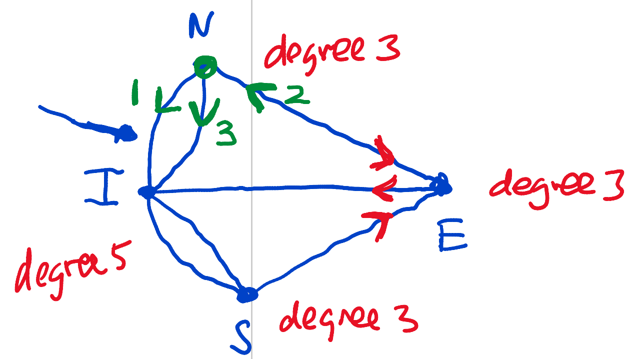
Lecture 23 Eulerian Trails and Circuits

Grimaldi 11.3





Question: Is it possible walk around the city and cross each bridge once?
and end up where you started?

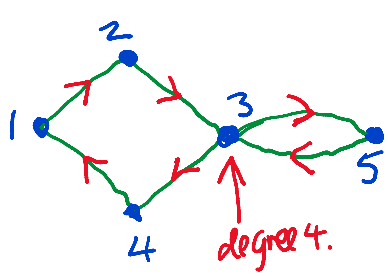
Definition (Eulerian circuit)

An **Euler ciruit** of a multi-graph $G = (V, E)$ is a circuit

$$W = v_1, e_1, v_2, e_2, \ldots, e_n, v_1$$

such that every edge in E appears once in W.

Examples.



1 2 3 5 3 4 1.

Notice every vetex has even degree.

Question. If all vertexes in G have even degree is there an Euler circuit?

Lemma

Let $G = (V, E)$ be a multigraph with $|E| \geq 1$.
If $\deg(v) \geq 2$ for all $v \in V$, then G contains a cycle of length ≥ 1.

Proof. If G has a loop  then it has a cycle of length 1.

degree $= 2$.

If G does not have a loop then G has a path with at least one edge. 

Let P be a path starting at $v_0 \in V$ of maximum length i.e. P cannot be enlarged.

$P =$ 

Since $\deg(v_m) \geq 2$, v_m must be incident with another edge e.
Since P is maximal e must be incident with one of the vertices on P.

Proof (cont).

(i) P 

loop (cycle of length 1)

(ii) P 

parallel edges. (cycle of length 2).

(iii) P 

$0 \leq k < m-1$

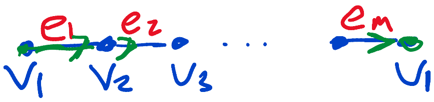
Notice there is a cycle in all cases.

Theorem (Euler)

A connected multigraph $G = (V, E)$ which is not the singleton vertex, has an Euler circuit if ~~and only~~ if every vertex in V has even degree.

G • $deg = 0$

Proof. ($\Rightarrow$) G is connected, G is not • and G has an Euler circuit.

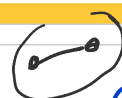


Each time we walk through a vertex we use two edges. Since a Euler circuit walks over every edge once the degree of each vertex must be even.

($\Leftarrow$) G is connected, G is not •, all vertices have even degree. The proof is by induction on $|E|$.

$|E|$ 1 2 (and even degrees)

Proof (cont.)

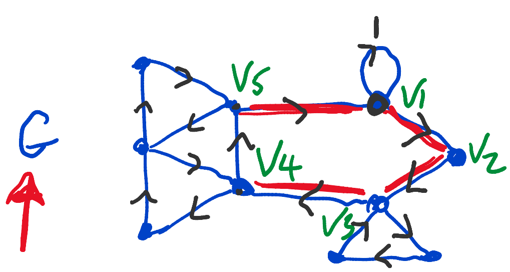


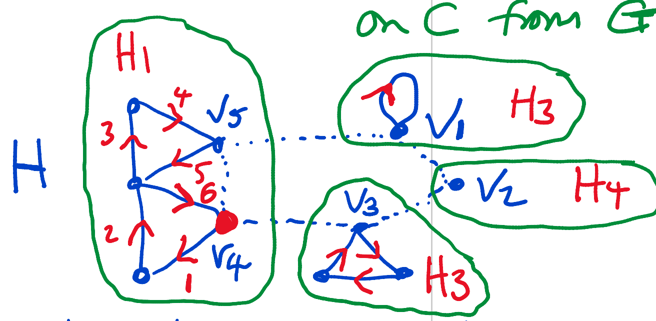
Ind Base $|E| = 1$ $G =$  has an Euler circuit $v e v$.

even degrees

Ind Step $|E| = n > 1$. Ind. Hyp: Assume the theorem is true for graphs with $1 \le |E| < n$ edges.

Since G has even degree vertices and is connected $deg(v) \ge 2$. By the Lemma G has a cycle C of length $m \ge 1$. Let $v_1, v_2, \ldots, v_m$ be the vertices on C. Idea: Remove all edges on C from G.





Observe every vertex in H also has even degree.

Proof (cont.) H may or may not be connected.

Let $H_1, H_2, \ldots, H_\ell$ be the connected components of H which have at least one edge. In the example H_1, H_2, H_3.

Ind. Hyp. $\Rightarrow$ $H_1, H_2, \ldots, H_\ell$ all have an Euler circuit. as #edges in $H_i < |E|$.

Starting at v_1, if $v_1 \in H_i$ with more than 1 edge, walk around the Euler circuit in H_i back to v_1 then walk to v_2. If $v_2 \in H_j$ with more than one edge, then walk around H_j's Euler circuit back to v_2, then walk to v_3. Repeat this until we get back to vertex v_1 and voila, we have constructed an E.C. for G.

The proof gives a recursive algorithm for finding an Euler circuit!

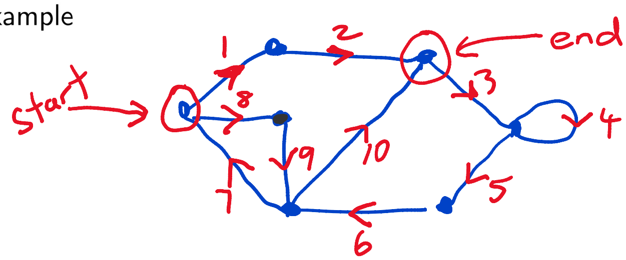
Definition

An **Euler trail** of a multi-graph $G = (V, E)$ is a trail (open walk, start and end at different vertexes).

$$T = v_0, e_1, v_1, e_2, \ldots, e_n, v_n$$

such that every edge in E appears once in T.

Example



Corollary (of Euler's theorem)

A connected multigraph $G = (V, E)$ has an Euler trail if and only if there are exactly two vertices in G of odd degree.

Proof. Exercise.