Lec 27 Hamiltonian Paths and Cycles Michael Monagan

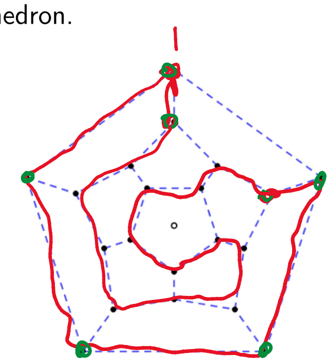
March 26, 2021   11:23 AM

Lecture 27: Hamiltonian Paths and Cycles

Grimaldi 11.5

Midterm # 3 is on Monday
Covers lectures 17-24 which is
assignments 5 and 6.

In 1856 a mathematician William Hamilton invented a game in which the ojbect is to find a cycle along the edges of a dodecahedron.

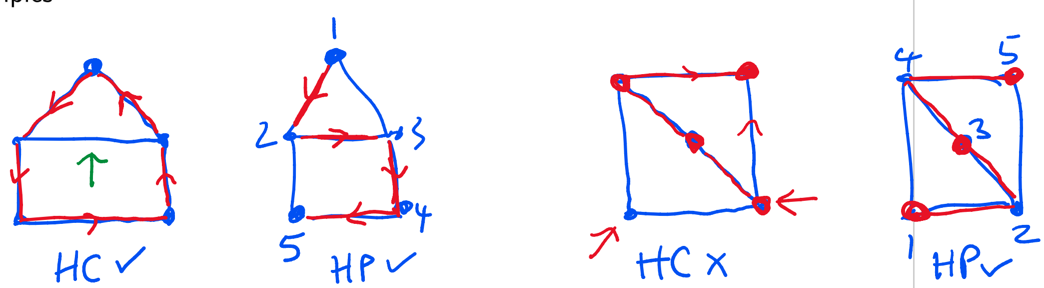


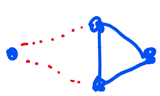
Problem: Can you find a cycle in the graph that includes all 20 vertices?

Definition

Let G be a graph. A path of G is a **Hamiltonian path** if it contains every vertex of G. A cycle of G is a **Hamiltonian cycle** if it contains every vertex of G.

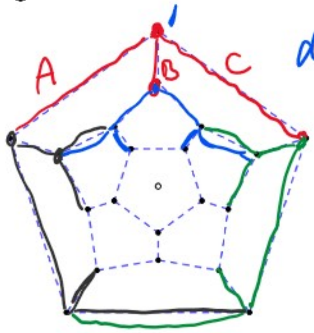
Examples



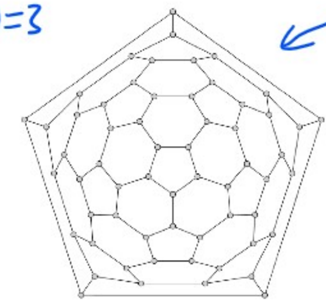


If G has a HP or a HC it must be connected.

Algorithm Exhaustive Search: try all possible paths.



$deg(v)=3$



Soccer Ball Graph.

$|V| = 60$

$|E| = 90$

$deg(v) = 3.$

$3 \cdot 2^{59}$ ← $n-1$ too big.

exponential in the $|V|$.

$V_1 \; V_2 \; V_3 \; \ldots \; V_{20}$

$3 \cdot 2 \cdot 2 \cdot \cdots \cdot 2 \to V_1 = 3 \cdot 2^{19} \approx 1.5 \cdot 10^6.$

Hamiltonian vs. Eulerian

The definition of Hamiltonian is very similar to Eulerian. In Hamiltonian each vertex appears exactly once. In Eulerian each edge appears exactly once. Although they look similar, having a Hamiltonian cycle and Having an Euler circuit is very different.

(1) There is a fast algorithm to test if a graph $G = (V, E)$ has an Euler circuit where the running time is a linear function of $|V| + |E|$, namely, test if G is connected and all vertices have even degree.

(2) No such fast test is known for a Hamiltonian ~~circuit~~ cycle. The problem of deciding if a graph has a Hamiltonian path/cycle is **NP-complete**. So it is widely believed that there does not exist an algorithm which takes as input an arbitrary graph $G = (V, E)$ and determines if G has a Hamiltonian path/cycle where the running time is bounded by a polynomial function of $|V| + |E|$.

Definition (Necessary and sufficient conditions)

Let P be a property of graphs and C be a set of condtions.

(1) C is **necessary** for P if every graph satisfying P also satisfies C. $P \Rightarrow C$.

(2) C is **sufficient** for P if every graph satisfying C also satisfies P. $C \Rightarrow P$

(3) If C is both **necessary and sufficient** for P, then a graph G satisfies P if and only if G satisfies C. We say C characterize P. $C \Leftrightarrow P$

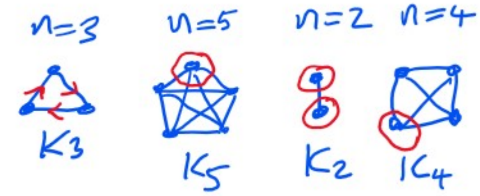
Examples

(1) Is is necessary for a graph to be connected to have a H.P.

$\overbrace{G \text{ has H.P.}}^{P} \overset{\Rightarrow}{\Rightarrow} \overbrace{G \text{ is connected.}}^{C}$

(2) Being a complete graph is a sufficient condition to have a H.P.

K_4  $\overset{C}{G \text{ is } K_n} \overset{\Rightarrow}{\Rightarrow} \overset{P}{G \text{ has a H.P.}}$



(3) n is odd is a necessary and sufficient condition for K_n to have an Euler circuit.

In K_n $\deg(v) = n-1$. So for n odd, $n-1$ is even. and there is an Euler circuit.

A sufficient condition for G to have an H.P.

G has a H.P. $\Rightarrow$ G is connected.

Theorem

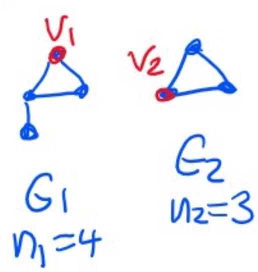
Let $G = (V, E)$ be a graph with $|V| = n$. If

$$\deg(x) + \deg(y) \geq n - 1 \quad \text{for all } x, y \in V \text{ with } x \neq y$$

then G has Hamiltonian path. $C \Rightarrow P$

Proof. Lemma $\deg(x) + \deg(y) \geq n-1 \Rightarrow G$ is connected.

Suppose G is not connected. Let G_1 and G_2 be connected components in G with n_1 and n_2 vertices. First $n_1 + n_2 \leq n$. If v_1 is a vertex in G_1 and v_2 is a vertex in G_2 Then.



$$\deg(v_1) + \deg(v_2) \leq \overbrace{n_1 - 1 + n_2}-1 \leq n - 2.$$

A contradiction. Therefore G is connected.

Proof (cont.) Now we show G has a H.P.

Let P be a path in G of maximal length $m-1$ edges.

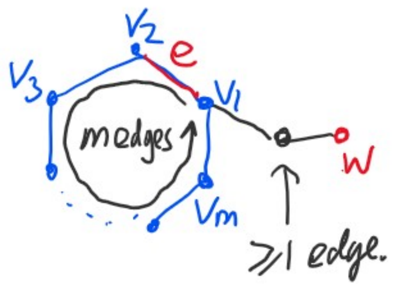
P 

If $m=n$ then P is a H.P.

Case $m<n$: I claim there is a cycle C in G with vertices $v_1, v_2, \dots, v_m$.

Since — Since $m<n$ there is a vertex w in G not on C.

Moreover G is connected so there is a path from w to C. If we remove the edge e what's left is a path of length $\geq m$.



This contradicts the maximality of P.

Therefore $m \not< n$. Therefore $m=n$.

Proof (cont.) To prove there is a cycle in $\{v_1, v_2, \dots, v_m\}$.

Since the path $P =$ 

is maximal, the vertices in G adjacent to v_1 (and v_m) are on P.

Let $A = \{1 \leq i \leq m-1$ s.t. v_{i+1} is adjacent to $v_1\}$

Let $B = \{1 \leq i \leq m-1$ s.t. v_i is adjacent to $v_m\}$

$|A| \leq m-1$

$A = \{4, \dots$

$B = \{4, \dots$

$|B| \leq m-1$.

Since $|A| + |B| = \deg(v_1) + \deg(v_2) \geq n-1 > m-1$, $A \cap B$ is not empty.

Let $j \in A \cap B$. So $\{v_{j+1}, v_1\}$ and $\{v_j, v_m\}$ are in G. and G contains P

Observe if we delete $e = \{v_j, v_{j+1}\}$ we have a cycle.

Corollary

If $G = (V, E)$ is a graph with $|V| = n$ and $\deg(v) \geq \frac{n-1}{2}$ holds for every $v \in V$, then G has a Hamiltonian path.

$\frac{n-2}{2}$? No we can't improve the bound.

Proof. So here for $x, y \in V$ $\deg(x) + \deg(y) \geq \frac{n-1}{2} + \frac{n-1}{2} = n-1$.

By the theorem G has a H.P.

Note the corollary is "tight". Consider

G.  Here $n=6$ and $\deg(v) = 2 \geq \frac{6-2}{2} = 2$.

But G does not have a H.P.

It is necessary & sufficient to pass a midterm

To pass MACM 201 it is neither necessary nor sufficient to pass midterm 3.

Is it sufficient to pass the final exam?

Let G be a simple graph.

Necessary & sufficient conditions to have an Euler circuit.

G is connected, $\deg(v)$ is even, G is not
for all $v \in V$