Lecture 14 First Order Non-homogenous Recurrences, Michael Monagan

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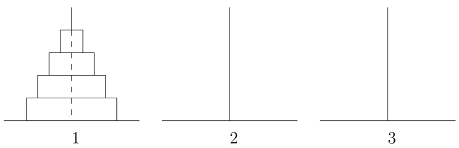
Lecture 14 Solving first order non-homogeneous RRs

Assignment #3 due tonight.
Assignment #4 posted by tomorrow morning.

Grimaldi 10.3



Can you move the disks on pole 1 to pole 3 using pole 2 as needed?
Rule 1: move one disk at a time.
Rule 2: do not put a disk on top of a smaller disk.
Question: how many moves is necessary?

Solving first order non-homogeneous recurrences

Consider the non-homogenous recurrence relations

(1) $a_n + c_1 a_{n-1} = f(n)$ where $c_1 \neq 0$ and $f(n) \neq 0$
(2) $x_n + c_1 x_{n-1} + c_2 x_{n-2} = f(n)$ where $c_2 \neq 0$ and $f(n) \neq 0$

How do we solve them?

Case (1) $a_n + c_1 a_{n-1} = f(n)$ where $c_1 = -1$.

$$a_n = a_{n-1} + f(n)$$

$$\cancel{a_1} = a_0 + f(1)$$
$$\cancel{a_2} = \cancel{a_1} + f(2)$$
$$\vdots$$
$$\cancel{a_{n-1}} = \cancel{a_{n-2}} + f(n-1)$$
$$a_n = \cancel{a_{n-1}} + f(n)$$
$$a_n = a_0 + f(1) + f(2) + \cdots + f(n)$$
$$= a_0 + \sum_{i=1}^{n} f(i).$$

We just need to calculate $\sum_{i=1}^{n} f(n)$.

Example 1. Solve $a_n - a_{n-1} = 3n^2$ with $a_0 = 7$.

$$a_n = a_0 + \sum_{i=1}^{n} 3 \cdot i^2 = 7 + 3\sum_{i=1}^{n} i^2$$
$$= 7 + \frac{n(n+1)(2n+1)}{2}.$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

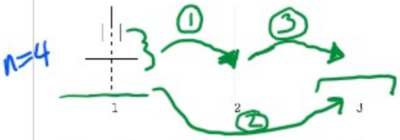
Example 2. Solve $a_n - 3a_{n-1} = 5 \cdot 3^n$ with $a_0 = 2$.

$$3^0\, a_n = 3\cancel{a_{n-1}} + 5 \cdot 3^n$$
$$3^1\, \cancel{a_{n-1}} = 3\,(3a_{n-2} + 5 \cdot 3^{n-1}) = 3^2 \cancel{a_{n-2}} + 5 \cdot 3^n$$
$$3^2\, \cancel{a_{n-2}} = 3^2(3a_{n-3} + 5 \cdot 3^{n-2}) = 3^3 \cancel{a_{n-3}} + 5 \cdot 3^n$$
$$\vdots$$
$$3^{n-1}\, \cancel{a_1} = 3^{n-1}(3a_0 + 5 \cdot 3^1) = \cancel{3^n a_0} + 5 \cdot 3^n$$

Adding
$$3^n\, \cancel{a_0} = 3^n\, 2$$
$$a_n = 3^n \cdot 2 + 5 \cdot 3^n \cdot n$$

Check:
$$a_n = 3a_{n-1} + 5 \cdot 3^n$$
$$3^n \cdot 2 + 5n \cdot 3^n = 3(3^{n-1} \cdot 2 + 5 \cdot (n-1) \cdot 3^{n-1}) + 5 \cdot 3^n$$
$$= 3^n \cdot 2 + 5 \cdot n \cdot 3^n - \cancel{5 \cdot 3^n} + \cancel{5 \cdot 3^n}$$

Example 3 – The Towers of Hanoi.



3 moves.

Move the disks from pole 1 to pole 3 using pole 2 as needed.
Move one disk at a time. Do not put a bigger disk on top of a smaller one. (recursively)

(1) Move the top n-1 disks from pole 1 to pole 2 using pole 3 as needed.
(2) Move the bottom disk from pole 1 to pole 3.
(3) Move the top n-1 disks from pole 2 to pole 3 using pole 1 as needed.

Let m_n be the **number of moves**.
Determine and solve a recurrence relation for m_n. $m_1 = 1$ $m_2 = 3$ moves

$$m_n = \underbrace{m_{n-1}}_{\#\text{moves (1)}} + \underbrace{1}_{\text{for bottom disk}} + \underbrace{m_{n-1}}_{\#\text{moves for (3)}}$$

$$m_n = 2m_{n-1} + 1$$

$$m_2 = 2 \cdot m_1 + 1 = 2 \cdot 1 + 1 = 3$$

Example 3 – The Towers of Hanoi (cont.)

$$2^0 \cdot m_n = \cancel{2m_{n-1}} + 1$$

$$\cancel{2^1 m_{n-1}} = 2(2m_{n-2} + 1) = \cancel{2^2 m_{n-2}} + 2$$

$$\cancel{2^2 m_{n-2}} = 2^2(2m_{n-3} + 1) = \cancel{2^3 m_{n-3}} + 2^2$$

$$\vdots$$

$$\cancel{2^{n-2} m_2} = 2^{n-2}\, 2 \quad + 1) = \cancel{2^{n-1} m_1} + 2^{n-2}$$

$$\cancel{2^{n-1} m_1} = 2^{n-1} 1 = 2^{n-1}$$

$$m_n = 1 + 2 + \cdots + 2^{n-1} = 2^n - 1 \qquad \sum_{i=0}^{n-1} 2^i = 2^n - 1$$

$$m_4 = 2^4 - 1 = 15 \text{ moves.}$$

Example 4 – Interest on a loan.

Pauline takes out a bank loan for \$$S$ dollars. She pays bak \$$P$ every month and the bank charges her $r\%$ interest per month. Let a_n be the amount she owes after n months. Determine, and solve, a recurrence relation for a_n.

$$a_0 = \$S.$$

$$a_n = \underbrace{1\cdot a_{n-1}}_{\text{what she owed at the end of the previous month.}} + \underbrace{r\cdot a_{n-1}}_{\text{interest}} \underbrace{-P}_{\text{payment.}} = \cancel{(1+r)a_{n-1}} - P$$

$$\cancel{(1+r)^1 a_{n-1}} = (1+r)\big((1+r)a_{n-2} - P\big) = \cancel{(1+r)^2 a_{n-2}} - (1+r)P$$

$$\cancel{(1+r)^2 a_{n-2}} = (1+r)^2\big((1+r)a_{n-3} - P\big) = \cancel{(1+r)^3 a_{n-3}} - (1+r)^2 P$$

$$\vdots$$

$$\cancel{(1+r)^{n-1} a_1} = \cancel{(1+r)^{n-1}}\big((1+r)a_0 - P\big) = \cancel{(1+r)^n a_0} - (1+r)^{n-1}P$$

$$\cancel{(1+r)^n a_0} = (1+r)^n S$$

Example 4 – Interest on a loan (cont.)

$$a_n = -P - (1+r)P - \cdots - (1+r)^{n-1}\cdot P + (1+r)^n\cdot S$$

$$= (1+r)^n\cdot S - P\Big[\underbrace{1 + (1+r) + (1+r)^2 + \cdots + (1+r)^{n-1}}_{\text{a geometric sum}}\Big]$$

$$\sum_{i=0}^{n-1} a^i = \frac{a^n - 1}{a-1} \text{ for } a \neq 1.$$

$$= (1+r)^n S - P\left[\frac{(1+r)^n - 1}{1+r-1} = \frac{(1+r)^n - 1}{r}\right]$$

$$a_n = (1+r)^n S - \frac{P}{r}\big((1+r)^n - 1\big).$$

For which n is $a_n \leq 0$.

Summation Formulas

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{k=0}^{n}\binom{n}{k} = 2^n \qquad \sum_{k=0}^{n} k\cdot\binom{n}{k} = n\cdot 2^{n-1}.$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{k=0}^{n-1} a^k = \frac{a^n - 1}{a-1} \quad a \neq 1. \qquad \sum_{k=0}^{n-1} k\cdot 2^k = (n-1)2^n + 1$$