Lecture 22 Multigraphs
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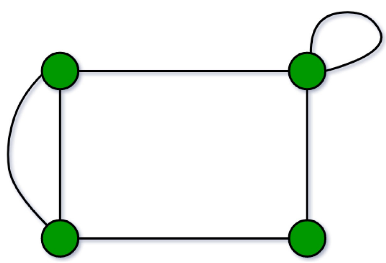
November 6, 2020 11:19 AM

Lecture 22 Graphs: Multigraphs

Grimaldi 11.1

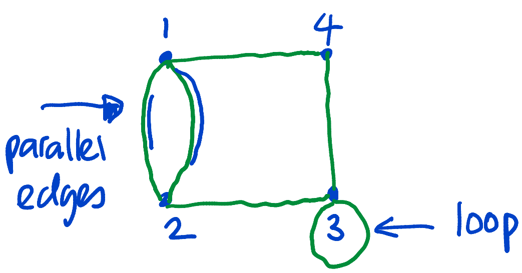
Assignment #5 solutions posted
Assignment #6 posted.



Definition (multigraph)

A **multigraph** $G = (V, E)$ is a set V of vertices, and a multiset E of edges where each edge is in $V \times V$.

Example: $V = \{1, 2, 3, 4\}$, $E = \{(1, 2), (1, 2), (2, 3), (3, 3), (3, 4), (4, 1)\}$.



Vertexes = Cities
Edges = Roads, Airlines.

Simple graphs are multigraphs with no loops and no parallel edges.

Definition (walks in multigraphs)

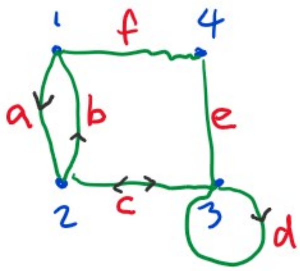
Let x and y be two vertices in a multigraph $G = (V, E)$. A **walk** in G is a finite alternating sequence

$$x\, e_1\, x_1\, e_2\, x_3\, e_3 \ldots e_{n-1}\, x_{n-1}\, e_n\, y$$

of vertices $x_i \in V$ and edges $e_i \in E$ with $n \geq 0$ edges. The **length of the walk** is n, the number of edges. A walk from x to y is called a **closed walk** if $x = y$ and an **open walk** if $x \neq y$. Note, vertices and edges in walks need not be distinct.

Convention: Grimaldi allows walks to have length 0 which he calls **trivial walks**.

Examples



1 a 2 c 3 d 3 c 2 b 1

This is a closed walk of length 5

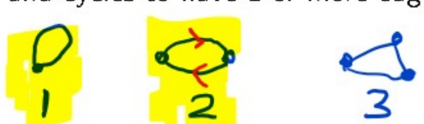
Definition (trails and circuits)

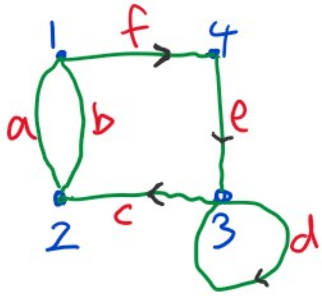
Let G be a multigraph and x and y be vertices in G.
A **trail** from x to y is an open walk in G that has no repeated edges. (can visit a vertex more than once)
A **circuit** from x to x is a closed walk in G that has no repeated edges.

Convention: Grimaldi says circuits must have at least 1 edge and cycles 3 edges. We will allow both circuits and cycles to have 1 or more edges.

Examples

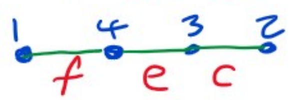
Cycles
Length





A trail 1 f 4 e 3 d 3 c 2. has length 4.

A path



Theorem (trails and paths)

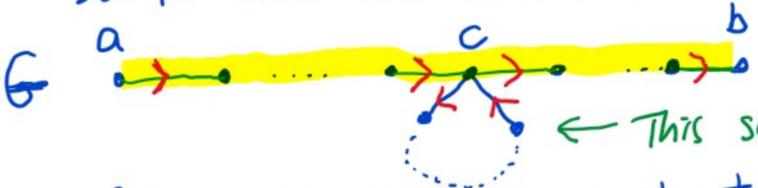
Let $G = (V, E)$ be a multigraph with vertices a and b. If there is a trail in G from a to b then there is a path in G from a to b.

Proof. Among the trails in G from a to b let T be a shortest one.

Either T is a path (no repeated vertices) or its not.

Suppose not. There must be a repeated vertex say c.

So the trail must look like



← This subtrail from c back to c has ≥1 edge.

This means there is a shorter trail in G from a to b contradicting our choice for T (a shortest trail).

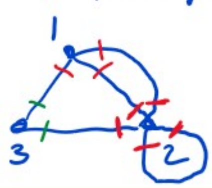
Therefore (Suppose not cannot be the case) so T has a path.

Definition (degree of a vertex in a multigraph)

If $G = (V, E)$ is a multigraph and $v \in V$, the **degree** of v, denoted $\deg(v)$, is the number of edges incident to v. Here a loop at v counts as two incident edges.

touching v

Example



$\deg(1) = 3$
$\deg(2) = 5$
$\deg(3) = 2$

$3+5+2 = 10 = 2 \cdot |E| = 2 \cdot 5$

Theorem

Every multigraph $G = (V, E)$ satisfies $\sum_{v \in V} \deg(v) = 2|E|$.

Poof. Each edge contributes a count of 1 to the degree of each incident vertex, so each edge contributes a count of 2 to the sum hence

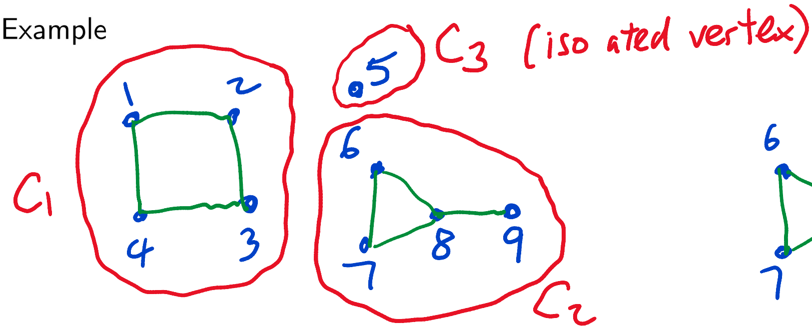
$$\sum_{v \in V} \deg(v) = 2|E|$$

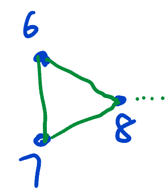
The definitions **subgraph**, **induced subgraph** and **spanning subgraph** that we made for simple graphs also work for multigraphs.

Definition (connected graph and connected components)

Let $G = (V, E)$ be a multigraph. We say G is **connected** if for all pairs $u, v \in V$ there is a path from u to v. The **connected components** of G are the maximal connected subgraphs of G.

Example



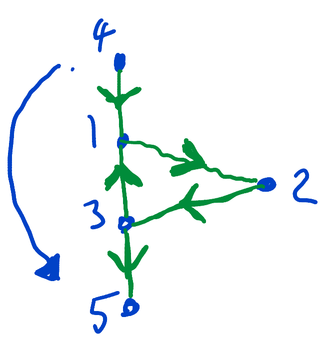


is a connected subgraph of G but not maximal because we can enlarge it by adding vertex 9 and the edge {8,9}.

Definition (directed graphs)

A **directed graph** or **digraph** $G = (V, E)$ is a set V of vertices and a set E of edges where edges are ordered pairs of vertices. We draw arrows on edges to indicate direction. If a graph is not directed, we say it is an **undirected graph**.

Example. $V = \{1, 2, 3, 4, 5\}$, $E = \{(1, 2), (2, 3), (3, 1), (4, 1), (3, 5)\}$.



Edges — Vertices.
Water Pipes — Junctions.
One way roads — intersections.

We will not study directed graphs in MACM 201.

Graph Terminology

- Graphs are classified as either directed graphs or undirected graphs.
- Simple graphs are graphs with no parallel edges and no loops.
- Adjacent vertices are also called neighbors.
- Graphs are also called networks. Usually a network refers to a real physical object whereas a graph could be abstract. Mathematicians and Computer Scientists usually use "graphs" whereas Engineers usually use "networks". Some terminology is different, for example

Graph Theory	Network Science
graph	network
vertex	node
edge	link