Lecture 25 Planar Graphs continued
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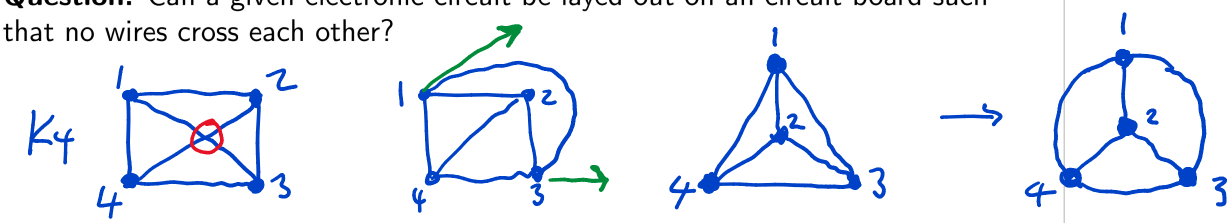
Lecture 25 Planar Graphs continued

Grimaldi 11.4

Assigment #7 is posted.
Midterm #3 Average = 76.8%

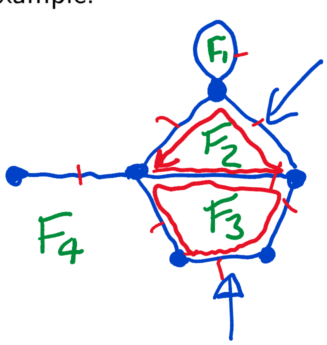
Question: Can a given electronic circuit be layed out on an circuit board such that no wires cross each other?



Review:

A graph $G = (V, E)$ is **planar** if G has a drawing (in the plane) where the edges intersect only at the vertices of G. Such a drawing is called a **planar embedding** of G. The embedding partitions the plane into at set F of regions called **faces**. For a face f we defined $\deg(f)$ to be to be the number of edges in a walk around the face.

Example.



$\deg(F_1) = 1$
$\deg(F_2) = 3$
$\deg(F_3) = 4$
$\deg(F_4) = 8$

$|V| = 6 \quad |E| = 8$

$|V| - |E| + |F| = 6 - 8 + 4 = 2.$

We proved Euler's formula $|V| - |E| + |F| = 2$ and $\sum_{f \in F} \deg(f) = 2|E|$.

It's easy to show that a graph is planar: find a planar embedding.
How can we show that a graph is NOT planar?

Theorem (Bound 1 for the number of edges) no loops no parallel edges

If $G = (V, E)$ is a connected planar simple graph with $|V| \geq 3$ then

$$|E| \leq 3|V| - 6 \quad \text{and} \quad 2|E| \geq 3|F|.$$

Proof. Since G is a simple graph all cycles in G have length ≥ 3. Their facial walks have length at least 3 so the face degrees ≥ 3. The smallest graph  Here $\deg(F_1) = 4$.

Let $F_1, F_2, \ldots, F_k$ be the faces. We have

$$2|E| = \sum_{i=1}^{k} \deg(F_i) \geq 3 \cdot k \Rightarrow 2|E| \geq 3 \cdot k. \Rightarrow k \leq \tfrac{2}{3}|E|$$

By Euler $2 = |V| - |E| + |F| = |V| - |E| + k \leq |V| - |E| + \frac{2}{3}|E| = |V| - \frac{1}{3}|E|$.

$$2 \leq |V| - \tfrac{1}{3}|E| \Rightarrow 6 \leq 3|V| - |E| \Rightarrow |E| \leq 3|V| - 6.$$

Corollary (to bound 1 for the number of edges)

The graph K_5 is not planar.

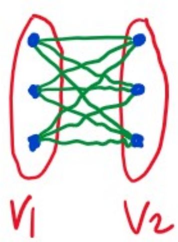
Proof. K_5 

$|V| = 5$
$|E| = 10$

If K_5 is planar then

$|E| \leq 3|V| - 6$
$10 \leq 3 \cdot 5 - 6 = 9$

So K_5 is not planar.

$K_{3,3}$ 

$|V| = 6$
$|E| = 3 \cdot 3 = 9$

$|E| \leq 3|V| - 6$
$9 \leq 3 \cdot 6 - 6 = 12$ ✓

We don't know if $K_{3,3}$ is planar or not.

Theorem (Bound 2 for the number of edges)

If $G = (V, E)$ is a connected planar simple graph with $|V| \geq 3$ and with no cycle of length 3 or less then

$$|E| \leq 2|V| - 4 \quad \text{and} \quad |E| \geq 2|F|.$$

Proof. This time the cycles have length ≥ 4 edges. So their face degrees are ≥ 4. And the infinite face also has degree ≥ 4. Let $F_1, F_2, \ldots, F_k$ be the faces.

$$2|E| = \sum \deg(F_i) \geq 4k \Rightarrow |E| \geq 2k. \qquad k \leq \tfrac{1}{2}|E|$$

By Euler

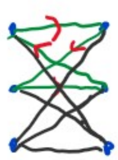
$$2 = |V| - |E| + |F| \leq |V| - |E| + \tfrac{1}{2}|E| = |V| - \tfrac{1}{2}|E|$$

$$\Rightarrow 4 \leq 2|V| - |E| \Rightarrow \boxed{|E| \leq 2|V| - 4}.$$

Corollary (to bound 2 for the number of edges)

The graph $K_{3,3}$ is not planar.

Proof. $K_{3,3}$ $\qquad |V| = 6 \quad |E| = 9$



$$\boxed{|E| \leq 2 \cdot |V| - 4}$$

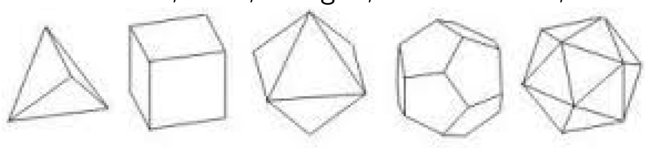
$$9 \leq 2 \cdot 6 - 4 = 8.$$

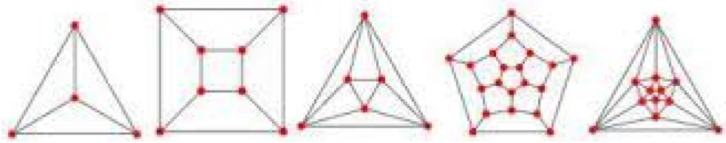
This proves $K_{3,3}$ is not planar.

The 5 Platonic Solids

SKIP.

The tetrahedron, cube, octagon, dodecahedron, and icosahedron.



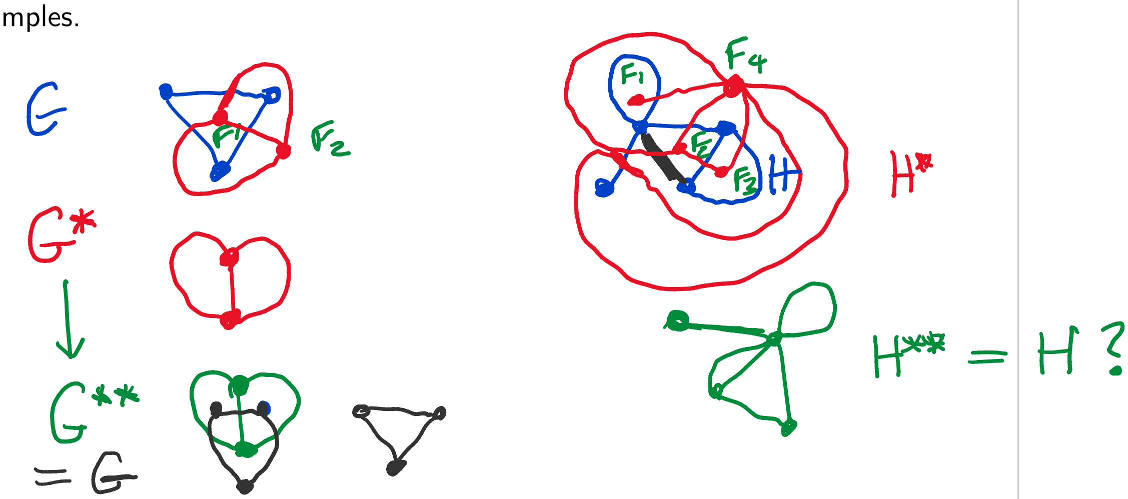


The 5 Platonic Solids - continued

Definition (Dual graphs)

Let $G = (V, E)$ be a connected multigraph embedded in the plane. The vertices of the **dual** multigraph G^* are the faces of G. If two faces f_i and f_j share an edge e then $e^* = \{f_i, f_j\}$ is an edge in G^*. This may be done so that e^* crosses e and G^* also ends up embedded in the plane.

Examples.



Features of duals

(1) Duals only exist for planar graphs
(2) If G^* is a dual of G then G is a dual of G^* ! $G^{**} = G$
(3) The degree of a vertex in G^* is the degree of the corresponding face of G.
(4) The dual of a simple graph may be a multigraph.

Examples