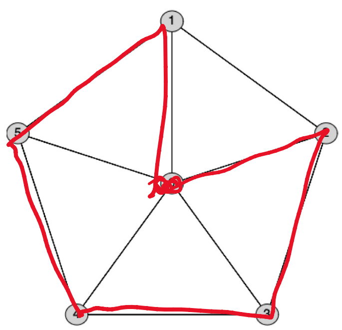
Lecture 5 Counting in Graphs

January 22, 2021 10:30 AM

Lecture 5: Counting in Graphs

Grimaldi 11.1, 11.4 (bipartite)

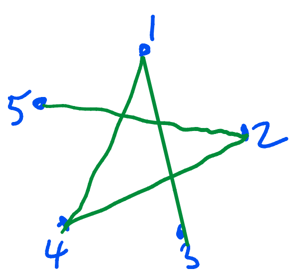
Assignment #1 due Mon Jan 25.
Assignment #2 due Mon Feb 1.
Midterm #1 Mon Feb 8.

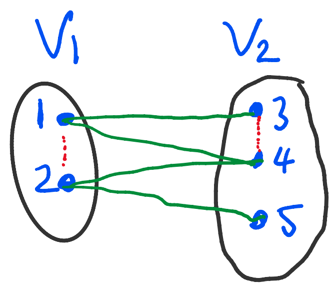


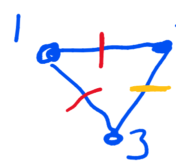
The Wheel graph W_5.

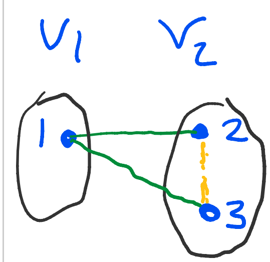
Problem: How many cycles does W_5 have?

Draw the graph $G = (V, E)$ where $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1,3\}, \{1,4\}, \{2,4\}, \{2,5\}\}$.









The triangle graph is not bipartite.

Definition (Bipartite graph)

A graph $G = (V, E)$ is **bipartite** if we can partition the vertices into two sets $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$ such that

(1) $V_1 \cap V_2 = \emptyset$

(2) $V_1 \cup V_2 = V$

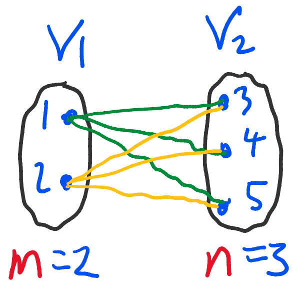
(3) every edge in E is incident (touches) with one vertex in V_1 and one vertex in V_2.

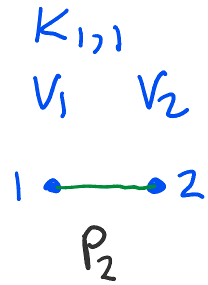
Definition ($K_{m,n}$)

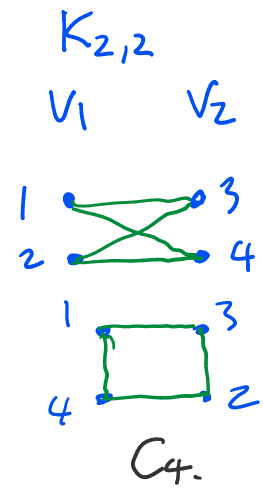
For integers $n \geq 1$ and $m \geq 1$ we define the **complete bipartite graph** $K_{n,m}$ to be the bipartite graph with $|V_1| = n$, $|V_2| = m$ and

$$E = \{\{v_1, v_2\} \mid v_1 \in V_1 \text{ and } v_2 \in V_2\}.$$

Example $K_{2,3}$

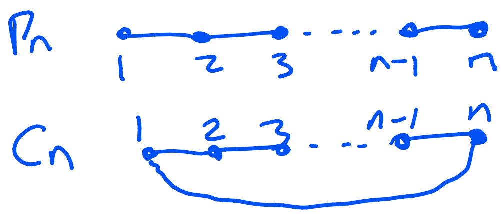






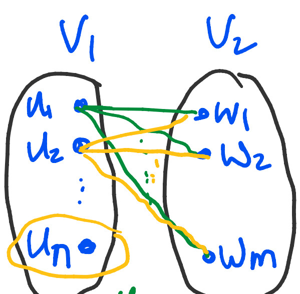
Question 1: How many edges are in a path on n vertices? $n-1$

Question 2: How many edges are in a cycle on n vertices? n



Question 3: How many edges are in K_n ? $\binom{n}{2} = \frac{n(n-1)}{2}$

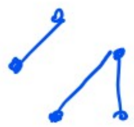
Question 4: How many edges are in $K_{n,m}$? $n \cdot m$



$$\underbrace{m}_{u_1} + \underbrace{m}_{u_2} + \cdots + m = n \cdot m$$

Question 5: How many graphs are there with n vertices? $2^{\binom{n}{2}}$

Question 6: How many graphs have n vertices and m edges? $\binom{\binom{n}{2}}{m}$



$n=5$

Max # edges $= \binom{n}{2}$.

Each edge may be present or not

$\binom{5}{2} = 10$ $\quad e_1 \; e_2 \; e_3 \; e_4 \cdots e_{10}$

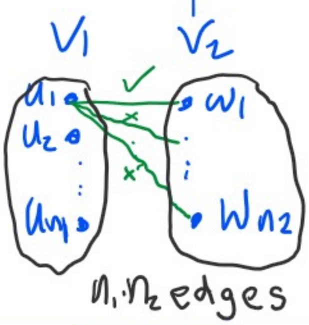
$\quad\quad\quad\quad$ ✓ × ✓ ✓ $\quad$ ×

Q6. Choose m edges from $\binom{n}{2}$ edges.

Let V_1, V_2 be disjoint sets with $|V_1| = n_1$ and $|V_2| = n_2$. $2^{n_1 \cdot n_2}$

Question 7: How many graphs have bipartition (V_1, V_2)?

Question 8: How many graphs have bipartition (V_1, V_2) with m edges? $\binom{n_1 \cdot n_2}{m}$



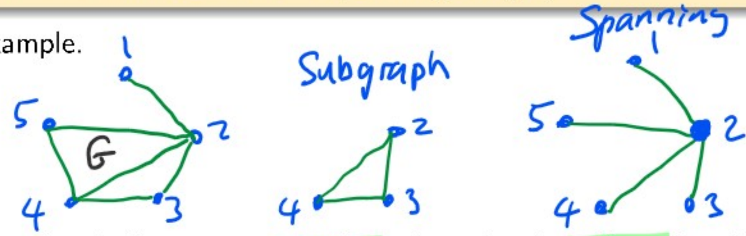
Q7. At most $n_1 \cdot n_2$ edges (K_{n_1,n_2}) and each edge may be there or not.

Q8. Choose m edges from $n_1 \cdot n_2$.

Definition (Subgraph)

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs.
G' is a **subgraph** of G if $V' \subseteq V$ and $E' \subseteq E$.
If $V' = V$ then we call G' a **spanning** subgraph of G.

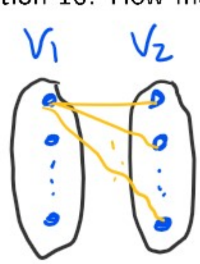
Example.



This is also a tree.
This is a spanning tree.

Question 9: How many spanning subgraphs does K_{n_1,n_2} have? $2^{n_1 \cdot n_2}$

Question 10: How many spanning subgraphs of K_{n_1,n_2} have exactly m edges? $\binom{n_1 \cdot n_2}{m}$

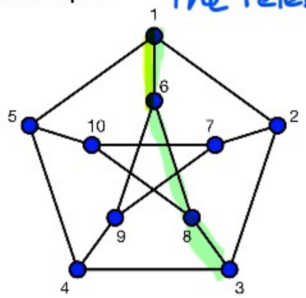


Q9. A spanning subgraph must include all $n_1 + n_2$ vertices.
We can include any of the $n_1 \cdot n_2$ edges.

Definition (Paths and Cycles)

If P is a subgraph of G that is a path we call P **a path of** G.
If C is a subgraph of G that is a cycle we call C **a cycle of** G.

Example. The Petersen Graph.

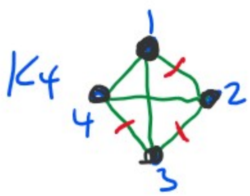


P = 1 6 8 3

C = 5 10 8 3 4

$P_4 =$

Question 11: How many 4-vertex paths does the graph K_n have?



1 2 3 4

Same Subgraphs

4 3 2 1

n choices n-1 choices n-2 choices n-3 choices

This overcounts by a factor of 2.

1 2 3 4

$= n(n-1)(n-2)(n-3)/2$ paths

Definition (induced subgraph)

Let $G = (V, E)$ be a graph and let $V' \subseteq V$. The subgraph of G **induced** by V' is the graph $G' = (V', E')$ where

$$E' = \{\{x, y\} \mid x \in V', y \in V' \text{and } \{x, y\} \in E\}.$$

For the graph below determine the induced subgraph for the vertex sets $\{1, 3, 4\}$ and $\{1, 0, 3, 4\}$.

$V' = \{3, 4, 5\}$