Lecture 31: Articulation Points and Biconnected Components

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An application of the depth-first search spanning tree.

Articulation Points

Definition (Articulation Point)

Let $G = (V, E)$ be a graph. A vertex v in G is an **articulation point** (AP) if removing v from G increases the number of connected components of G.

Lemma (12.3)

Let $G = (V, E)$ be a graph. A vertex $v \in V$ is an articulation point of G if and only if there are two vertices x and y in V such that $x \neq y \neq v$ and every path between x and y includes v.

Definition (Biconnected Component)

Let $G = (V, E)$ be a graph. A subgraph of G is **biconnected** if it is connected and has no articulation points. A maximal biconnected subgraph of G is called a **biconnected component** of G.

Lemma

Let $G = (V, E)$ be a graph. If G has a Hamiltonian cycle then G must have no APs, equivalently, G must be biconnected.

Exercise: Find a biconnected graph which does not have a Hamiltonian cycle.

How can we find the Articulation points in a **connected** graph G?

Algorithm 1.

set AP $= \phi$.
for each $v \in V$ do
  if removing v from G disconnects G **then**
    set AP $=$ AP $\cup\{v\}$.
  end if
end for
output AP.

Algorithm 2

Step 1: Construct a DFS spanning tree T for G and number the edges in T in the order visited during the DFS.

Step 2: Traverse T in pre-order. If a vertex v has a backedge e_n from u to v, number all edges on the walk $v\, e_1\, x_1\, e_2 \ldots x_{n-2}\, e_{n-1}\, u\, e_n\, v$ from v down to u and back to v with the edge number on e_1.

What are the articulation points?

What are the biconnected components?

Why is this algorithm better than Algorithm 1?
If implemented carefully, can be done in time proportional to $|V| + |E|$.