Lecture 18 Calculating with Generating Functions

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Definition (Generating Function)

Let $a_0, a_1, a_2, a_3, \ldots$ be a sequence of real numbers (or integers). The function

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

is called the **generating function** for the sequence.

Note: we are interested in the coefficients of $A(x)$ not the values of $A(x)$.
All polynomials may be viewed as generating functions.

Example 1

Example 2. How many ways can we make 30 cents from nickels, dimes and quarters?

Example 3. How many integer solutions does $x_1 + x_2 + x_3 = n$ have if $x_i \geq 0$?

Definition (Arithmetic for Generating Functions)

Let $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$
and $B(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \dots$ and c be a constant. Then

(1) Sum: $A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots = \sum_{n=0}^{\infty}(a_n + b_n)x^n.$

(2) Scalar product: $cA(x) = ca_0 + ca_1x + \cdots = \sum_{n=0}^{\infty}(ca_n)x^n.$

(3) Product:

$$A(x) \cdot B(x) = (a_0b_0) + (a_0b_1 + a_1b_0)x + \cdots = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n.$$

(4) Derivative: $A'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{n=1}^{\infty}(na_n)x^{n-1}.$

Example. Let $A(x) = 1 + x + x^2 + x^3 + \ldots$ and $B(x) = 2 + 2x + 2x^2 + 2x^3 + \ldots$.

$2A(x) + B(x) =$

$A(x) \cdot B(x) =$

$A'(x) =$

What about inverses? Let x be a real number.
The number 1 has the property $1 \cdot x = x$ for all x. [identity]
If x is non-zero it has an inverse $1/x$ so that $x \cdot \frac{1}{x} = 1$. [inverses]

Example 1 Let $A(x) = 1 + x + x^2 + \ldots$ and $B(x) = 1 - x$.
Verify that $A(x) \cdot B(x) = 1$ and conclude that $A(x) = 1/B(x) = 1/(1-x)$.

Example 2. Find the inverse of $(1-x)^k$.

Let a_n be the number of integer solutions of $x_1 + x_2 + \cdots + x_k = n$ where $x_i \geq 0$.

Example 3 Let $C(x) = 1 + 2x + 4x^2 + 8x^3 + \dots$ and
$N(x) = 1 + x^5 + x^{10} + x^{15} + \dots$ (the GF for nickels).
Using $2xC(x)$ and $x^5N(x)$ find the inverse of $C(x)$ and $N(x)$.

Express $C(x)$ and $N(x)$ in terms of $A(x) = 1 + x + x^2 + x^3 + \cdots = 1/(1-x)$.