Lecture 28 Planar Graphs cont. Michael Monagan

March 24, 2021   10:55 AM

Lecture 26: Planar Graphs continued

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Grimaldi 11.4

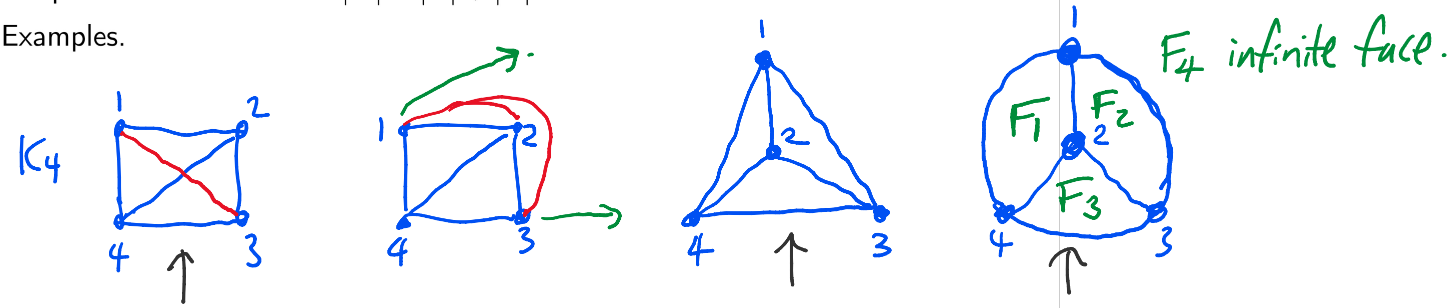
Midterm 3 is on Monday March 29th.
Review problems are posted.
Same rules/procedure as Midterm 2.
Midterm 3: Lectures 17–24 Assignments 5 & 6.

Question: Can a given electronic circuit be layed out on an circuit board such that no wires cross each other?

Review:

A graph $G = (V, E)$ is **planar** if G has a drawing (in the plane) where the edges intersect only at the vertices of G. Such a drawing is called a **planar embedding** of G. The embedding partitions the plane into at set F of regions called **faces**. We proved Euler's formula $|V| - |E| + |F| = 2$.

Examples.

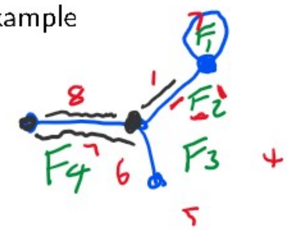


$$|V| - |E| + |F| = 4 - 6 + 4 = \underline{\underline{2}}$$

Definition (Face degrees)

Let $G = (V, E)$ be a connected multigraph embedded in the plane and let f be a face of this embedding. We define the **degree** of f, denoted $\deg(f)$, to be the number of edges in a facial walk of f.

Example



$\deg F_1 = 1$
$\deg F_2 = 3$
$\deg F_3 = 4$
$\deg F_4 = 8$

$\sum \deg(f_i) = 1+3+4+8 = 16 = 2|E|.$
$|E| = 8$

Theorem

If G has faces $f_1, f_2, \ldots, f_k$ then $\displaystyle\sum_{i=1}^{k} \deg(f_i) = 2|E|.$

Proof each edge is counted twice in $\sum \deg(f_i)$.

It's easy to show that a graph is planar: find a planar embedding.
How can we show that a graph is NOT planar?

Theorem (Bound 1 for the number of edges)

no loops, no parallel edges

If $G = (V, E)$ is a connected planar simple graph with $|V| \geq 3$ then

$$|E| \leq 3|V| - 6 \quad \text{and} \quad 2|E| \geq 3|F|.$$

Proof. Since G is a simple graphs all cycles in G have length ≥ 3.
Their facial walks have ≥ 3 edges so their degrees ≥ 3.
? Infinite face. The smallest graphs are  and .
Their infinite faces have degree ≥ 3.
Let $F_1, F_2, \ldots, F_k$ be the faces. We have

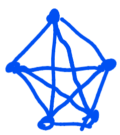
$$2|E| = \sum_{i=1}^{k} \deg F_i \geq 3k \Rightarrow 2|E| \geq 3 \cdot k \Rightarrow k \leq \tfrac{2}{3}|E|$$

By Euler $2 = |V| - |E| + |F| \leq |V| - |E| + \tfrac{2}{3}|E| = |V| - \tfrac{1}{3}|E|$
So $2 \leq |V| - \tfrac{1}{3}|E| \Rightarrow 6 \leq 3|V| - |E| \Rightarrow |E| \leq 3|V| - 6$

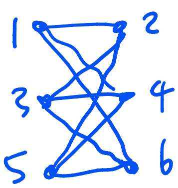
Corollary (to bound 1 for the number of edges)

The graph K_5 is not planar.

Proof.

K_5  $|V|=5$, $|E|=10$

If K_5 is planar
$|E| \leq 3|V|-6$
$10 \leq 3 \cdot 5 - 6 = 9$.
This proves K_5 is not planar.

$K_{3,3}$  $|V|=6$, $|E|=9$.

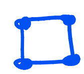
$|E| \leq 3|V|-6$
$9 \leq 3 \cdot 6 - 6 = 12$
We don't know if $K_{3,3}$ is planar or not?

Theorem (Bound 2 for the number of edges)

If $G=(V,E)$ is a connected planar simple graph with $|V| \geq 3$ and with no cycle of length 3 or less then

$$|E| \leq 2|V| - 4 \quad \text{and} \quad |E| \geq 2|F|.$$

no triangles.

Proof. This time the cycles have length ≥ 4 edges.
So their facial degrees ≥ 4. ? Infinite face
The smallest graphs are   which have degree 4 for the infinite face.

$$2|E| = \sum_{i=1}^{k} \deg(f_i) \geq 4 \cdot k \Rightarrow |E| \geq 2k \Rightarrow k \leq \tfrac{1}{2}|E|.$$

By Euler

$$2 = |V| - |E| + |F| \leq |V| - |E| + \tfrac{1}{2}|E| = |V| - \tfrac{1}{2}|E|.$$

$$\Rightarrow 4 \leq 2|V| - |E| \Rightarrow |E| \leq 2|V| - 4.$$

Corollary (to bound 2 for the number of edges)

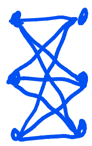
The graph $K_{3,3}$ is not planar.

Proof. $K_{3,3}$ $\quad |V|=6$, $|E|=9$

If $K_{3,3}$ is planar then

$$|E| \le 2|V|-4$$
$$9 \le 2\cdot 6-4=\underline{8}$$

Therefore $K_{3,3}$ is not planar.



no triangles.

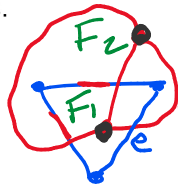
Exercise? What if G has no cycles of length ≤ 4?

Definition (Dual graphs)

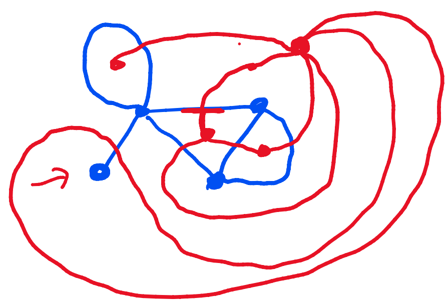
Let $G=(V,E)$ be a connected multigraph embedded in the plane. The vertices of the **dual** multigraph G^* are the faces of G. If two faces f_i and f_j share an edge e then $e^* = \{f_i, f_j\}$ is an edge in G^*. This may be done so that e^* crosses e and G^* also ends up embedded in the plane.

Examples.

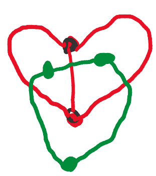
G



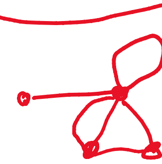
$H.$



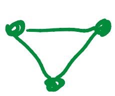
G^*



H^*



G^{**}



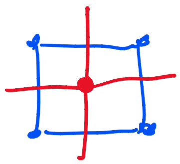
Features of duals

(1) Duals only exist for planar graphs
(2) If G^* is a dual of G then G is a dual of G^* ! $(G^*)^* = G$
(3) The degree of a vertex in G^* is the degree of the corresponding face of G.
(4) The dual of a simple graph may be a multigraph.

must?

Examples

(3)



• has degree 4.

face has degree 4