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Lecture 11 Recurrence Relations

Grimaldi Chapter 10 Recurrence Relations

The Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$ is generated by the **recurrence**

$$f_{n+1} = f_n + f_{n-1} \text{ for } n \geq 2$$

The next Fibonacci number is the sum of the previous ones.

and **initial values**

$$f_1 = 1, \; f_2 = 1.$$

$$f_{n+1} = f_n + f_{n-1}$$

$n=2$ $\quad f_3 = f_2 + f_1 = 1+1 = 2.$

$n=3$ $\quad f_4 = f_3 + f_2 = 2+1 = 3$

$n=4$ $\quad f_5 = f_4 + f_3 = 3+2 = 5$

Example 2. Let b_n be the number of binary strings of length n bits. Require $n \geq 1$.

0 or 1

bit $\underline{\quad}_1 \; \underline{\quad}_2 \; \underline{\quad}_3 \; \ldots \; \underline{\quad}_{n-1} \; \underline{\quad}_n$

2^n binary strings of length n

$b_n = 2^n$

A new way: To construct a binary string of length n first construct one of length $n-1$ bits.

two choices 0, 1.

bit $\left[\underset{1}{0} \; \underset{2}{1} \; \underset{3}{1} \; \ldots \; \underset{n-1}{0}\right] \underset{n}{\underline{\quad}}$

So for each b.s. of length $n-1$ we get two b.s.' of length n.

There are b_{n-1} b.s.' of length $n-1$. So $2 \cdot b_{n-1}$ of length n.

R.R. $\longrightarrow$ $b_n = 2 \cdot b_{n-1}$.

initial value. $b_1 = 2$.

$b_2 = 2 \cdot b_1 = 2 \cdot 2 = 4$

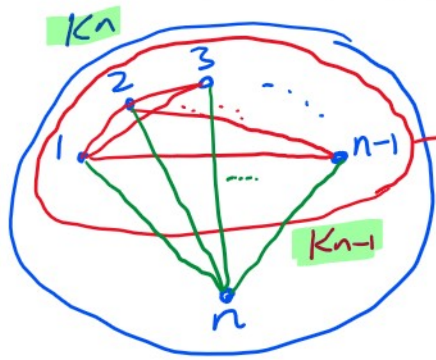
$b_3 = 2 \cdot b_2 = 2 \cdot 4 = 8$

Example 3. Let k_n be the number of edges in K_n the complete graph on n vertices.

K_1 • $k_1 = 0$

K_2 •—• $k_2 = 1$

K_3 △ $k_3 = 3$



K_{n-1} has k_{n-1} edges.

$$k_n = \underbrace{k_{n-1}}_{\#\text{edges in } K_{n-1}} + \underbrace{n-1}_{\text{additional green edges}}$$

$n=2$

$$k_2 = k_1 + 2-1 = 0+1 = \underline{\underline{1.}}$$

$$k_3 = k_2 + 3-1 = 1+2 = \underline{\underline{3}}$$

Definition

A **linear** recurrence relation (RR) of **order** k for a sequence $a_1, a_2, a_3, \ldots$ is an equation of the form

$$\overset{\neq 0}{c_0} a_n + c_1 a_{n-1} + \cdots + \overset{\neq 0}{c_k} a_{n-k} = f(n) \quad \text{for } n \geq k$$

where $c_0, c_1, \ldots, c_k$ are constants and $c_0 \neq 0, c_k \neq 0$. If $f(n) = 0$ the RR is said to be **homegeneous**, otherwise it is **non-homogeneous**.

Examples

$$b_n = 2 \cdot b_{n-1} \Rightarrow 1 \cdot b_n - 2 \cdot b_{n-1} = 0.$$

linear homogeneous recurrence of order 1

$$k_n = k_{n-1} + (n-1) \Rightarrow 1 \cdot k_n - 1 \cdot k_{n-1} = n-1.$$

linear non-homogeneous recurrence of order 1

$n \to n-1$

$$f_{n+1} = f_n + f_{n-1}$$

$$\Rightarrow f_n = f_{n-1} + f_{n-2}$$

$$\Rightarrow f_n - f_{n-1} - f_{n-2} = 0$$

linear homogeneous second order recurrence.

Note: We can **shift** a RR up or down without changing the solutions. For example

$$\begin{aligned} a_{n+1} &= 2a_n + n \\ a_n &= 2a_{n-1} + n - 1 \\ a_{n+2} &= 2a_{n+1} + n + 1 \end{aligned}$$

$n \to n-1$

$n \to n+1$

Initial Values: To fix the values of a recurrence of order k we need k consecutive initial values.

E.g. $f_{n+1} = f_n + f_{n-1}$ either $\begin{cases} f_1 = 1 \\ f_2 = 1 \end{cases}$ or $\begin{cases} f_0 = 0 \\ f_1 = 1 \end{cases}$ or $\begin{cases} f_2 = 1 \\ f_3 = 2 \end{cases}$

E.g. $f_2 = 1$, $f_3 = 2$.

$n=3$ $f_4 = f_3 + f_2 = 2+1 = 3$.

$n=2$ $f_3 = f_2 + f_1 \Rightarrow 2 = 1 + f_1 \Rightarrow f_1 = 1$.

$n=1$ $f_2 = f_1 + f_0 \Rightarrow 1 = 1 + f_0 \Rightarrow f_0 = 0$.

$n=0$ $f_1 = f_0 + f_{-1} \Rightarrow 1 = 0 + f_{-1} \Rightarrow f_{-1} = 1$.

Example 4. Let S_n be the set of all binary strings of length n with the property that every 1 is followed by a 0 (so 1 cannot be the last bit).

(1) List S_1, S_2, S_3.
(2) Let $c_n = |S_n|$. Give a recurrence relation for c_n.

$S_1 = \{0, \cancel{1}\}$ $c_1 = 1$

$S_2 = \{00, 10, \cancel{01}, \cancel{11}\}$ $c_2 = 2$

$S_3 = \{000\ \cancel{001}\ \cancel{011}\ 010\ 100\ \cancel{101}\ \cancel{110}\ \cancel{111}\}$ $c_3 = 3$.

A b.s. in S_n either starts with a 1 or a 0. So it must look like

$0\,\underline{x}$ where $x \in S_{n-1}$ $c_n = |S_n|$

$1\,0\,\underline{y}$ where $y \in S_{n-2}$ $c_{n-1} = |S_{n-1}|$

Therefore $c_n = \underset{x}{c_{n-1}} + \underset{y}{c_{n-2}}$ which is the Fibonacci RR!!

Check $c_3 = c_2 + c_1 = 2 + 1 = 3$.

Example 5. Let D_n be the set of all strings of length n over the alphabet $\Sigma = \{A, B, C, D\}$ such that every A is followed by a C and every B is followed by DD.

(1) List D_0, D_1, D_2.
(2) Let $d_n = |D_n|$. Give a recurrence relation for d_n.

$D_0 = \{ "" \}$ $d_0 = 1$ $D_1 = \{ \cancel{A}, \cancel{B}, C, D \}$ $d_1 = 2.$

$D_2 = \{ AC, CC, CD, DC, DD \}$ $d_2 = 5.$

Every string in D_n must start with an $A, B, C,$ or D.

1: ACx where $x \in D_{n-2}$

2: $BDDy$ where $y \in D_{n-3}$

3: Cz

4: Dz where $z \in D_{n-1}$.

$d_n = 2d_{n-1} + d_{n-2} + d_{n-3}$

$d_3 = 2d_2 + d_1 + d_0 = 2 \cdot 5 + 2 + 1 = \underline{13}$

Additional space

$d_n - 2d_{n-1} - d_{n-2} - d_{n-3} = 0$

This is a 3rd order linear homogeneous recurrence.