Lecture 7: Basics of Discrete Probability

 1cm
Grimaldi 3.4 and 3.5

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Example 1. Suppose we pick a binary string x of length 6 at **random**. What is the **probability** that x has two 1's in it?

Definition (Probability of an Event)

Hypothesis:
S is a **set** of possible outcomes called the **sample space**, all having equal likelihood. Each subset $A \subseteq S$ is called an **event**, i.e. a set of considered outcomes. Each element of S determines an **outcome**.

Experiment:
We generate an event by "drawing" at random an outcome x from S.
Note: Other words used for "drawing" are "choosing", "selecting" and "picking".

Event: Let $Pr(A)$ denote the probability that $x \in A$.

Question: What is $Pr(A)$?
Answer: If each outcome is equally likely and $|S|$ is finite then

$$Pr(A) = \frac{|A|}{|S|}$$

Fundamental Principle. Calculating $Pr(A)$ requires defining the two sets S and A. If all outcomes are equally likely, we just need to calculate $|S|$ and $|A|$.

Example 2. What is the probability that a random binary string of size $n \geq 2$ starts with 11 ?

Example 3. What is the probability that the sum of two rolls of a dice is 7 ?

Definition (Axioms of Probability)

Let S be a sample space and let A and B be subsets of S.

1. $0 \leq Pr(A) \leq 1$
2. $Pr(S) = 1$
3. If $A \cap B = \phi$ then $Pr(A \cup B) = Pr(A) + Pr(B)$.

Note: These axioms hold whether the outcomes of S have equal likelihood or not.

Theorem (the rule of complement)

Let $\bar{A} = S - A$ *be the* **complement** *of* A*. Then* $Pr(\bar{A}) = 1 - Pr(A)$*.*

Proof:

Example 3 (illustrating the third axiom) What is the probability that a random binary string of size $n \geq 3$ has exactly two 1's or exactly three 1's ?

Example 4. Let $S = \{1, 2, 3, \ldots, 12\}$. If x is chosen from S at random what is the probability that x is divisible by 2 OR 3 ?

Theorem (the additive rule)

Let S be a sample space and $A, B \subseteq S$ be two events from S. Then

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B).$$

Proof:

Exercise. Let $S = \{1, 2, 3, \ldots, 60\}$. If x is chosen from S at random what is the probability that x is divisible by 2 **or** divisible by 3 **or** divisible **or** 5 ?
Generalize the additive rule to three subsets A, B, C.