Lecture 20 Rational GFs Copyright Michael Monagan

March 10, 2021 11:37 AM

Lecture 20 Generating Functions continued.

Grimaldi 9.2 Calculation Techniques

Midterm 2 average = 71.6 median = 73.8
Midterm 1 average = 70.5 median = 72.5
Assignment #5 posted. Due Monday.
(Lectures 17—20).

We have been working with the two basic GF's

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n$$

We already proved the more general generating function:

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$$

Combining this formula with **substitution** allows us to determine the coefficients of any rational function of the form

$$[x^n] \quad \frac{p(x)}{(ax+b)^k}.$$

On page 422 the textbook uses a natural generalization of binomial coefficients, called the extended binomial theorem to get these coefficients. We will use substitutions instead.

Example 1. For $A(x) = \dfrac{x}{(1-x)^2}$ find $[x^n](A(x))$.

$$\frac{1}{(1-x)^2} = 1 + 2\cdot x + 3\cdot x^2 + \cdots + \underline{(n+1)\, x^n} + \cdots$$

$$\frac{x}{(1-x)^2} = 1\cdot x + 2\cdot x^2 + 3\cdot x^3 + \cdots + n\cdot x^n + (n+1)\cdot x^{n+1} + \cdots$$

$$[x^n]A(x) = n.$$

Example 2. Find the coefficient of x^5 of $A(x) = \dfrac{1}{(1-2x)^7}$

$k=7$

$$\frac{1}{(1-x)^7} = \sum_{n=0}^{\infty} \binom{n+7-1}{n} \cdot x^n$$

$x \to 2x$ $\qquad$ $x \to 2x$

$$\frac{1}{(1-2x)^7} = \sum_{n=0}^{\infty} \binom{n+6}{n} 2^n \cdot x^n$$

$n=5$

$$[x^5]A(x) = \binom{5+6}{5} \cdot 2^5.$$

Partial Fractions

Question: How can we deterine the coefficients of GFs of the form

$$\frac{p(x)}{ax^2+bx+c} \quad \text{and} \quad \frac{p(x)}{ax^3+bx^2+cx+d} \quad \text{etc?}$$

If $ax^2+bx+c = a(x-\alpha)(x-\beta)$ and $\alpha \neq \beta$ solve

$$\frac{1}{(x-\alpha)(x-\beta)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta}$$

for A, B. If $ax^3+bx^2+cx+d = a(x-\alpha)(x-\beta)(x-\gamma)$ and $\alpha \neq \beta \neq \gamma$ solve

$$\frac{1}{(x-\alpha)(x-\beta)(x-\gamma)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma}$$

for A, B, C. If $ax^3+bx^2+cx+d = a(x-\alpha)(x-\beta)^2$ and $\alpha \neq \beta$ solve

$$\frac{1}{(x-\alpha)(x-\beta)^2} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{(x-\beta)^2}$$

for A, B, C. Then use the formula for $1/(1-x)^k$ with substitutions.

Example 1. Find the coefficient of x^n of $C(x) = \frac{3x}{x^2 - 3x + 2}$.

Factor $x^2 - 3x + 2 = (x-2)(x-1)$

PF: $\frac{1}{(x-2)(x-1)} = \frac{A}{(x-2)} + \frac{B}{(x-1)} = \frac{1 \,/ -2}{(x-2)\,/ -2} - \frac{1}{x-1}$

$= \frac{-1/2}{1 - x/2} + \frac{1}{1-x}$

Clear Fractions:

$1 = A(x-1) + B(x-2)$

$1 = (A+B)x - (A+2B)$

$[x^1]$ $(0 = A+B)$

$[x^0]$ + $(1 = -A-2B)$

$1 = -B \Rightarrow B = -1$

$0 = A - 1 \Rightarrow A = 1$

$[x^n]\left(\frac{1}{1-x}\right) = 1$

$[x^n]\left(\frac{-1/2}{1-x/2}\right) = -\frac{1}{2^{n+1}}$

$A(x) = \frac{1}{1-x}$

$= 1 + x + x^2 + \cdots$

$A\left(\frac{x}{2}\right) = \frac{1}{1-x/2}$

$= 1 + \frac{x}{2} + \frac{x^2}{2^2} + \cdots + \frac{x^n}{2^n} + \cdots$

$[x^n]C(x) = [x^{n-1}]\, \frac{3}{x^2 - 3x + 2} = \begin{cases} 0 & n = 0 \\ 3\left(1 - \frac{1}{2^n}\right) & \text{for } n \geq 1 \end{cases}$

Example 2. Find values for A, B, C so that the expression below is true.

$$D(x) = \frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

Clear fractions.

$1 = A(x-2)^2 + B(x-3)(x-2) + C(x-3)$

$x=2$ $\quad 1 = A \cdot 0^2 + B \cdot 0 + C \cdot (-1) \Rightarrow C = -1.$

$x=3$ $\quad 1 = A \cdot 1^2 + B \cdot 0 + C \cdot 0 \Rightarrow A = 1$

$x=4$ $\quad 1 = A \cdot 2^2 + B \cdot 1 \cdot 2 + C \cdot 1$

$\Rightarrow 1 = \underline{4} + 2B - \underline{1} \Rightarrow 1 - 3 = 2B \Rightarrow B = -1.$

$D(x) = \frac{1}{(x-3)} - \frac{1}{x-2} - \frac{1}{(x-2)^2}.$

Series Division

To find the series for the quotient

$$C(x) = \frac{A(x)}{B(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots}{b_0 + b_1 x + b_2 x^2 + \cdots}$$

$\neq 0$

let $C(x) = c_0 + c_1 x + c_2 x^2 + \cdots$ and write $A(x) = B(x)C(x)$ so that

$$(a_0 + a_1 x + a_2 x^2 + \cdots) = (b_0 + b_1 x + b_2 x^2 + \cdots)(c_0 + c_1 x + c_2 x^2 + \cdots)$$

In this equation the a_i and b_i are known coefficients, the c_i are unknown. Equating coefficients in x^i for $i = 0, 1, 2, \cdots$ and solving for c_i we obtain

$[x^0]$ $a_0 = b_0 c_0 \implies c_0 = a_0/b_0 \implies b_0 \neq 0$
$[x^1]$ $a_1 = b_0 c_1 + b_1 c_0 \implies c_1 = (a_1 - b_1 c_0)/b_0$
$[x^2]$ $a_2 = b_0 c_2 + b_1 c_1 + b_2 c_0 \implies c_2 = (a_2 - b_1 c_1 - b_2 c_0)/b_0$
$\cdots$
$[x^n]$ $a_n = b_0 c_n + b_1 c_{n-1} + \cdots + b_n c_0 \implies c_n = (a_n - b_1 c_{n-1} - \cdots - b_n c_0)/b_0$

Example 1. Calculate $A(x) = (1+x)/(1-x)^2 = (1+x)/(1-2x+x^2)$ to x^3.

$A(x) = \dfrac{(1+x)}{(1-2x+x^2)} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$(1+x) = (1-2x+x^2)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{n-2} x^{n-2} + a_{n-1} x^{n-1} + a_n x^n + \cdots$

$[x^0]$ $1 = 1 \cdot a_0 \Rightarrow a_0 = 1$

$[x^1]$ $1 = 1 \cdot a_1 - 2 \cdot a_0 = a_1 - 2 \Rightarrow a_1 = 3.$

$[x^2]$ $0 = 1 a_2 - 2 \cdot a_1 + 1 \cdot a_0 = a_2 - 6 + 1 \Rightarrow a_2 = 5.$

$[x^3]$ $0 = 1 \cdot a_3 - 2a + a_1 = a_3 - 10 + 3 \Rightarrow a_3 = 7.$

$A(x) = 1 + 3x + 5x^2 + 7x^3 + \cdots$

$[x^n]$ $n \geq 2$ $0 = 1 \cdot a_n - 2a_{n-1} + a_{n-2} \Rightarrow a_n = 2a_{n-1} - a_{n-2}.$

$\Rightarrow a_3 = 2 \cdot 5 - 3 = 7$ ✓

Example 2. Find the series for $x/(1-x-x^2)$ using series division.

Let $A(x) = \frac{x}{(1-x-x^2)} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$x = (1-x-x^2)(a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-2}x^{n-2} + a_{n-1}x^{n-1} + a_n x^n + \cdots)$

$[x^0]$ $0 = a_0 \Rightarrow a_0 = 0$

$[x^1]$ $1 = 1 \cdot a_1 - 1 \cdot a_0 = a_1 \Rightarrow a_1 = 1$

$[x^2]$ $0 = 1 \cdot a_2 - a_1 - a_0 = a_2 - 1 - 0 \Rightarrow a_2 =$

$[x^3]$ $0 = 1 \cdot a_3 - a_2 - a_1 = a_3 - 1 - 1 = a_3 - 2$ $\quad a_3 = 2$

$[x^n]$ $0 = 1 \cdot a_n - 1 \cdot a_{n-1} - 1 \cdot a_{n-2} \Rightarrow a_n = a_{n-1} \; a_{n-2}$ (Fibonacci)

$n \geq 2$

So $x/(1-x-x^2)$ is the Fibonacci generating nction.

Exercise. Find the series for $(1+x)/(1-3x+3x^2-x^3)$ to x^4 using series division and determine a recurrence for the nth coefficient.