# Lecture 27 Trees



Grimaldi 12.1

Assignment #7 is due Wednesday.
Upto today's lecture.



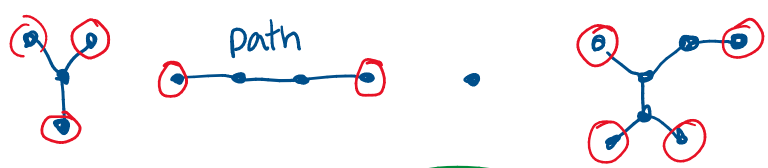
## Definition ( tree, forest and leaf )

Let $G = (V, E)$ be a multigraph. $G$ is a **tree** if $G$ is connected and $G$ does not contain a cycle. $G$ is a **forest** if $G$ does not contain a cycle. A vertex of degree 1 is called **leaf** or **pendant vertex**.

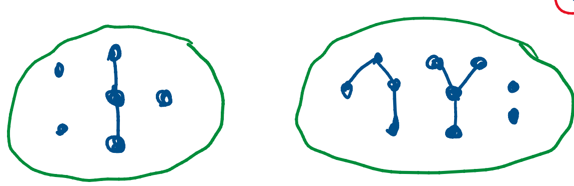
A tree is a forest.
A forest may not be a tree.

Examples

Trees




Forrests



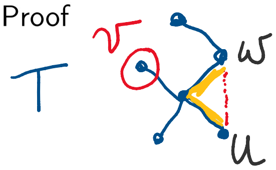
Since a tree cannot have loops or parallel edges, it is a simple graph.
We previously showed that every graph with all vertices of degree $\geq 2$ must have a cycle. Therefore every tree with $\geq 2$ vertices must contain a leaf. Later we will see that trees have $\geq 2$ leaves.

## Lemma

*If $T = (V, E)$ is a tree with leaf $v$ then $T - v$ is a tree.*

Show [ $T-v$ is connected and acyclic ]

Proof




First observe if $T-v$ has a cycle. then $T$ has a cycle which would contradict $T$ is a tree.

Let $u, w \in V$ with $u \neq w \neq v$. There is a path from $u$ to $w$ in $T$ (as $T$ is connected) The only two vertices of degree 1 in a path are the end vertices. All other vertices have degree $\geq 2$. So $v$ (has degree 1) is not on the path. Therefore in $T-v$ there is a path from $u$ to $w$. Therefore $T-v$ is connected.

This observation gives us a powerful tool for proving properties of trees. Try using induction on the number of vertices and, for the inductive step, deleting a leaf then applying the inductive hypothesis.

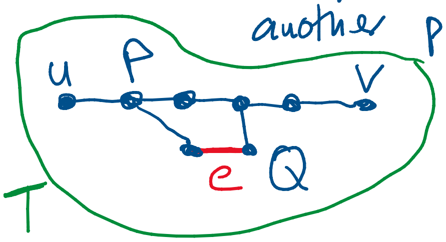


## Theorem ( unique paths )

*If $T = (V, E)$ is a tree and $u, v \in V$ are distinct, there is a unique path in $T$ with ends $u, v$.*

Proof. (is a path from $u$ to $v$) A tree is connected so there is a path from $u$ to $v$.

(path is unique). Towards a contradiction suppose there is another path $Q$ from $u$ to $v$. Since $Q \neq P$ there must be at least one edge $e \in E$ that is on $Q$ but not on $P$. Let $e = \{a, b\}$. Notice $T-e$ is connected. Therefore $e$ is in a cycle in $T$. This contradicts $T$ is a tree.




So there is a unique path from $u$ to $v$.

**Theorem ( main property of trees )**

*If $T = (V, E)$ is a tree then $|V| = |E| + 1$.* $\Rightarrow |E| = |V| - 1$.
*If $G = (V, E)$ is a forest with $k$ trees then $|V| = |E| + k$.*

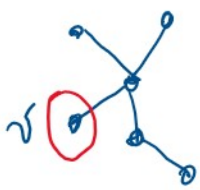
Proof. By induction on $|V|$. Let $n = |V|$ in $T$.

Base: $n=1$ $T$ is the singleton vertex $\bullet$

Here $|V| = 1$, $|E| = 0$ and $|V| = |E| + 1$ ✓

Ind. Step: $n \geq 2$ :

Ind. Hypothesis. Assume $|V| = |E| + 1$ holds for any tree with $< n$ vertices.




Let $v$ be a leaf in $T$.

Then $T - v$ is a tree with $|V| - 1$ vertexes and $|E| - 1$ edges. By induction

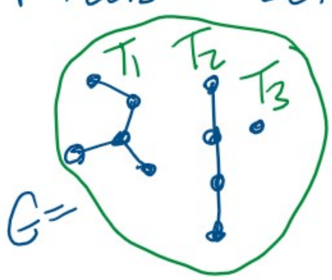
$T - v$ satisfies $(|V| - 1) = (|E| - 1) + 1 \Rightarrow |V| = |E| + 1$.



Proof (cont).

By induction on $n$, $|V| = |E| + 1$ holds for all trees with $n$ vertices.

Forrests. Let $T_1, T_2, \ldots, T_k$ be the trees in the forest $G$.




By the first part of the Theorem

$|V_i| = |E_i| + 1$ for $1 \leq i \leq k$ where $T_i = (V_i, E_i)$

Summing over $i$ we get

$$\underbrace{|V_1| + |V_2| + \cdots + |V_k|}_{= |V|} = \underbrace{(|E_1| + 1) + (|E_2| + 1) + \cdots + (|E_k| + 1)}_{|E| + k}$$

So $|V| = |E| + k$ in a forrest of $k$ trees.

## Lemma

*If $G = (V, E)$ satisfies $|V| = |E| + 1$ then $G$ must have a vertex of degree 0 or at least two of degree 1.* $\Rightarrow |E| = |V| - 1.$

Proof. Let $n = |V|$ and let $k_0$ be the # vertices of degree 0 and $k_1$ be the # of vertices of degree 1 in G. We want to show that $k_0 \geq 1$ or $k_1 \geq 2$ (or both).

Consider the degree sum formula from 11-4

$$2|E| = \sum_{v \in V} \deg(v)$$

at least degree 2.

$$2 \cdot |E| = 2(|V| - 1) \geq 0 \cdot k_0 + 1 \cdot k_1 + 2\underbrace{(n - k_0 - k_1)}_{\text{\# vertices of degree } \geq 2}$$

$$2(n-1) \geq k_1 + 2n - 2k_0 - 2k_1$$

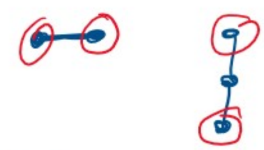
$$\cancel{2n} - 2 \geq \cancel{2n} + k_1 - 2k_0 - 2k_1$$

$$k_1 + 2k_0 \geq 2. \Rightarrow k_1 \geq 2 \text{ or } k_0 \geq 1.$$



## Lemma

*Every tree $T = (V, E)$ with $|V| \geq 2$ has at least two leaves.*



Proof. T is a tree $\Rightarrow$ T is connected

and $T \neq \bullet$ $\Rightarrow$ every vertex has degree $\geq 1$.

$\Rightarrow$ there are no vertexes of degree 0

$\Rightarrow$ there are $\geq 2$ vertices of degree 1 (previous Lemma)

$\Rightarrow$ there are two leaf vertices.

Summary. If T is a tree with $n = |V|$ vertices

- If $n \geq 2$ then T has $\geq 2$ leaves
- If T has 2 leaves then T is a path
- T has $n-1$ edges.

$|V| = |E| + 1$

$|E| = |V| - 1$

$|E| = n - 1$