Lecture 8: Conditional Probability and Independence

Grimaldi 3.6

Example 1. Let S be the set of binary sequences of length 8.
Let $A \subset S$ be the sequences starting with 111.
Let B be the sequences in S with five 1's.
Suppose we pick x from A at random. What is $Pr(B)$?

Definition (Conditional Probability)

Let S be a sample space and A and B two subsets of S. The **conditional probability** of B given/knowing A, denoted by

$$Pr(B|A)$$

is the probability that a random outcome from A also belongs to B. It can be obtained by the formula

$$Pr(B|A) = \frac{Pr(B \cap A)}{Pr(A)}.$$

Example 2. Assume two dice are rolled. What is the probability that, if they sum up to at least 9 (event A) that both dice have the same value (event B).

Four consequences of $Pr(B|A) = Pr(B \cap A)/Pr(A)$.

1. Switching A and B.

2. Multiplicative rule.

3. Law of total probability.

4. Bayes' Theorem.

Example 3. Suppose 10% of olympic cyclists use steroid Z and the IOC develops a test for Z with the following properties.

1. If a cyclist is taking Z the probability they test positive is 0.99.
2. If they are not taking Z the probability they test positive is 0.05.

Question: If a randomly chosen cyclist tests positive for Z, what is the probability they are taking steroid Z.

Definition (Independent Events)

Two events A and B are **independent** if either one of them has probability 0 or both have positive probabily and

$$Pr(B|A) = Pr(B) \text{ and } Pr(A|B) = Pr(A).$$

For example, if we toss a coin twice, the first toss is independent of the second.

Theorem

Two events A and B are independent if and only if

$$Pr(A \cap B) = Pr(A)Pr(B).$$

Proof:

Example 4. Suppose Alex tosses a fair coin 3 times. Here the sample space $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. Consider the events

A: The first toss is a H: $A = \{HHH, HHT, HTH, HTT\}$.
B: The second toss is a H: $B = \{HHH, HHT, THH, THT\}$.
C: There are 2 or 3 heads: $C = \{HHH, HHT, HTH, THH\}$.

Are A and B independent?

Are A and C independent?

Are A and $\bar{B}$ independent?