Lec 3 The Binomial Coefficients
Copyright Michael Monagan

January 18, 2021 11:45 AM

Lecture 3: Combinations and the Binomial Theorem

Assignment #1 is due next Monday. Jan 25th.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Grimaldi Section 1.3

The quantity $\binom{n}{k}$ is the number of ways of choosing a set of size k from a set of size n. We also saw that it is the number of binary strings of length n with k 1's so

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Theorem ($\binom{n}{k} = \binom{n}{n-k}$)

$k \to n-k$

0 0 1 1 1

1 1 0 1 0

$n=5$
$k=3$

Proof.

$$\binom{n}{n-k} = \frac{n!}{(n-k)!\,(n-(n-k))!} = \frac{n!}{(n-k)!\cdot k!}$$

A second "combinatorial proof":
The # of binary strings of length n with k 1 bits is $\binom{n}{k}$
The # of " " " " " " $(n-k)$ 0 bits in $\binom{n}{n-k}$
For each binary string with k 1 bits there is
one binary string with n-k 0 bits
So $\binom{n}{k} = \binom{n}{n-k}$.

Theorem ($\sum_{k=0}^{n}\binom{n}{k} = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n} = 2^n$)

Let S be the set of binary strings of length n.
Let S_k be the set of binary strings of length n with k 1 bits.

$n=3$ $\quad S = S_0 \cup S_1 \cup S_2 \cup S_3$

$\{000\} \quad \{100, 010, 001\} \quad \{110, 101, 011\} \quad \{111\}$

size of S

$|S| = |S_0| + |S_1| + |S_2| + |S_3|$

0 1's

$= \binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1+3+3+1 = 8 = 2^3$

The total # of binary strings of length n is 2^n.

$2^n = |S| = \sum_{k=0}^{n} |S_k| = \sum_{k=0}^{n}\binom{n}{k}.$

Expanding $(x+y)^n$

$(x+y)^1 = 1\cdot x + 1\cdot y$

$(x+y)^2 = (x+y)(x+y) = 1\cdot x^2 + 2\cdot xy + 1\cdot y^2$

$(x+y)^3 = (x+y)\cdot(x+y)(x+y) = 1x^3 + 3x^2y + 3xy^2 + 1y^3$

There are 3 ways to choose 2 x's and 1 y from the three $(x+y)$ factors.

$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$

$= 1\cdot x^4 + \binom{4}{3}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{1}x^1y^3 + \binom{4}{0}x^0y^4$

ways of choosing 3 x's from 4 factors.

$(x+y)^4 = \sum_{k=0}^{4}\binom{4}{k}x^k y^{4-k}.$

Theorem (The Binomial Theorem)

If n is a positive integer then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots + \binom{n}{n} x^n y^0.$$

Because of this theorem the numbers $\binom{n}{k}$ are called binomial coefficients
We now have three equivalent ways to think of $\binom{n}{k}$:

① $\binom{n}{k}$ is the # of subsets of size k from a set of size n.

② $\binom{n}{k}$ is the # of binary strings of length n with k 1 bits.

③ $\binom{n}{k}$ is the coefficent of $x^k \cdot y^{n-k}$ in $(x+y)^n$.

Using the Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Let $x=1$, $y=1$

$$\underbrace{(1+1)^n}_{=2^n} = \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}.$$

Exericse. Find the coefficient of $x^5 y^{95}$ in $(3x-y)^{100}$.

Let $X = 3x$, $Y = -y$

$$(X+Y)^{100} = X^{100} + \cdots + \binom{100}{5} X^5 Y^{95} + \cdots + Y^{100}$$

$$(3x-y)^{100} = (3x)^{100} + \cdots + \binom{100}{5}(3x)^5(-y)^{95} + \cdots + (-y)^{100}$$

$$= \binom{100}{5} 3^5 (-1)^{95} x^5 y^{95}$$

$$= -3^5 \binom{100}{5}.$$

Theorem (The Multinomial Theorem)

If $x_1, x_2, \ldots, x_m$ are variables and n a positive integer, then,

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$$

Proof: The monomial $x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$ comes from a product of

k_1 x_1's
$+k_2$ x_2's
$\vdots$
$+k_m$ x_m's
$= n$

e.g. $\underbrace{\underline{x_1}\ \underline{x_2}\ \underline{x_m} \ldots \underline{x_1}\ \underline{x_m} \ldots \underline{x_2}\ \underline{x_1}}_{n \text{ variables.}}$

The # of permutations of length n with k_1 x_1's, k_2 x_2's, ..., k_m x_m's.

$$= \binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!\, k_2! \cdots k_m!}$$

Example. What is the coefficient of xy^2z^2 in $(w + x + y + z)^5$? $n=5$

$(w+x+y+z)(w+x+y+z)(w+x+y+z)(w+x+y+z)(w+x+y+z)$

$= 1 \cdot w^5 \cdot x^0 \cdot y^0 \cdot z^0 + \cdots + \binom{5}{0,1,2,2} w^0 \cdot x^1 \cdot y^2 \cdot z^2 + \cdots$

$w^0 x^1 y^2 z^2$

$$= \frac{5!}{0!\,1!\,2!\,2!} = \frac{5 \cdot \cancel{4} \cdot 3 \cdot 2}{\cancel{2 \cdot 2}} = 5 \cdot 3 \cdot 2 = 30$$

$z \cdot x \cdot z \cdot y \cdot y$ 2 z's, 2 y's
$x \cdot y \cdot y \cdot z \cdot z$