Lecture 13 Solving Second Order Recurrences, Michael Monagan

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Lecture 13 Second Order Recurrence Relations

Grimaldi 10.2

Assignment #3 due Monday Feb 22nd
I will have my office hours next week.

$f_{n+1} = f_n + f_{n-1}$

No classes next week. :)

$a_n = a_{n-1} + 2a_{n-2}$

For constants a, b, c consider a recurrence relation of the form

$$a\,x_n + b\,x_{n-1} + c\,x_{n-2} = 0 \quad \text{for } n \geq 2. \tag{1}$$

Second order homogeneous constant coefficents

Suppose that $x_n = r^n$ is a solution to equation (1). In this case we have

$X_n = r^n$

$X_{n-1} = r^{n-1}$

$$ar^n + br^{n-1} + cr^{n-2} = 0 \quad \text{for all } n \geq 2. \tag{2}$$

$\Rightarrow r^{n-2}(ar^2 + br + c) = 0$

$n = 2 \Rightarrow 1\,(\underline{a}r^2 + \underline{b}r + \underline{c}) = 0$

Observe that the $n \geq 2$ condition is redundant in equation (2). If this holds for $n = 2$, then it holds for all larger values (multiplying by powers of r gives the other equations). This reduces us to a familiar equation

$$ar^2 + br + c = 0$$

Conclusion: A number r satisfies $ar^2 + br + c = 0$ if and only if $x_n = r^n$ is a solution to our recurrence.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Definition

The homogeneous second order linear recurrence relation

$$ax_n + bx_{n-1} + cx_{n-2} = 0$$

has **characteristic equation**

characteristic polynomial

$$(ar^2 + br + c) = 0.$$

The roots of $ar^2 + br + c$ are precisely those numbers r for which $x_n = r^n$ satisfies the above recurrence.

Exercise. Find all real numbers r so that $x_n = r^n$ is a solution to the recurrence

$$1 \cdot x_n - 5x_{n-1} + 6x_{n-2} = 0$$

$r^2 - 5r + 6 = 0$

$(r-3)(r-2) = 0$

$r = \frac{5 \pm \sqrt{25-24}}{2} = \frac{5}{2} \pm \frac{1}{2} = \underline{3, 2}.$

$x_n = 3^n$ and $x_n = 2^n$ are solutions.

Theorem (Linearity)

Both of the properties below hold for the recurrence relation

$$ax_n + bx_{n-1} + cx_{n-2} = 0 \qquad (3)$$

(A) *If $x_n = r^n$ is a solution of (3) then Cr^n is a solution to (3) for any constant C.*

(B) *If $x_n = s^n$ and $x_n = t^n$ are solutions of (3) then $s^n + t^n$ is a solution.*

It follows from (A) and (B) that $Cs^n + Dt^n$ is a solution for any constants C, D.

Proof: (A). Given $x^n = r^n$ is a solution $\Rightarrow a \cdot r^n + b \cdot r^{n-1} + c \cdot r^{n-2} = 0.$

? $x_n = Cr^n$: $a \cdot C \cdot r^n + b \cdot C \cdot r^{n-1} + C r^{n-2} = C[a \cdot r^n + b \cdot r^{n-1} + c \cdot r^{n-2}] = 0.$

$x_{n-1} = C \cdot r^{n-1}$

(B) Given $a \cdot s^n + b \cdot s^{n-1} + c \cdot s^{n-2} = 0$ and $a \cdot t^n + b \cdot t^{n-1} + c \cdot t^{n-2} = 0$

$x_n = s^n + t^n$: $a(s^n + t^n) + b \cdot (s^{n-1} + t^{n-1}) + c(s^{n-2} + t^{n-2})$

$x_{n-1} = s^{n-1} + t^{n-1}$ $= as^n + bs^{n-1} + cs^{n-2} + at^n + bt^{n-1} + ct^{n-2} = 0 + 0 = 0.$

Exercise. The recurrence relation

$$x_n - 5x_{n-1} + 6x_{n-2} = 0$$

has the solutions $x_n = 3^n$ and $x_n = 2^n$. Check that $C2^n + D3^n$ is a solution.

$X_n = C2^n + D3^n$: $(C2^n + D3^n) - 5(C2^{n-1} + D3^{n-1}) + 6(C \cdot 2^{n-2} + D3^{n-2})$

$= \ldots$

$= \ldots$

$= 0.$

How do we determine what C and D are? With two consecutive initial values. Find the solution with the initial values $x_0 = 6$ and $x_1 = 13$.

$n=0$: $X_0 = C \cdot 1 + D \cdot 1 = 6$ (1)

$n=1$: $X_1 = C \cdot 2 + D \cdot 3 = 13$ (2)

$2(1) - (2)$ $0 \cdot C - D = 12 - 13 = -1 \Rightarrow D = +1$

$C + 1 = 6 \Rightarrow C = 5.$

$\boxed{X_n = 5 2^n + 1 \cdot 3^n}$

General solutions

Theorem

Let a, b, c be fixed constants with $a \neq 0$ and consider the recurrence

$$ax_n + bx_{n-1} + cx_{n-2} = 0. \tag{4}$$

If the characteristic equation,

$$ar^2 + br + c = 0$$

has two **distinct** *real roots, say r_1 and r_2, then every sequence satisfying this recurrence has the form*

$$x_n = Cr_1^n + Dr_2^n \quad \text{general solution.} \tag{5}$$

where C and D are fixed constants. Accordingly, we will call equation (5) the ***general solution*** *to the recurrence.*

Example. The Fibonacci sequence $(f_1, f_2, f_3, \ldots)$ is generated by the recurrence

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for } n \geq 2$$

together with the initial values $f_0 = 0$ and $f_1 = 1$.

(1) Find the general solution to the above recurrence.

(2) Find a closed form (a formula in n) for the Fibonacci sequence.

$$1 \cdot f_n - f_{n-1} - f_{n-2} = 0$$

$$\Rightarrow \; 1 \cdot r^2 - 1 \cdot r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{1+4}}{2 \cdot 1}$$

$$= \frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

$$f_n = C \cdot \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^n + D\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^n$$

$$n=0: \quad f_0 = C \cdot 1 + D \cdot 1 = 0 \Rightarrow C = -D$$

$$n=1: \quad f_1 = C \cdot \left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) + D\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right) = 1$$

$$f_1 = -D\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) + D\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)$$

$$= \cancel{-\frac{1}{2}D} - \frac{\sqrt{5}}{2}D + \cancel{\frac{1}{2}D} - \frac{\sqrt{5}}{2}D = 1$$

$$\Rightarrow -\sqrt{5}D = 1 \Rightarrow D = -1/\sqrt{5}$$

$$C = 1/\sqrt{5}$$

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^n$$

$$f_1 = \frac{1}{\sqrt{5}}\left(\cancel{\frac{1}{2}} + \frac{\sqrt{5}}{2}\right)^1 - \frac{1}{\sqrt{5}}\left(\cancel{\frac{1}{2}} - \frac{\sqrt{5}}{2}\right)^1$$

$$= \frac{1}{2} - -\frac{1}{2} = 1$$

Solving $ar^2 + br + c = 0$ using the quadratic formula we get

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Leftarrow \text{discriminant}$$

✓ If $b^2 - 4ac = 0$ then we have two repeated real roots.

✗ If $b^2 - 4ac < 0$ we have two complex roots.

Theorem (Repeated real roots case)

Let a, b, c be real constants with $a \neq 0, c \neq 0$ and consider the recurrence

$$a\,x_n + b\,x_{n-1} + c\,x_{n-2} = 0.$$

If the characteristic polynomial $ar^2 + br + c$ has a repeated root r then every sequence satisfying this recurrence has the form

$$x_n = Cr^n + D\,nr^n \tag{6}$$

where C and D are constants. Equation (6) is the **general solution** *to the recurrence.*

Example. Solve the following recurrence

$$\underline{1}\, x_n - \underline{6}x_{n-1} + \underline{9}x_{n-2} = 0 \quad \text{with} \quad x_0 = 2,\ x_1 = 3.$$

$$1 \cdot r^2 - 6 \cdot r + 9 = 0$$

$$\Rightarrow \quad (r-3)^2 = 0 \quad \Rightarrow \quad r = 3, 3.$$

$$\Rightarrow \quad x_n = C \cdot 3^n + D \cdot n \cdot 3^n$$

$$n=0: \ x_0 = C \cdot 1 + D \cdot 0 \cdot 1 = 2 \ \Rightarrow \ C = 2$$

$$n=1: \ x_1 = C \cdot 3 + D \cdot 1 \cdot 3 = 3 \ \Rightarrow \ 6 + 3D = 3 \ \Rightarrow \ D = -1.$$

$$x_n = 2 \cdot 3^n - n \cdot 3^n$$

Check:

$$(2 \cdot 3^n - n \cdot 3^n) - 6(2 \cdot 3^{n-1} - (n-1) \cdot 3^{n-1}) + 9(2 \cdot 3^{n-2} - (n-2)3^{n-2})$$

$$= 3^{n-2}(2 \cdot 9 - 9 \cdot n - 6(6 - 3(n-1)) + 9(2 - (n-2)))$$

$$= 3^{n-2}(\underbrace{18 - 36 + 18}_{=0} + (\underbrace{-9 + 18 - 9}_{=0})(n) \underbrace{- 6(-3) \cdot (-1) + 9 \cdot (+2)}_{=0}) = 0.$$