Lecture 26 Hamiltonian Cycles and Paths
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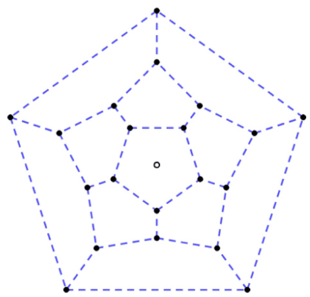
November 20, 2020 11:50 AM

Lecture 26 Hamiltonian Paths and Cycles

Grimaldi 11.5

In 1856 a mathematician William Hamilton invented a game in which the ojbect is to find a cycle along the edges of a dodecahedron.

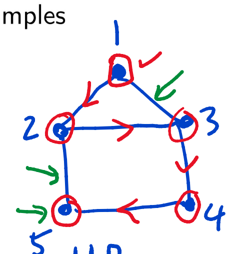


Problem: Can you find a cycle in the graph that includes all 20 vertices?

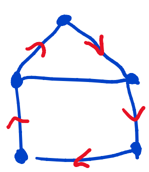
Definition

Let G be a graph. A path of G is a **Hamiltonian path** if it contains every vertex of G. A cycle of G is a **Hamiltonian cycle** if it contains every vertex of G.

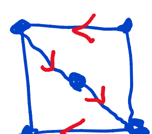
Examples



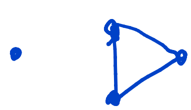
H.P.



H.C



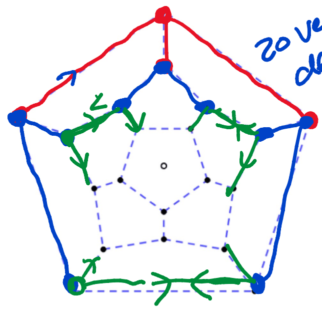
H.P or H.C.?

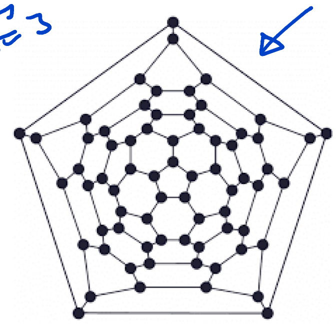


If G has a H.P. or a H.C. then G must be connected.

Algorithm Exhaustive Search: try all possible paths.

20 vertices $deg(v)=3$





Soccerball Graph

$|V|=60$

$|E|=90$

$deg(v)=3$

$3\cdot 2^{59}$ too big.

exponential in $|V|=60$.

$$\underset{v_1}{3}\cdot \underset{v_2}{2}\cdot 2\cdot 2\cdots \cdot \underset{v_{20}}{2} = 3\cdot 2^{19} \approx 1.5\, 10^6$$

Hamiltonian vs. Eulerian

The definition of Hamiltonian is very similar to Eulerian. In Hamiltonian each vertex appears exactly once. In Eulerian each edge appears exactly once. Although they looks similar, having a Hamiltonian cycle and Having an Euler circuit is very different.

(1) There is a fast algorithm to test if a graph $G=(V,E)$ has an Euler circuit where the running time is a linear function of $|V|+|E|$, namely, test if G is connected and all vertices have even degree.

(2) No such fast test is known for a Hamiltonian circuit. The problem of deciding if a graph has a Hamiltonian path/cycle is **NP-complete**. So it is widely believed that there does not exist an algorithm which takes as input an arbitrary graph $G=(V,E)$ and determines if G has a Hamiltonian path/cycle where the running time is bounded by a polynomial function of $|V|+|E|$.

(is not exponential.)

Error

Definition (Necessary and sufficient conditions)

Let P be a property of graphs and C be a set of condtions.

(1) C is **necessary** for P if every graph satisfying P also satisfies C. $P \Rightarrow C$

(2) C is **sufficient** for P if every graph satisfying ~~P~~ C also satisfies ~~C~~ P. $C \Rightarrow P$

(3) If C is both **necessary and sufficient** for P, then a graph G satisfies P if and only if G satisfies C. We say C characterize when p is satisfied. $P \Leftrightarrow C$

Examples

(1) ~~Is~~ It is necessary for G to be connected to have a H.P.

G has a H.P. $\Rightarrow$ G is connected.

(2) Being a complete graph is a sufficient condition to have a H.P.

K_n K_4  G is $K_n \Rightarrow G$ has a H.P.

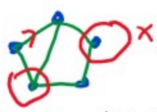
(3) ~~Being connected and having all vertices of even degree are necessary and sufficient conditions to have an Euler circuit.~~

$K_{n,n}$: n is even is a necessary and sufficient condition for $K_{n,n}$ to have an E.C.

Necessary conditions

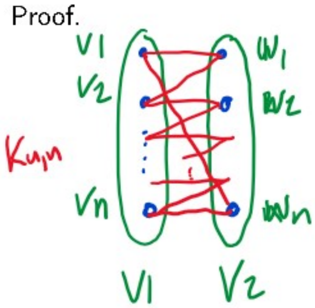
Theorem If $G = (V, E)$ is a graph with a Hamiltonian cycle, then $G - v$ is connected for every vertex $v \in V$.

Proof.

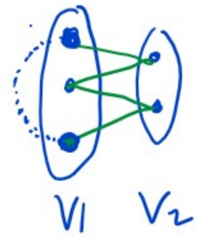


Theorem Let $G = (V, E)$ be a bipartite graph with bipartition $V = V_1 \cup V_2$. If G has a Hamiltonian cycle, then $|V_1| = |V_2|$.

Proof.



So $v_1 w_1 v_2 w_2 \cdots v_n w_n v_1$.



$|V_1| = |V_2|$ is a necessary cond. for a bipartite graph to have a H.C.

A sufficient condition

Idea. G. with enough edges will have a H.C.

Theorem

Let $G = (V, E)$ be a graph with $|V| = n$. If

$\deg(x) + \deg(y) \geq n - 1$ *for all $x, y \in V$ with $x \neq y$* ~~*and x not adjacent to y*~~

then G has Hamiltonian path.

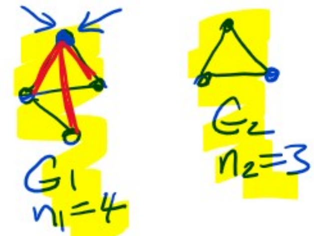
Proof. Lemma. $\deg(x)+\deg(y) \geq n-1 \Rightarrow G$ is connected.

Suppose G is not connected. Let G_1 and G_2 be connected components of G with n_1 and n_2 vertices.

First $n_1+n_2 \leq n$. If v_1 is a vertex in G_1 and v_2 is a vertex in G_2 then

$$\deg(v_1) + \deg(v_2) \leq n_1 - 1 + n_2 - 1 = n_1 + n_2 - 2 \leq n-2$$

A contradiction. Therefor G is connected.



Proof (cont.)

Now we show that G has a H.P.

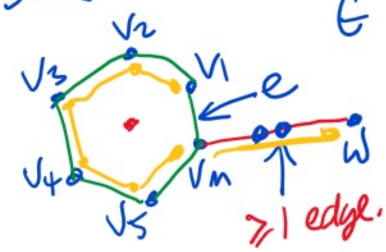
Let P be a path in G of maximal length $m-1$.

P v_1 — v_2 ...... v_{m-1} — v_m

If $m=n$ then P is a H.P. and we are done.

Case $m<n$. I claim there is a cycle with vertices $\{v_1, v_2, \ldots, v_m\}$.

Since $m<n$ there is a vertex w in G not on P. Moreover G is connected by the Lemma.



If we remove e, what's left is a path of length $\geq m$ which contradicts the maximality of P.

Therefor $m \not< n$.

Proof (cont.) To prove: there is a cycle in $\{v_1, \dots, v_m\}$.

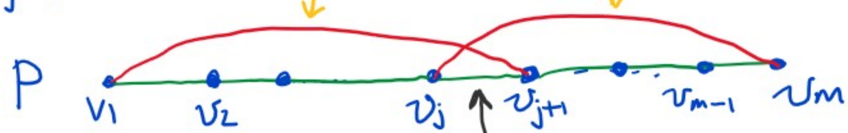
Since the path $P = v_1 \text{---} \cdots \text{---} v_m$ is maximal, the vertices in G adjacent to v_1 are on P. Similarly for v_m.

Let $A = \{1 \le i \le m-1 : v_{i+1} \text{ is adjacent to } v_1\}$ so $|A| \le m-1 \le n-2$

Let $B = \{1 \le i \le m-1 : v_i \text{ is adjacent to } v_m\}$ so $|B| \le m-1 \le n-2$.

Since $|A| + |B| = \deg(v_1) + \deg(v_2) \ge n-1 > m-1$, $A \cap B$ cannot be empty.

Let $j \in A \cap B$. So $\{v_{j+1}, v_1\}$ and $\{v_j, v_m\}$ are in G and G contains the subgraph



Observe if we delete the edge $\{v_j, v_{j+1}\}$ we have a cycle $v_1, v_2, \dots, v_j, v_m, v_{m-1}, \dots, v_{j+1}, v_1$.

Corollary

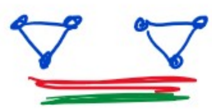
$\frac{n-2}{2}$?

If $G = (V, E)$ is a graph with $|V| = n$ and $\deg(v) \ge \frac{n-1}{2}$ holds for every $v \in V$, then G has a Hamiltonian path.

Proof. Here for $x, y \in V$ $\deg(x) + \deg(y) \ge \frac{n-1}{2} + \frac{n-1}{2} \ge n-1$.

which satisfies the degree condition of the theorem.

Note. The corollary is "tight".

G: 

Here $n=6$ and $\deg(v) = 2 \ge \frac{6-2}{2} = 2$.
but G does not have a H.P.