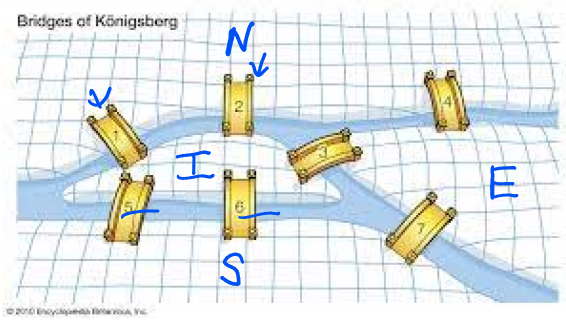
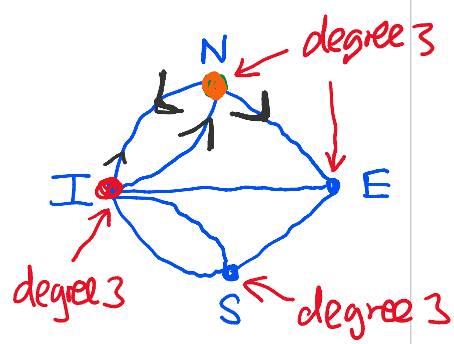
Lecture 24: Eulerian Trails and Circuits

Assignment #6 due Monday @ 11pm.

Grimaldi 11.3





Question: Is it possible walk around the city, cross each bridge once, and end up where you started? NO.

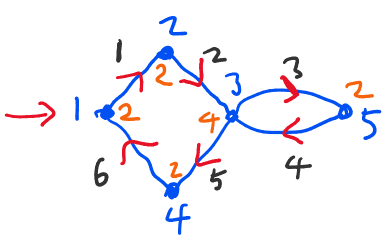
Definition (Eulerian circuit)

An **Euler ciruit** of a multi-graph $G = (V, E)$ is a circuit

$$W = v_1, e_1, v_2, e_2, \ldots, e_n, v_1$$

such that every edge in E appears once in W.

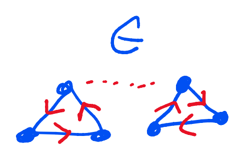
Examples.



1 2 3 5 3 4 1.

Notice every vertex has even degree.

Question: If all vertices of G have even degree, (and G is connected) is there an Euler circuit?

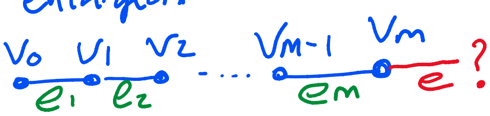


Lemma

Let $G = (V, E)$ be a multigraph with $|E| \geq 1$.
If $\deg(v) \geq 2$ for all $v \in V$, then G contains a cycle of length ≥ 1.

Proof. If G has a loop v then it has a cycle of length 1.
degree $v \geq 2$

If G does not have a loop then G has a path of length at least 1 edge since $|E| \geq 1$.
Let P be a path starting at $v_0 \in V$ of maximum length
i.e. P cannot be enlarged.

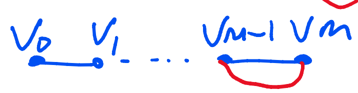
$P =$ 

Since $\deg(v_m) \geq 2$, v_m must be incident to another edge e.
Since P is maximal e must be incident to one of the vertices in P.

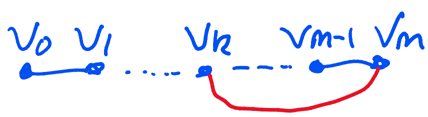
Proof (cont).

(i) $P =$ 

loop Ignore this case

(ii) $P =$ 

parallel edge

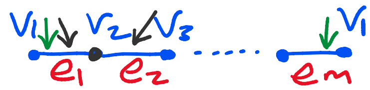
(iii) $P =$ 

Notice G has a cycle in all cases.

Theorem (Euler)

A connected multigraph $G = (V, E)$ which is not the singleton vertex, has an Euler circuit if and only if every vertex in V has even degree.

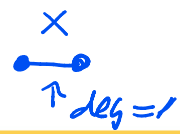
Proof. (⇒). G is connected, G is not • and G has an Euler circuit.



Each time we walk through a vertex we use two edges.
Since an Euler circuit walks over every edge once the degree of every vertex must be even.

(⇐) G is connected, G is not • (singleton vertex.) and all vertices have even degree.
The proof is by induction on $\underline{|E|}$ in G.

Base: $|E| = 1$. $G =$



$G =$

has a Euler circuit

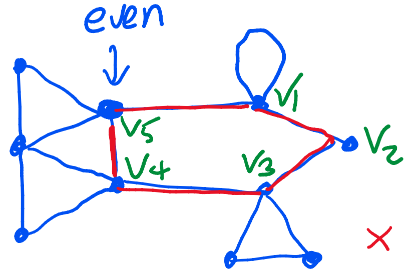
Proof (cont.)

Step: $\underline{n = |E| > 1}$. Ind. Hyp. Assume the Theorem is true for graphs with $1 \le |E| < \underline{n}$ edges.

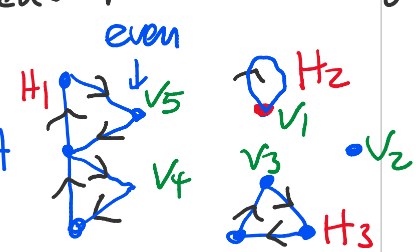
Since G has even degree vertices and is connected $\deg(v) \ge 2$ for all $v \in V$. By the Lemma G has a cycle C of length $m \ge 1$. Let $v_1, v_2, \ldots, v_m$ be the vertices on C.

Idea: Remove all edges on C from G.

G



H



→ Observe every vertex in H has even degree!! ←

Proof (cont.) H may or may not be connected.

Let $H_1, H_2, \ldots, H_\ell$ be the connected components of H with at least one edge.

Ind Hypothesis $\Rightarrow$ $H_1, H_2, \ldots, H_\ell$ have Euler circuits. because # edges in them $< |E| = n$.

Starting at vertex v_1, if $v_1 \in H_i$ where H_i has ≥ 1 edge walk around the Euler circuit in H_i back to v_1 then walk to v_2 on C. If $v_2 \in H_j$ where H_j has ≥ 1 edge walk around the Euler circuit in H_j back at v_2. Repeat this until we arrive back at vertex v_1 and we have an Euler circuit for G.

The proof gives a recursive algorithm for finding an Euler circuit!

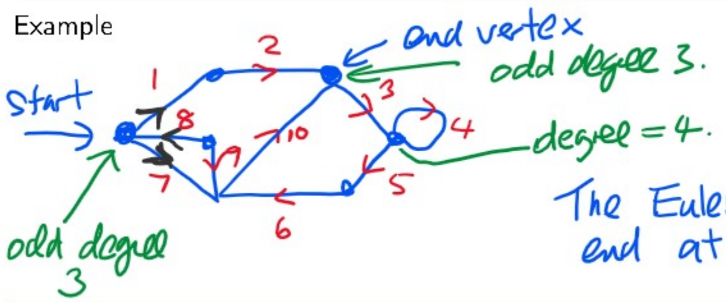
Definition

An **Euler trail** of a multi-graph $G = (V, E)$ is a trail

$$T = v_0, e_1, v_1, e_2, \ldots, e_n, v_n$$

such that every edge in E appears once in T.

Example



The Euler trail must start and end at the odd degree vertices

Corollary (of Euler's theorem)

A connected multigraph $G = (V, E)$ has an Euler trail if and only if there are exactly two vertices in G of odd degree.

Proof. Exercise.