Lecture 10: Applications of Discrete Random Variables

The bins and balls problem

Suppose we throw m balls into n bins randomly.
On average, how many bins will be empty?
On average, how many bins will have one ball in them?

The coupon collectors problem

Suppose we have a bin containing n types of coupons and we draw coupons one at a time from the bin at random. Assume the probability of drawing each type of coupon is $1/n$ and the bin has a very large number of coupons. On average, how many draws do we need to make till we get all n coupons?

Definition

Let S be a sample space and X a random variable on S. Let x be a value from the range of X. The probability of x, denoted by $Pr(X = x)$ is the sum of the probabilities of all outcomes s of S such that $X(s) = x$.

Example 1 Let S be the set of all binary sequences of size $n = 3$ bits.
Let $X(s)$ be the number of 1 bits in a binary string $s \in S$.
Here the range of X denoted $r(X)$ is $\{0, 1, 2, 3\}$.

$\Pr(X = 0) =$

$\Pr(X = 1) =$

$\Pr(X = 2) =$

$\Pr(X = 3) =$

$\Pr(X = k) =$

Definition

The **expected value** of a random variable X on a sample space S is defined by

$$E(X) = \sum_{x \in r(X)} x Pr(X = x) = \sum_{s \in S} X(s) Pr(s).$$

Example 1 (cont.)

x	0	1	2	3
$Pr(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$E[X] = \sum_{x \in r(X)} x Pr(X = x) =$$

Theorem (Linearity of Expection)

Let X and Y be two random variables on the same sample space S and $a \in \mathbb{R}$. Then

(1) $E(aX) = aE(X)$ *and*

(2) $E(X + Y) = E(X) + E(Y)$.

Proof

The bins and balls problem

Suppose we throw m balls into n bins randomly.
Question 1: What is the probability that bin i has k balls?

Question 2: On average, how many bins are empty?

Exercise: On average, how many bins have one ball?

The coupon collectors problem

Suppose a large bin contains many copies of $n = 10$ coupons. Assuming there are an equal number of each coupon, if we draw coupons at random from the bin, on average, how many draws will it take to get all n coupons?

The coupon collectors problem continued.

Exercise: On average, how many times must you toss a fair coin before you get a head and a tail?