Lecture 28: Trees

Grimaldi 12.1

Definition (tree, forest and leaf)

Let $G = (V, E)$ be a multigraph. G is a **tree** if G is connected and G does not contain a cycle. G is a **forest** if G does not contain a cycle. A vertex of degree 1 is called **leaf** or **pendant vertex**.

Examples

Since a tree cannot have loops or parallel edges, it is a simple graph.
We previously showed that every graph with all vertices of degree ≥ 2 must have a cycle. Therefore, every tree with ≥ 2 vertices must have a leaf. Later we will see that all trees with at least two vertices have at least two leaves.

Lemma

If $T = (V, E)$ is a tree with leaf v then $T - v$ is a tree.

Proof

This observation gives us a powerful tool for proving properties of trees. Try using induction on the number of vertices and, for the inductive step, deleting a leaf then applying the inductive hypothesis.

Theorem (unique paths)

If $T = (V, E)$ is a tree and $u, v \in V$ are distinct, there is a unique path in T with ends u, v.

Proof.

Theorem (main property of trees)

If $T = (V, E)$ is a tree then $|V| = |E| + 1$.
If $G = (V, E)$ is a forest with k trees then $|V| = |E| + k$.

Proof.

Proof (cont).

Lemma

If $G = (V, E)$ satisfies $|V| = |E| + 1$ then G must have a vertex of degree 0 or at least two of degree 1.

Proof.

Lemma

Every tree $T = (V, E)$ with $|V| \geq 2$ has at least two leaves.

Proof.

Exercise. Let T be a tree. Show that removing any edge from T disconnects T.