Lecture 1 S21

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Lecture 1: Fundamental Combinatorial Objects

Assignment 1 is posted.

We will study four combinatorial objects

1. sets and subsets
2. strings and permutations
3. graphs
4. trees

Example 1: Sets and Subsets $S = \{A, B, C, D\}$

How many ways can we choose subsets of size 3 from S?

$\binom{4}{3} = \binom{4}{1} = 4$ $\binom{n}{k} = \binom{n}{n-k}$

Choosing 3 elements is the same as removing 1. 4 ways to choose 1

→ A B C D

→ $\{B, C, D\}$ $\{A, C, D\}$ $\{A, B, D\}$ $\{A, B, C\}$

Strings

Definition (alphabet and string)

An **alphabet** Σ is a set of n elements called **letters**.
A **string** S of size n is an ordered sequence of n letters from Σ.

Examples $\Sigma = \{0, 1\}$ $\Sigma = \{A, C, G, T\}$

↑ bits

A = Adenine
C = Cytosine
G = Guanine
T = Thymine

01101 has length $n=5$

ACCT ⇄ TCCA

binary strings.

How many binary strings of length n are there? 2^n

Exercise How many DNA sequences are there of length n ? is not 4^n? $\neq \frac{4^n}{2}$

Exercise: Find all strings of length 6 over $\{0,1\}$ that don't have 10 as a substring.

111111
011111
001111
000111
000011
000001
000000

$\left.\right\} 7$

Permutations

Definition (permutation)

A **permutation** P over an alphabet Σ is a string over Σ where every letter occurs exactly once.

Excercise For $\Sigma = \{1, 2, 3\}$ find all permutations.

$n = |\Sigma| = 3$

123 213 312
132 231 321

6

$3! = 3 \cdot 2 \cdot 1 = 6.$

Proof.

n choices, n-1 choices, n-2 choices, 1 choices

$\Sigma = \{1, 2, \ldots, n\}$

letters 1 2 3 .... n

permutations is $n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1 = n!$

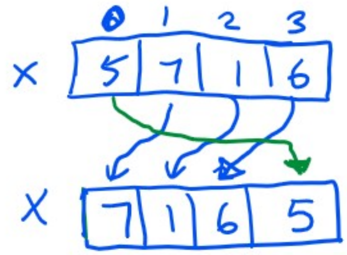
Theorem

The number of permutations of a set of n objects is $n!$.

Permutations are used in cryptography as functions.

array of 4 integers

```
void P( int[4] x ) {
  int t;
  t = x[0];
  x[0] = x[1];
  x[1] = x[2];
  x[2] = x[3];
  x[3] = t;
  return;
}
```



5 7 1 6

$\boxed{7\ 1\ 6\ 5}$

$t = \underline{5}$

This P permutes the array x.

Graphs

Loop. No loops. $\{1,1\}$ X

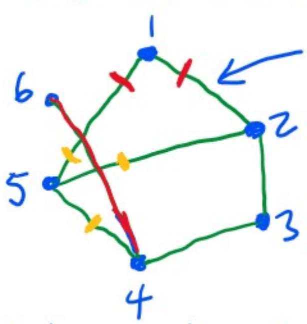
Definition (graph)

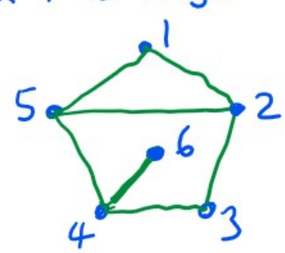
A (simple) **graph** G is a pair (V, E) where V is a set of **vertices** and E is a set of unordered pairs of vertices called **edges**. If $e = \{i, j\} \in E$ we say vertices i and j are **adjacent**. The **degree** of a vertex is the number of adjacent vertices.

Example $V = \{1, 2, 3, 4, 5, 6\}$, 6 vertices

$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}, \{2, 5\}, \{4, 6\}\}$ 7 edges

Vertex 1 is adjacent to vertex 2.

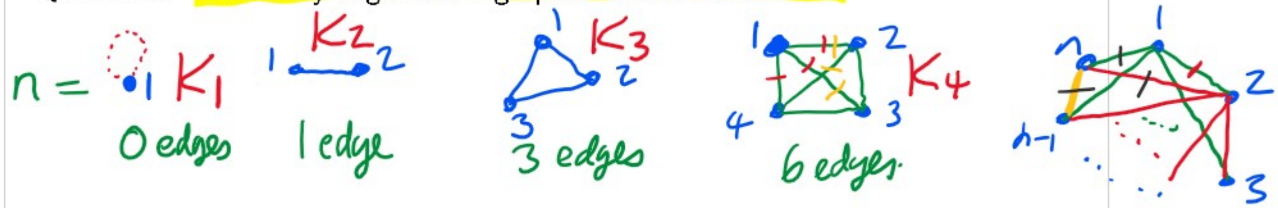




Vertices	Edges
Cities	Roads
Elec. Comps.	Wires
People	Relationships.

Vertex 1 has degree 2
Vertex 5 has degree 3

Question. How many edges can a graph with n vertices have?



vertex 1, 2, 3, ..., n−1, n

$$\#\text{edges} = (n-1) + (n-2) + (n-3) + \cdots + 1 + 0 = \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2} = \binom{n}{2}.$$

Definition (complete graph)

A graph $G = (V, E)$ is **complete** if $|V| \geq \mathbf{1}$ and for all $i, j \in V$ ($i \neq j$) the edge $\{i, j\} \in E$. The complete graph with n vertices is denoted K_n.

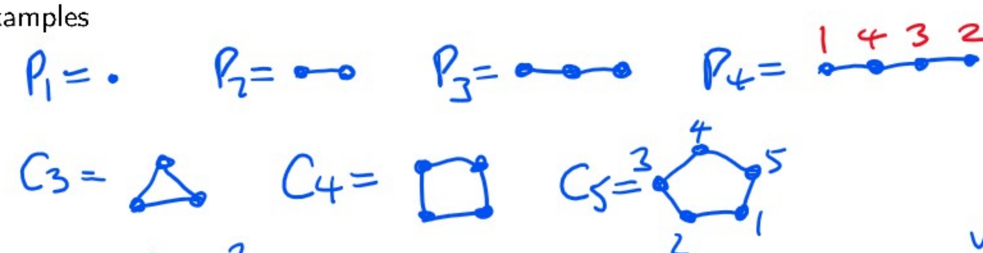
Definition (path graph)

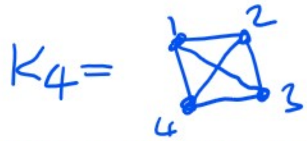
A graph $G = (V, E)$ is a **path** if $|V| \geq \mathbf{1}$ and V may be ordered $v_1, v_2, \ldots, v_n$ so that $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\}$. The path graph with n vertices is denoted P_n.

Definition (cycle graph)

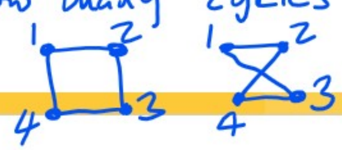
A graph $G = (V, E)$ is a **cycle** if $|V| \geq \mathbf{3}$ and V may be ordered $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$. The cycle graph with n vertices is denoted C_n.

Examples



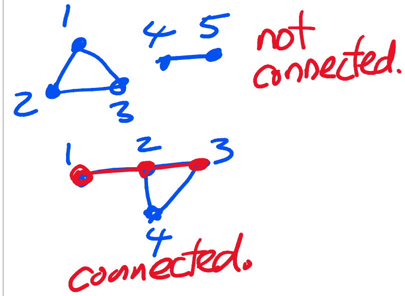


How many cycles of length $\underline{4}$ (vertices) are in K_4.



Definition (connected graph)

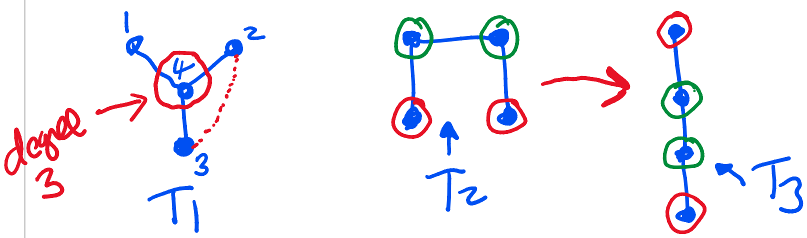
A graph $G = (V, E)$ is **connected** if there is a path in G from vertex $i \in V$ to vertex j for all $i \neq j$.



Definition (tree)

A graph $G = (V, E)$ is a **tree** if it is connected and has no cycles.

Example. All (unlabelled) trees with 4 vertices.



T_2 and T_3 are the same trees, just drawn differently.

Exercise. Draw all (unlabelled) trees with 5 vertices.

Exercise. If G is a tree with $n > 0$ vertices, how many edges must G have?