Lecture 8 Conditional Probability and Independence
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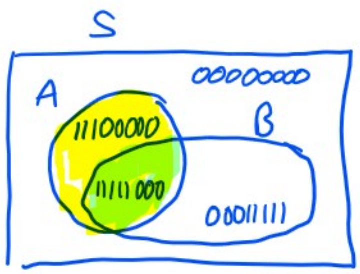
January 29, 2021 11:51 AM

Lecture 8: Conditional Probability and Independence

Assignment #2 due Monday @ 11pm.
Covers Lectures 5,6,7,8 so upto today.
My office hour Tomorrow (Saturday) @ 8pm.

Grimaldi 3.6

Example 1. Let S be the set of binary sequences of length 8.
Let $A \subset S$ be the sequences starting with 111.
Let B be the sequences in S with five 1's.
Suppose we pick x from A at random. What is $Pr(B)$?



$|S| = 2^8 \qquad |A| = 2^5$ (111 _ _ _ _ _) $\qquad |B| = \binom{8}{5}$

2 1 bits

$|A \cap B| = |\{111\,\_\,\_\,\_\,\_\,\_\}| = \binom{5}{2} = \frac{5!}{2!\,3!} = \frac{5 \cdot 4}{2} = 10.$

Let $Pr(B \mid A)$ = probability $x \in B$ given $x \in A$.

$$= \frac{|A \cap B|}{|A|} = \frac{10}{2^5} = \frac{5}{16}.$$

$$= \frac{|A \cap B| / |S|}{|A| / |S|} = \frac{Pr(A \cap B)}{Pr(A)}.$$

Definition (Conditional Probability)

Let S be a sample space and A and B two subsets of S. The **conditional probability** of B given/knowing A, denoted by

$$Pr(B|A)$$

is the probability that a random outcome from A also belongs to B. It can be obtained by the formula

$$Pr(B|A) = \frac{Pr(B \cap A)}{Pr(A)}.$$

Example 2. Assume two dice are rolled. What is the probability that, if they sum up to at least 9 (event A) that both dice have the same value (event B). (a "double")

The sample space $S = \{ (i,j) : 1 \le i,j \le 6 \}$ $|S| = 6 \cdot 6 = 36$.

$A = \{ 3+6, 6+3, 4+5, 5+4, 4+6, 6+4, 5+5, 6+5, 5+6, 6+6 \}$. $|A| = 10$

$B = \{ (1,1), (2,2), (3,3), (4,4), (5,5), (6,6) \}$ $|B| = 6$.

$A \cap B = \{ 5+5, 6+6 \}$ $|A \cap B| = 2$

$$\underset{\text{double}}{Pr(B} | \underset{\geq 9}{A)} = \frac{Pr(A \cap B)}{Pr(A)} = \frac{2/36}{10/36} = \frac{1}{5}$$

Four consequences of $Pr(B|A) = Pr(B \cap A)/Pr(A)$.

1. Switching A and B. $Pr(A|B) = Pr(A \cap B)/Pr(B)$.

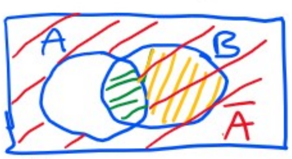
2. Multiplicative rule. $Pr(A) \cdot Pr(B|A) = Pr(B \cap A) = Pr(A \cap B) = Pr(B) \cdot Pr(A|B)$

3. Law of total probability.

disjoint.

$B = (B \cap A) \cup (\bar{A} \cap B)$

S



$Pr(B) = Pr((B \cap A) \cup (\bar{A} \cap B)) = Pr(B \cap A) + Pr(\bar{A} \cap B)$.

$\Rightarrow$ $\boxed{Pr(B) = Pr(A) Pr(B|A) + Pr(\bar{A}) \cdot Pr(B|\bar{A})}$ by (2).

4. Bayes' Theorem.

$Pr(B|A) = Pr(A \cap B)/Pr(A) \underset{(1)}{=} Pr(B) \cdot Pr(A|B) / Pr(A)$

$$\boxed{Pr(B|A) = \frac{Pr(A|B) \cdot Pr(B)}{Pr(A)}}$$

Example 3. Suppose 10% of olympic cyclists use steroid Z and the IOC develops a test for Z with the following properties. drug

1. If a cyclist is taking Z the probability they test positive is 0.99.
2. If they are not taking Z the probability they test positive is 0.05. false positive.

Question: If a randomly chosen cyclist tests positive for Z, what is the probability they are taking steroid Z.

$S = \{$ all olympic cyclists $\}$

$A = \{$ olympic cyclists taking $Z \}$ 10%

$\rightarrow B = \{$ olympic cyclists which test positive $\}$

$\boxed{Pr(B) = 0.1 \times .99 + .9 \times .05 = 0.144}$ = Law of Total probability.

taking Z — not taking Z

We want to know

$$Pr(A|B) = \frac{Pr(B|A) \cdot Pr(A)}{Pr(B)} = \frac{.99 \times .10}{0.144} = \boxed{.6875}$$

very low

taking Z — tested +ve — test +ve — taking Z

Definition (Independent Events)

Two events A and B are **independent** if either one of them has probability 0 or both have positive probabily and

$$Pr(B|A) = Pr(B) \text{ and } Pr(A|B) = Pr(A).$$

For example, if we toss a coin twice, the first toss is independent of the second.

Theorem (Test for independence).

Two events A and B are independent if and only if

$$Pr(A \cap B) = Pr(A)Pr(B).$$

$\boxed{Pr(A) \neq 0.}$ $Pr(B) \neq 0$?

Proof: Suppose $Pr(B|A) = Pr(B)$ (A and B are independent)

$\times Pr(A)$

$\Rightarrow Pr(B|A) \cdot Pr(A) = Pr(A) \cdot Pr(B)$

use $Pr(B|A) = Pr(A \cap B)/Pr(A)$ from slide 1 & 2. $Pr(A) \neq 0$

$\Rightarrow \frac{Pr(A \cap B)}{Pr(A)} \cdot Pr(A) = Pr(A) \cdot Pr(B)$

$\Leftrightarrow Pr(A \cap B) = Pr(A) \cdot Pr(B)$

Example 4. Suppose Alex tosses a fair coin 3 times. Here the sample space $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. Consider the events

A: The first toss is a H: $A = \{HHH, HHT, HTH, HTT\}$. $|A| = 4$ $Pr(A) = |A|/|S| = 4/8 = \frac{1}{2}$
B: The second toss is a H: $B = \{HHH, HHT, THH, THT\}$. $|B| = 4$ $Pr(B) = 1/2$
C: There are 2 or 3 heads: $C = \{HHH, HHT, HTH, THH\}$. $|C| = 4$ $Pr(C) = 1/2$.

Are A and B independent? IS $\underbrace{Pr(A \cap B)}_{= 2/8 = 1/4} = \underbrace{Pr(A)}_{= 1/2} \cdot \underbrace{Pr(B)}_{= 1/2} = 1/4$ INDEP ✓

$A \cap B = \{HHT, HHH\}$

Are A and C independent? IS $\underbrace{Pr(A \cap C)}_{= 3/8} = \underbrace{Pr(A)}_{= 1/2} \cdot \underbrace{Pr(C)}_{= 1/2}$ NOT INDEP ✗

$A \cap C = \{HHT, HTH, HHH\}$ 3/8

Are A and $\bar{B}$ independent? IS $\underbrace{Pr(A \cap \bar{B})}_{= 2/8} = \underbrace{Pr(A)}_{= 1/2} \cdot \underbrace{Pr(\bar{B})}_{= 1/2} = 1 - Pr(B)$ INDEP. ✓

$A \cap \bar{B} = \{HTH, HTT\}$

2nd toss = T