Lecture 19 Rational Genertng Functions

Grimaldi 9.2

$$A(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$

$$A'(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

We have already seen that the generating function

$$A(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

has a compact represenation as the rational function $\frac{1}{1-x}$. Generating functions which can be compactly represented as rational functions will be our main subject.

Definition

A generating function $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ is **rational** if it can be expressed as

$$A(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

(1) Given a sequence of numbers express it as a rational GF ?

(2) Given a rational GF, find the associated sequence (coefficient extraction)

Two useful generating functions

$$A(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

$$A'(x) = 1 + 2x + 3x^2 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}.$$

Using just these two GF's with basic arithmetic operations gives us the ability to describe many other GF's.

Example 1. Determine the sequence for the GF

$$\frac{x^3 - 2}{1 - x}$$

Example 2. Determine the sequence for the GF

$$\frac{2x^2+5}{(1-x)^2}+7x$$

Definition (Substitution)

Let $A(x) = a_0 + a_1 x + a_2 x^2 + \dots$ be a GF and c be a constant. Define

$$A(cx^m) = a_0 + a_1(cx^m) + a_2(cx^m)^2 + a_3(cx^m)^3 \cdots = \sum_{n=0}^{\infty} a_n c^n x^{mn}.$$

Example 1. The GF for nickels is $N(x) = 1 + x^5 + x^{10} + \cdots = \sum_{n-0}^{\infty} x^{5n}$.
Express $N(x)$ as a rational function.

Example 2. What is the GF for $1, -1, 1, -1, \ldots$?

Example 3. Express $C(x) = 1 - 2x + 4x^2 - 8x^3 + 16x^4 - \ldots$ as a rational function.

Example 4. Find a rational GF for the sequence $1, -2, 3, -4, 5, -6, \ldots$?

Example 5. Express $D(x) = -x + 2x^2 - 3x^3 + 4x^4 - \ldots$ as a rational function.

Finding Coefficients

Using substitution and our two basic GF's $A(x) = 1 + x + x^2 + \ldots$ and $A'(x)$ we can now determine the coefficients for any GF that has the form

$$\frac{p(x)}{ax+b} \quad \text{or} \quad \frac{p(x)}{(ax+b)^2}$$

Problem 1. Find the coefficient of x^k in the GF

$C(x) = \dfrac{x^2}{2x+3}$

Problem 2. Find the coefficient of x^k in the GF

$$D(x) = \frac{x^2}{(x+2)^2}$$