Lecture 24 Planar Graphs
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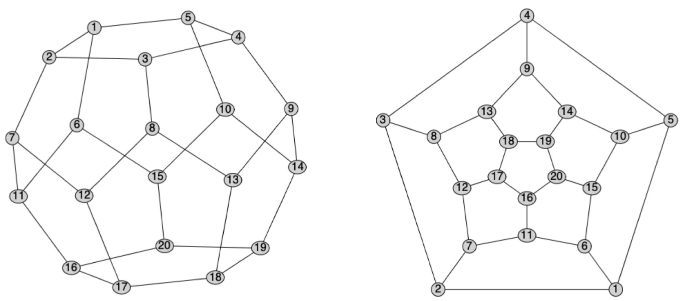
Lecture 24 Planar Graphs

Grimaldi 11.4

Midterm 3 is on Monday.
Same procedure and rules as for Midterm 2.



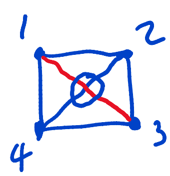
These are both drawings of the same graph. To see this locate the cycles $1-2-3-4-5-1$ and $16-17-18-19-20$ in both graphs.

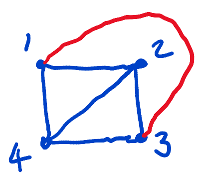
Definition (planar graph)

A graph G is **planar** if G has a drawing (in the plane) so that the edges intersect only at the vertices of G. Such a drawing is called a **planar embedding** of G.

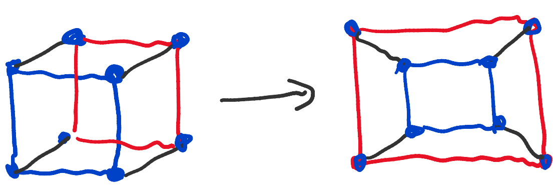
Examples

K_4



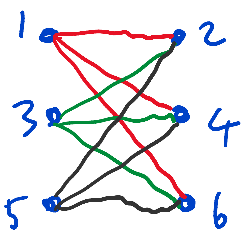


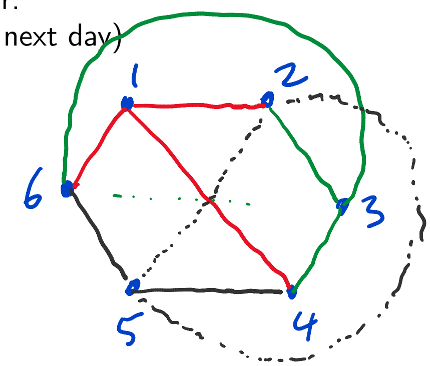
is a planar embedding of K_4.



a planar embedding of the cube.

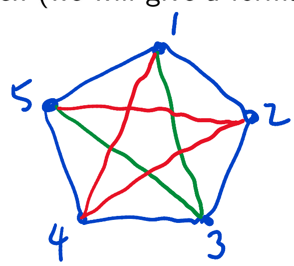
Observation: The graph $K_{3,3}$ is not planar.
Proof sketch (we will give a formal proof next day)

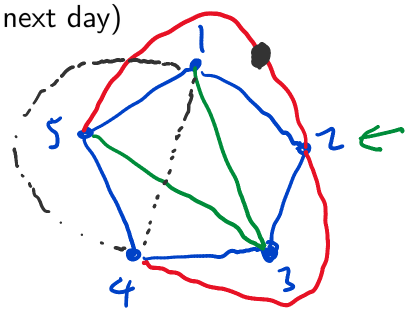




Get Stuck.

Observation: The graph K_5 is not planar.
Proof sketch (we will give a formal proof next day)

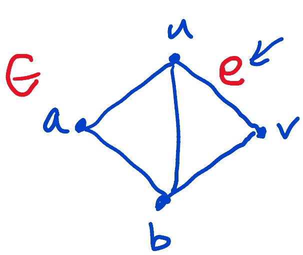


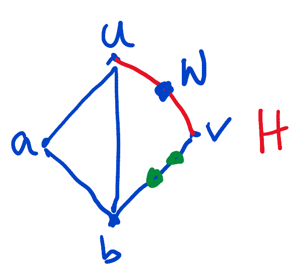


Definition (subdivision)

Let $G = (V, E)$ be a multigraph and let $e = \{u, v\}$ be an edge in E. To **subdivide** the edge e is to delete e and add a new vertex w and two new edges $e_1 = \{u, w\}$ and $e_2 = \{w, v\}$ to G. If the graph H is obtained from G by a sequence of subdivions, then H is called a subdivison of G.

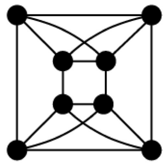
Example





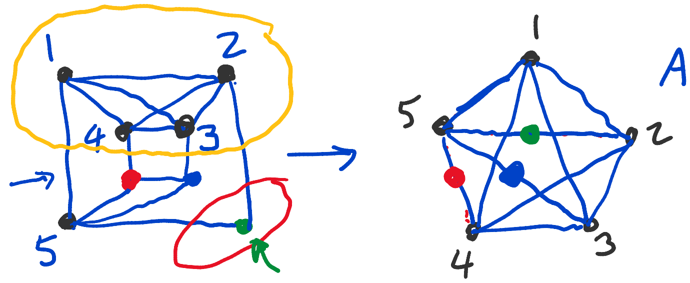
Observation. If H is a subdivision of G then H is planar if and only if G is planar.
This means that every subdivision of $K_{3,3}$ and K_5 is nonplanar.

Example: Is this graph planar? I.e. can you find a planar embedding?



K_5?

If we think G is planar then we can try to find a planar embedding. If not we look for a subdivision of K_5 or $K_{3,3}$ in G.



A subdivision of K_5 inside the graph.

Exercise: Find a subdivision of $K_{3,3}$ in G.

Question: Which graphs are planar ?

Definition

Let G and H be multigraphs. We say that G **contains a subdivision** of H if there is a subgraph of G isomorphic to some subdivision of H.

Theorem (Kuratowski-Wagner)

A multigraph G is planar if and only if G does not contain a subdivision of $K_{3,3}$ or a subdivision of K_5.

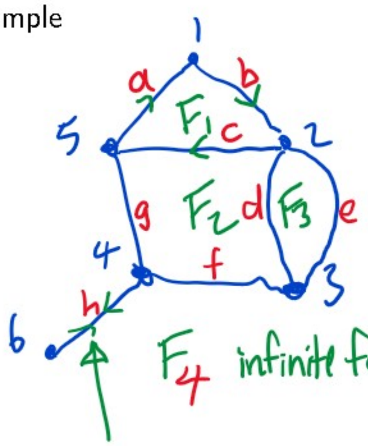
Notes.

1. If G is planar then every subgraph of G is planar.
2. The Hopcroft–Tarjan planarity test (1974) takes linear time in $|V|+|E|$.

Definition (Faces)

Let G be a planar graph embedded in the plane. The embedding partitions the plan into connected regions called **faces**. There is one unbounded region called the **infinite face**. All other faces are **internal faces**. If G is connected, every face has vertices and edges on its boundary. They form a closed walk called a **facial walk**

Example



F_4 infinite face.

Facial walks

F_1 1b2c5a1.

F_3 2d3e2.

F_4 6h4f3e2b1a5g4h6

$$|V| - |E| + |F| = 6 - 8 + 4 = 2$$

Euler's Th.

This graph has $|V| = 6$, $|E| = 8$, $|F| = 4$.

Theorem (Euler's formula)

If $G = (V, E)$ is an connected multigraph embedded in the plane and F is the set of faces, then

$$|V| - |E| + |F| = 2.$$

This implies all embeddings of a planar graph have the same number of faces.

~~Example~~

By induction on $|E|$ in G.

Base: $|E| = 0$ $G = \bullet$ $|V| - |E| + |F| = 1 - 0 + 1 = 2$ ✓

Proof.

Ind. Step: $|E| = n \geq 1$. Assume $|V| - |E| + |F| = 2$ holds for graphs with $|E| < n$. (Ind. Hyp).

Case (i) G has a cycle. Let e be an edge on the cycle. Notice e separates two faces.

Let $H = G - e$. Then H has $|E| - 1$ edges and $|F| - 1$ faces, $|V|$ vertices. and H is connected. By the induction hypoth.

Proof (cont.) H: $|V| - (|E|-1) + (|F|-1) = 2$

$\Rightarrow |V| - |E| + |F| = 2.$ ✓

Case (ii) If G has no cycle but is connected it's a tree, and it has one face.

Let's delete a leaf vertex v from G.

Observe G' is connected,

$|E'| = |E| - 1$, $|V'| = |V| - 1$, $|F'| = |F|$.

By Ind. Hyp. to G' we have

$|V'| - |E'| + |F'| = 2$

$\Rightarrow |V| - \cancel{1} - (|E| - \cancel{1}) + |F| = 2$

$\Rightarrow |V| - |E| + |F| = 2$!!

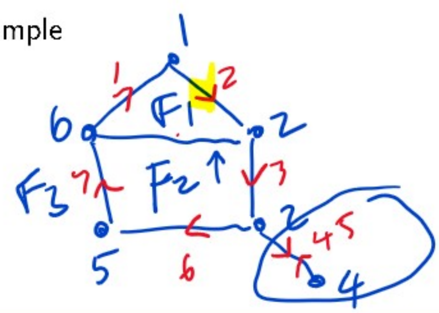
$G' = G - v.$

By induction $|V| - |E| + |F| = 2$ for all connected planar graphs.

Definition (Face degrees)

Let $G = (V, E)$ be a connected multigraph embedded in the plane and let f be a face of this embedding. We define the **degree** of f, denoted $\deg(f)$, to be the number of edges in a facial walk of f.

Example



$\deg(F_1) = 3$

$\deg(F_2) = 4$

$\deg(F_3) = 7$

Theorem

If G has faces $f_1, f_2, \ldots, f_k$ then $\sum_{i=1}^{k} \deg(f_i) = 2|E|$.

Proof Each edge in E counts 2 to the sum.