Discrete Mathematics II Lecture Outlines

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If you print them so to take notes, we suggest you print two per page.

Lecture 1: Fundamental Combinatorial Objects

We will study four combinatorial objects

1. sets and subsets
2. strings and permutations
3. graphs
4. trees

Example Sets and Subsets

Strings

Definition (alphabet and string)

An **alphabet** Σ is a set of n elements called **letters**.
A **string** S of size n is an ordered sequence of n letters from Σ.

Examples $\quad \Sigma = \{0, 1\} \qquad \Sigma = \{A, C, G, T\}$

Exercise How many DNA sequences are there of length n ?

Example Find all strings of length 6 over $\{0, 1\}$ that don't have 10 as a substring.

Permutations

Definition (permutation)

A **permutation** P over an alphabet Σ is a string over Σ where every letter occurs exactly once.

Example $\Sigma = \{1, 2, 3\}$ find all permutations.

Theorem

The number of permutations of a set of n objects is $n!$.

Graphs

Definition (graph)

A (simple) **graph** G is a pair (V, E) where V is a set of **vertices** and E is a set of unordered pairs of vertices called **edges**. If $e = \{i, j\} \in E$ we say vertices i and j are **adjacent**. The **degree** of a vertex is the number of adjacent vertices.

Example $V = \{1, 2, 3, 4, 5, 6\}$,
$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{1, 5\}, \{2, 5\}, \{4, 6\}\}$

Question. How many edges can a graph with n vertices have?

Definition (complete graph)

A graph $G = (V, E)$ is **complete** if $|V| \geq \mathbf{1}$ and for all $i, j \in V$ the edge $\{i, j\} \in E$. The complete graph with n vertices is denoted K_n.

Definition (path graph)

A graph $G = (V, E)$ is a **path** if $|V| \geq \mathbf{1}$ and V may be ordered $v_1, v_2, \ldots, v_n$ so that $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\}$. The path graph with n vertices is denoted P_n.

Definition (cycle graph)

A graph $G = (V, E)$ is a **cycle** if $|V| \geq \mathbf{3}$ and V may be ordered $E = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$. The cycle graph with n vertices is denoted C_n.

Examples

Definition (connected graph)

A graph $G = (V, E)$ is **connected** if there is a path in G from vertex $i \in V$ to vertex j for all $i \neq j$.

Definition (tree)

A graph $G = (V, E)$ is a **tree** if it is connected and has no cycles.

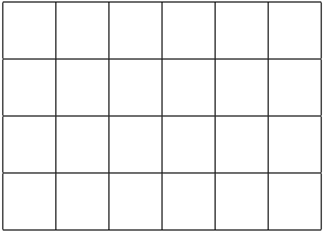
Example. All (unlabelled) trees with 4 vertices.

Exercise. Draw all (unlabelled) trees with 5 vertices.
Exercise. If G is a tree with $n > 0$ vertices, how many edges must G have?

Lecture 2: Basic Counting Principles

Reading: Grimaldi Sections 1.1, 1.2

Lattice paths arise in theoretical physics.



How many lattice paths are there from $(0,0)$ to $(6,4)$ if we are restricted to North steps and East steps only?

Definition (Rule of Sum)

If there are m ways to perform task X and n ways to perform task Y, there are $m + n$ ways to perform **either** X or Y.

Definition (Rule of Product)

If there are m ways to perform task X and n ways to perform task Y, there are $m\,n$ ways to perform **both** X and Y.

Examples.

Exercise. If there are 10 people at a party and all hug each other, how many hugs are there?

Theorem (Strings)

If Σ is an alphabet with k letters, the number of strings of length n over Σ is k^n.

Proof.

Theorem (Permutations)

The number of permutations of a set of n distinct objects is $n!$.

Proof.

Definition (Permultations with Repetition)

Suppose there k_1 objects of type A, k_2 of type B, ..., and k_r of type R and let $n = k_1 + k_2 + \cdots + k_n$ be the total number of objects. The number of distinct permutations is denoted by $\binom{n}{k_1, k_2, \ldots, k_r}$.

Example. Consider the letters M, E, E, N, N. How many permutations are there?

Theorem (Permutations with Repetition)

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1!, k_2!, \ldots, k_r!}.$$

Proof.

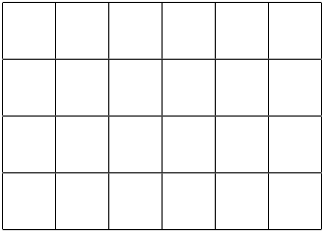
Exercise. How many binary strings of length 20 are there with exactly 13 1's?

Theorem (Subsets and Combinations)

If S is a set of size n, the number of subsets of size k $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

Proof.

Lattice paths arise in theoretical physics.



How many lattice paths are there from $(0, 0)$ to $(6, 4)$ if we are restricted to North steps and East steps only?

Lecture 3: Combinations and the Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Grimaldi Section 1.3

The quantity $\binom{n}{k}$ is the number of ways of choosing a set of size k from a set of size n. We also saw that it is the number of binary strings of length n with k 1's so

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Theorem ($\binom{n}{k} = \binom{n}{n-k}$)

Proof.

Theorem ($\displaystyle\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$)

Expanding $(x + y)^n$

Theorem (The Binomial Theorem)

If n is a positive integer then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots + \binom{n}{n} x^n y^0.$$

Because of this theorem the numbers $\binom{n}{k}$ are called binomial coefficients
We now have three equivalent ways to think of $\binom{n}{k}$:

Using the Binomial Theorem

Exericse. Find the coefficient of x^5y^{95} in $(3x - y)^{100}$.

Theorem (The Multinomial Theorem)

If $x_1, x_2, \ldots, x_m$ are variables and n a positive integer, then,

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$$

Proof:

Example. What is the coefficient of xy^2z^2 in $(w+x+y+z)^5$?

How many combinations of size 3 are there from $S = \{a, b, c\}$ if repetitions are allowed?

Theorem (combinations with repetitions)

Let S be a set with n elements. The number of ways to select k objects from S, with repetition allowed, is

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}.$$

Proof with binary strings.

Example. How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 10$ with $x_i \geq 0$?

Example. How many integer solutions are there to
$x_1 + x_2 \leq 7$ with $x_1 \geq 0$ and $x_2 \geq 0$?

Example. How many ways are their to distribute 5 apples, 4 oranges and 3 pears to three people?

Example. Consider the following code segments.
What is the value of counter after the loops have executed ?

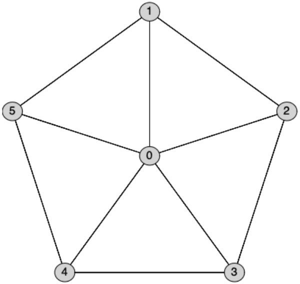
```
counter = 0;
for( i=1; i<=20; i++ )
   for( j=1; j<=20; j++ )
      for( k=1; k<=20; k++ )
         counter = counter + 1;
```

```
counter = 0;
for( i=1; i<=20; i++ )
   for( j=i; j<=20; j++ )
      for( k=j; k<=20; k++ )
         counter = counter + 1;
```

Example. A box contains 10 red balls, 10 green balls and 10 blue balls. Each set of balls is numbered 1 to 10. Suppose 7 balls are drawn at random from the box. In how many ways can there be 3 of one colour, 2 of a second colour and 2 of the 3rd colour.

Exercise. How many paths of length 2 edges are there in K_6?

Grimaldi 11.1, 11.3



The Wheel graph W_5.

Problem: How many cycles does W_5 have?

Draw the graph $G = (V, E)$ where $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}\}$.

Definition (Bipartite graph)

A graph $G = (V, E)$ is **bipartite** if we can partition the vertices in V into two non-empty sets V_1 and V_2 such that

(1) $V_1 \cap V_2 = \emptyset$

(2) $V_1 \cup V_2 = V$

(3) every edge in E is incident with one vertex in V_1 and one vertex in V_2.

Definition ($K_{m,n}$)

For integers $n \geq 1$ and $m \geq 1$ we define the **complete bipartite graph** $K_{m,n}$ to be the bipartite graph with $|V_1| = n$, $|V_2| = m$ and

$$E = \{\{v_1, v_2\} \mid v_1 \in V_1 \text{ and } v_2 \in V_2\}.$$

Example $K_{2,3}$

Question 1: How many edges are in a path on n vertices?
Question 2: How many edges are in a cycle on n vertices?

Question 3: How many edges are in K_n ?
Question 4: How many edges are in $K_{m,n}$?

Question 5: How many graphs are there with n vertices?
Question 6: How many graphs have n vertices and m edges?

Let V_1, V_2 be disjoint sets with $|V_1| = n_1$ and $|V_2| = n_2$.
Question 7: How many graphs have bipartition (V_1, V_2)?
Question 8: How many graphs have bipartition (V_1, V_2) with m edges?

Definition (Subgraph)

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs.
G' is a **subgraph** of G if $V' \subseteq V$ and $E' \subseteq E$.
If $V' = V$ then we call G' a **spanning** subgraph of G.

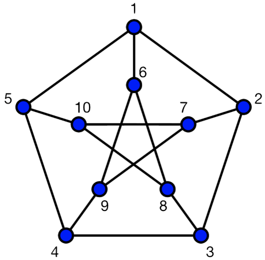
Example.

Question 9: How many spanning subgraphs does K_{n_1,n_2} have?
Question 10: How many spanning subgraphs of K_{n_1,n_2} have exactly m edges?

Definition (Paths and Cycles)

If P is a subgraph of G that is a path we call P **a path of** G.
If C is a subgraph of G that is a cycle we call C **a cycle of** G.

Example.



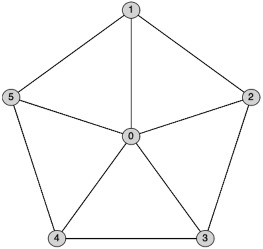
Question 11: How many 4-vertex paths does the graph K_n have?

Definition (induced subgraph)

Let $G = (V, E)$ be a graph and let $V' \subseteq V$. The subgraph of G **induced** by V' is the graph $G' = (V', E')$ where

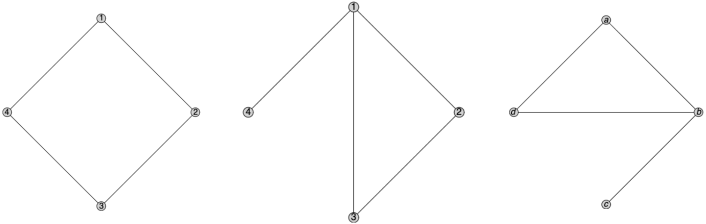
$$E' = \{\{x, y\} \mid x \in V', y \in V' \text{and } \{x, y\} \in E\}.$$

For the graph below determine the induced subgraph for the vertex sets $\{1, 3, 4\}$ and $\{1, 0, 3, 4\}$.



Grimaldi 11.2

Which of the following graphs are the “same” ?



Definition (isomorphic graphs)

Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be two graphs. Then G is **isomorphic** to H (has the same structure as) if there is a bijection $f : V_1 \to V_2$ such that

$$\{u, v\} \in E_1 \Longleftrightarrow \{f(u), f(v)\} \in E_2.$$

The function f is called an **isomorphism**.

Example.

Example. Draw all non-isomorphic graphs with $|V| = 3$ and $|V| = 4$.

Exercise. Draw all non-isomorphic graphs with 5 vertices and 4 edges.

How can we test if two graphs G and H are isomorphic?

An "efficient" graph isomorphism algorithm is not known.

Example. For $n \geq t$, how many subgraphs of K_n are isomorphic to K_t ?

Example. Let K_4^- be K_4 less one edge.
How many subgraphs of K_n are isomorphic to K_4^-?

Example. How many subgraphs of $K_{n,m}$ are isomorphic to $K_{3,4}$?

 1cm
Grimaldi 3.4 and 3.5

Suppose we pick a binary string x of length 6 at **random**.
What is the **probability** that x has two 1's in it?

Example 1. Suppose we pick a binary string x of length 6 at **random**. What is the **probability** that x has two 1's in it?

Definition (Probability of an Event)

Hypothesis:
S is a **set** of possible outcomes called the **sample space**, all having equal likelihood. Each subset $A \subseteq S$ is called an **event**, i.e. a set of considered outcomes. Each element of S determines an **outcome**.

Experiment:
We generate an event by "drawing" at random an outcome x from S.
Note: Other words used for "drawing" are "choosing", "selecting" and "picking".

Event: Let $Pr(A)$ denote the probability that $x \in A$.

Question: What is $Pr(A)$?
Answer: If each outcome is equally likely and $|S|$ is finite then

$$Pr(A) = \frac{|A|}{|S|}$$

Fundamental Principle. Calculating $Pr(A)$ requires defining the two sets S and A. If all outcomes are equally likely, we just need to calculate $|S|$ and $|A|$.

Example 2. What is the probability that a random binary string of size $n \geq 2$ starts with 11 ?

Example 3. What is the probability that the sum of two rolls of a dice is 7 ?

Definition (Axioms of Probability)

Let S be a sample space and let A and B be subsets of S.

1. $0 \leq Pr(A) \leq 1$
2. $Pr(S) = 1$
3. If $A \cap B = \phi$ then $Pr(A \cup B) = Pr(A) + Pr(B)$.

Note: These axioms hold whether the outcomes of S have equal likelihood or not.

Theorem (the rule of complement)

Let $\bar{A} = S - A$ *be the* **complement** *of* A*. Then* $Pr(\bar{A}) = 1 - Pr(A)$*.*

Proof:

Example 3 (illustrating the third axiom) What is the probability that a random binary string of size $n \geq 3$ has exactly two 1's or exactly three 1's ?

Example 4. Let $S = \{1, 2, 3, \ldots, 12\}$. If x is chosen from S at random what is the probability that x is divisible by 2 OR 3 ?

Theorem (the additive rule)

Let S be a sample space and $A, B \subseteq S$ be two events from S. Then

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B).$$

Proof:

Exercise. Let $S = \{1, 2, 3, \ldots, 60\}$. If x is chosen from S at random what is the probability that x is divisible by 2 **or** divisible by 3 **or** divisible **or** 5 ?
Generalize the additive rule to three subsets A, B, C.

Lecture 8: Conditional Probability and Independence

Grimaldi 3.6

Example 1. Let S be the set of binary sequences of length 8.
Let $A \subset S$ be the sequences starting with 111.
Let B be the sequences in S with five 1's.
Suppose we pick x from A at random. What is $Pr(B)$?

Definition (Conditional Probability)

Let S be a sample space and A and B two subsets of S. The **conditional probability** of B given/knowing A, denoted by

$$Pr(B|A)$$

is the probability that a random outcome from A also belongs to B. It can be obtained by the formula

$$Pr(B|A) = \frac{Pr(B \cap A)}{Pr(A)}.$$

Example 2. Assume two dice are rolled. What is the probability that, if they sum up to at least 9 (event A) that both dice have the same value (event B).

Four consequences of $Pr(B|A) = Pr(B \cap A)/Pr(A)$.

1. Switching A and B.

2. Multiplicative rule.

3. Law of total probability.

4. Bayes' Theorem.

Example 3. Suppose 10% of olympic cyclists use steroid Z and the IOC develops a test for Z with the following properties.

1. If a cyclist is taking Z the probability they test positive is 0.99.
2. If they are not taking Z the probability they test positive is 0.05.

Question: If a randomly chosen cyclist tests positive for Z, what is the probability they are taking steroid Z.

Definition (Independent Events)

Two events A and B are **independent** if either one of them has probability 0 or both have positive probabily and

$$Pr(B|A) = Pr(B) \text{ and } Pr(A|B) = Pr(A).$$

For example, if we toss a coin twice, the first toss is independent of the second.

Theorem

Two events A and B are independent if and only if

$$Pr(A \cap B) = Pr(A)Pr(B).$$

Proof:

Example 4. Suppose Alex tosses a fair coin 3 times. Here the sample space $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$. Consider the events

A: The first toss is a H: $A = \{HHH, HHT, HTH, HTT\}$.
B: The second toss is a H: $B = \{HHH, HHT, THH, THT\}$.
C: There are 2 or 3 heads: $C = \{HHH, HHT, HTH, THH\}$.

Are A and B independent?

Are A and C independent?

Are A and $\bar{B}$ independent?

Lecture 9: Discrete Random Variables

Grimaldi 3.7 (we will not cover variance)

Definition (Random Variable)

Let S be a sample space. A **random variable** X on S is a function $X : S \to \mathbb{R}$ that associates a numerical value to each possible outcome.
The **range** $r(X)$ of X is the set of all values it can take.

Example 1. If S is the set of all binary sequences of size $n = 4$.
The function that counts the number of 1's is a random variable.

Example 2. If S is the set of all rolls of two dice.
The function that adds the values of the dice is a random variable.

Example 3. Suppose we throw m balls into n bins randomly.
Let X be the number of empty bins.

Definition

Let S be a sample space and X a random variable on S. Let x be a value from the range of X. The probability of x, denoted by

$$Pr(X = x)$$

is the sum of the probabilities of all outcomes s of S such that $X(s) = x$.

Example 1 (cont.)
Let $X(s)$ be the number of 1 bits in a binary string with $n = 4$ bits.
Here $r(X) = \{0, 1, 2, 3, 4\}$

Pr(X = 0) =

Pr(X = 1) =

Pr(X = 2) =

Pr(X = 3) =

Pr(X = 4) =

Pr(X = k) =

Definition

The **expected value** of a random variable X on a sample space S is defined by

$$E(X) = \sum_{x \in r(X)} x Pr(X = x) = \sum_{s \in S} X(s) Pr(s).$$

Example 1 (cont.)

x	0	1	2	3	4
$Pr(X = x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$$E[X] = \sum_{x \in r(X)} x Pr(X = x) =$$

The Geometric Distribution

Reference Example 9.18 on page 428 of Grimaldi

Example 4. On average, how many times must we roll a fair die before we get a 6?

Example 4 cont.

Example 4 cont.

Summary: We say T is geometrically distributed with parameter p and $\Pr(T = k) = p(1-p)^{k-1}$ for $k \geq 1$ and $E(T) = 1/p$.

The Binomial Distribution

Example 5. Suppose we toss a biased coin n times.
Let the probability of getting heads be $p = 0.7$ and tails be $q = 0.3$.
Let H be the number of heads. What is $\Pr(H = k)$ and $E(H)$?

Summary: We say X is binomially distributed with parameters p and n and $\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $0 \leq k \leq n$ and $E(X) = np$.

Lecture 10: Applications of Discrete Random Variables

The bins and balls problem

Suppose we throw m balls into n bins randomly.
On average, how many bins will be empty?
On average, how many bins will have one ball in them?

The coupon collectors problem

Suppose we have a bin containing n types of coupons and we draw coupons one at a time from the bin at random. Assume the probability of drawing each type of coupon is $1/n$ and the bin has a very large number of coupons. On average, how many draws do we need to make until we get all n coupons?

Definition

Let S be a sample space and X a random variable on S. Let x be a value from the range of X. The probability of x, denoted by $Pr(X = x)$ is the sum of the probabilities of all outcomes s of S such that $X(s) = x$.

Example 1 Let S be the set of all binary sequences of size $n = 3$ bits.
Let $X(s)$ be the number of 1 bits in a binary string $s \in S$.
Here the range of X denoted $r(X)$ is $\{0, 1, 2, 3\}$.

$\Pr(X = 0) =$

$\Pr(X = 1) =$

$\Pr(X = 2) =$

$\Pr(X = 3) =$

$\Pr(X = k) =$

Definition

The **expected value** of a random variable X on a sample space S is defined by

$$E(X) = \sum_{x \in r(X)} x Pr(X = x) = \sum_{s \in S} X(s) Pr(s).$$

Example 1 (cont.)

x	0	1	2	3
$Pr(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$E[X] = \sum_{x \in r(X)} x Pr(X = x) =$$

Theorem (Linearity of Expection)

Let X and Y be two random variables on the same sample space S and $a \in \mathbb{R}$. Then

(1) $E(aX) = aE(X)$ *and*

(2) $E(X + Y) = E(X) + E(Y)$.

Proof

The bins and balls problem

Suppose we throw m balls into n bins randomly.
Question 1: What is the probability that bin i has k balls?

Question 2: On average, how many bins are empty?

Exercise: On average, how many bins have one ball?

The coupon collectors problem

Suppose a large bin contains many copies of $n = 10$ coupons. Assuming there are an equal number of each coupon, if we draw coupons at random from the bin, on average, how many draws will it take to get all n coupons?

The coupon collectors problem continued.

Exercise: On average, how many times must you toss a fair coin before you get a head and a tail?

Lecture 11 Recurrence Relations

Grimaldi Chapter 10 Recurrence Relations

The Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$ is generated by the **recurrence**

$$f_{n+1} = f_n + f_{n-1} \text{ for } n \geq 2$$

and **initial values**

$$f_1 = 1, \; f_2 = 1.$$

Example 2. Let b_n be the number of binary strings of length n bits.

A new way: To construct a binary string of length n first construct one of length $n - 1$ bits.

Example 3. Let k_n be the number of edges in K_n the complete graph on n vertices.

Definition

A **linear** recurrence relation (RR) of **order** k with **constant coefficients** for a sequence $a_1, a_2, a_3, \ldots$ is an equation of the form

$$c_0 a_n + c_1 a_{n-1} + \cdots + c_k a_{n-k} = f(n) \text{ for } n \geq k$$

where $c_0, c_1, \ldots, c_k$ are constants and $c_0 \neq 0, c_k \neq 0$. If $f(n) = 0$ the RR is said to be **homegeneous**, otherwise it is **non-homogeneous**.

Examples

Note: We can **shift** a RR up or down without changing the solutions. For example

$$\begin{aligned} a_{n+1} &= 2a_n + n \\ a_n &= 2a_{n-1} + n - 1 \\ a_{n+2} &= 2a_{n+1} + n + 1 \end{aligned}$$

Initial Values:

Example 4. Let S_n be the set of all binary strings of length n with the property that every 1 is followed by a 0 (so 1 cannot be the last bit).

(1) List S_1, S_2, S_3.
(2) Let $c_n = |S_n|$. Give a recurrence relation for c_n.

Example 5. Let D_n be the set of all strings of length n over the alphabet $\Sigma = \{A, B, C, D\}$ such that every A is followed by a C and every B is followed by DD.

(1) List D_0, D_1, D_2.
(2) Let $d_n = |D_n|$. Give a recurrence relation for d_n.

Additional space

Lecture 12 Solving First Order Recurrence Relations

Grimaldi Chapter 10.1

$p_n = n\, p_{n-1}$

$c_n = (n-1) + c_{n-1}$

$h_n = 1 + 2h_{n-1}$

```
 1: void Bubblesort( double A[], int n ) {
 2: // sort the array A of size n into ascending order
 3:    int i; double t;
 4:    if( n==1 ) return;
 5:    for( i=1; i<=n-1; i++ )
 6:       if( A[i-1] > A[i] ) {
 7:           t = A[i-1]; A[i-1] = A[i]; A[i] = t;
 8:       }
 9:    Bubblesort(A,n-1);
10:    return;
11: }
```

What should we count to determine the cost of the Bubblesort algorithm?
We will count the number of comparisons between elements of A in line 6.
Let c_n be the number of comparisons.

A First Look at Solving Recurrence Relations.

How can we solve RRs like

(1) $b_n = 2b_{n-1}$ for $n \geq 2$ and $b_1 = 2$.

(1) $c_n = c_{n-1} + (n-1)$ for $n \geq 2$ and $c_1 = 0$.

Example $b_n = 2b_{n-1}$

Example $c_n = c_{n-1} + (n-1)$

Theorem (A different way to solve first order RRs)

Every sequence $x_0, x_1, x_2, \ldots$ satisfying the recurrence $x_n = dx_{n-1}$ has the general solution $x_n = cd^n$ for some constant c. (The sequence is a geometric progression.)

Proof (substitution)

This suggests the following general strategy for solving RRs:

(1) Find the **general solution** to the RR. This will have one or more constants. Note: a RR of order k will has k constants.

(2) Use the k initial values to determine the constants. This gives a **unique** solution.

Example. Suppose $x_n = 5x_{n-1}$ and $x_0 = 7$. First find the general solution then the unique solution satisfying $x_0 = 7$.

Exercise 1. Solve $p_n = np_{n-1}$ where $p_1 = 1$.

Exercise 2. Solve $x_n = x_{n-1} + An + B$ for $x_1 = C$.

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$$f_{n+1} = f_n + f_{n-1}$$

$$a_n = a_{n-1} + 2a_{n-2}$$

For constants a, b, c consider a recurrence relation of the form

$$a\,x_n + b\,x_{n-1} + c\,x_{n-2} = 0 \quad \text{for } n \geq 2. \tag{1}$$

Suppose that $x_n = r^n$ is a solution to equation (1). In this case we have

$$ar^n + br^{n-1} + cr^{n-2} = 0 \quad \text{for all } n \geq 2. \tag{2}$$

Observe that the $n \geq 2$ condition is redundant in equation (2). If this holds for $n = 2$, then it holds for all larger values (multiplying by powers of r gives the other equations). This reduces us to a familiar equation

$$ar^2 + br + c = 0$$

Conclusion: A number r satisfies $ar^2 + br + c = 0$ if and only if $x_n = r^n$ is a solution to our recurrence.

Definition

The homogeneous second order linear recurrence relation

$$ax_n + bx_{n-1} + cx_{n-2} = 0$$

has **characteristic equation**

$$ar^2 + br + c = 0.$$

The roots of $ar^2 + br + c$ are precisely those numbers r for which $x_n = r^n$ satisfies the above recurrence.

Exercise. Find all real numbers r so that $x_n = r^n$ is a solution to the recurrence

$$x_n - 5x_{n-1} + 6x_{n-2} = 0$$

Theorem (Linearity)

Both of the properties below hold for the recurrence relation

$$ax_n + bx_{n-1} + cx_{n-2} = 0 \tag{3}$$

(A) *If $x_n = r^n$ is a solution of (3) then Cr^n is a solution to (3) for any constant C.*

(B) *If $x_n = s^n$ and $x_n = t^n$ are solutions of (3) then $s^n + t^n$ is a solution.*

It follows from (A) and (B) that $Cs^n + Dt^n$ is a solution for any constants C, D.

Proof:

Exercise. The recurrence relation

$$x_n - 5x_{n-1} + 6x_{n-2} = 0$$

has the solutions $x_n = 3^n$ and $x_n = 2^n$. Check that $C2^n + D3^n$ is a solution.

How do we determine what C and D are? With two consecutive initial values.
Find the solution with the initial values $x_0 = 6$ and $x_1 = 13$.

General solutions

Theorem

Let a, b, c be fixed constants with $a \neq 0$ and consider the recurrence

$$ax_n + bx_{n-1} + cx_{n-2} = 0. \tag{4}$$

If the characteristic equation,

$$ar^2 + br + c = 0$$

has two **distinct** *real roots, say r_1 and r_2, then every sequence satisfying this recurrence has the form*

$$x_n = Cr_1^n + Dr_2^n \tag{5}$$

where C and D are fixed constants. Accordingly, we will call equation (5) the ***general solution*** *to the recurrence.*

Example. The Fibonacci sequence $(f_1, f_2, f_3, \ldots)$ is generated by the recurrence

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for } n \geq 2$$

together with the initial values $f_0 = 0$ and $f_1 = 1$.

(1) Find the general solution to the above recurrence.

(2) Find a closed form (a formula in n) for the Fibonacci sequence.

Solving $ar^2 + br + c = 0$ using the quadratic formula we get

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac = 0$ then we have two repeated real roots.
If $b^2 - 4ac < 0$ we have two complex roots.

Theorem (Repeated real roots case)

Let a, b, c be real constants with $a \neq 0, c \neq 0$ and consider the recurrence

$$a\, x_n + b\, x_{n-1} + c\, x_{n-2} = 0.$$

If the characteristic polynomial $ar^2 + br + c$ has a repeated root r then every sequence satisfying this recurrence has the form

$$x_n = Cr^n + D\, nr^n \tag{6}$$

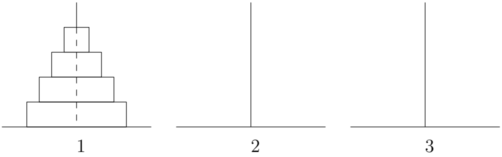
where C and D are constants. Equation (6) is the **general solution** *to the recurrence.*

Example. Solve the following recurrence

$$x_n - 6x_{n-1} + 9x_{n-2} = 0 \quad \text{with} \quad x_0 = 2,\ x_1 = 3.$$

Lecture 14 Solving first order non-homogeneous RRs

Grimaldi 10.3



Can you move the disks on pole 1 to pole 3 using pole 2 as needed?
Rule 1: move one disk at a time.
Rule 2: do not put a disk on top of a smaller disk.
Question: how many moves is necessary?

Solving first order non-homogeneous recurrences

Consider the non-homogenous recurrence relations

(1) $a_n + c_1 a_{n-1} = f(n)$ where $c_1 \neq 0$ and $f(n) \neq 0$

(2) $x_n + c_1 x_{n-1} + c_2 x_{n-2} = f(n)$ where $c_2 \neq 0$ and $f(n) \neq 0$

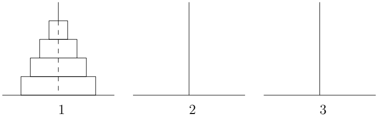
How do we solve them?

Case (1) $a_n + c_1 a_{n-1} = f(n)$ where $c_1 = -1$.

Example 1. Solve $a_n - a_{n-1} = 3n^2$ with $a_0 = 7$.

Example 2. Solve $a_n - 3a_{n-1} = 5 \cdot 3^n$ with $a_0 = 2$.

Example 3 – The Towers of Hanoi.



Move the disks from pole 1 to pole 3 using pole 2 as needed.
Move one disk at a time. Do not put a bigger disk on top of a smaller one.

(1)

(2)

(3)

Let m_n be the **number of moves**.
Determine and solve a recurrence relation for m_n.

Example 3 – The Towers of Hanoi (cont.)

Example 4 – Interest on a loan.

Pauline takes out a bank loan for \$$S$ dollars. She pays bak \$$P$ every month and the bank charges her r% interest per month. Let a_n be the amount she owes after n months. Determine, and solve, a recurrence relation for a_n.

Example 4 – Interest on a loan (cont.)

Lecture 15: Solving Non-Homogeneous Relations

Grimaldi 10.3

Definition

If $f(n) \neq 0$, a recurrence relation of the form

(1) $ax_n + bx_{n-1} = f(n) \qquad a \neq 0, b \neq 0$

(2) $ax_n + bx_{n-1} + cx_{n-2} = f(n) \qquad a \neq 0, c \neq 0$

is called a **non-homogeneneous** recurrence relation.
The **associated homogeneous relation** is obtained by setting f to be 0

(1) $ax_n + bx_{n-1} = 0$

(2) $ax_n + bx_{n-1} + cx_{n-2} = 0$

Definition

- A **particular solution** is a single sequence $x_n^{(p)}$ satisfying a recurrence without consideration of the initial condition.
- The **general solution** to a recurrence is the set of all sequences x_n satisfying it (without consideration of the initial condition)

Theorem

The general solution to a non-homogeneous recurrence is given by ***one particular solution****, $x_n^{(p)}$, plus the* ***general solution*** *to the associated homogeneous equation, $x_n^{(h)}$. That is, the solution has the form*

$$x_n = x_n^{(p)} + x_n^{(h)}.$$

Example 1 $x_n = 6x_{n-1} + 3^n$ for $n > 1$ and $x_0 = 7$.

Example 2 $x_n - 4x_{n-1} + 3x_{n-2} = \frac{2^n}{4}$ and $x_0 = 5, x_0 = 6.$

To find a particular solution to a non-homogeneous recurrence of the form

$$a x_n + b x_{n-1} = f(n) \qquad n \geq 1$$

$$a x_n + b x_{n-1} + c x_{n-2} = f(n) \qquad n \geq 2$$

(1) Exponential functions $f(n) = kr^n$

  (a) If r is not a root of the char. poly. of the homog. recurrence then look for a particular solution of the form $x_n^{(p)} = Cr^n$.

  (b) If r is a root of multiplicity m then look for a particular solution of the form $x_n^{(p)} = Cn^m r^n$

(2) Power functions $f(n) = kn^d$

  (a) Look for a solution of the form $x_n^{(p)} = a_d n^d + a_{d-1} n^{d-1} \ldots + a_1 n + a_0$

  (b) If n^t, for some $t \leq d$, is a solution to the homogeneous equation then multiply the trial solution $x_n^{(p)}$ by the smallest power of n, say n^s, for which no summand of $n^s f(n)$ is a solution of the homog. relation.

See Grimaldi page 479-481 for examples on how to determine the form of $x_n^{(p)}$.

Example 3. Find a particular solution to $x_n - 3x_{n-1} + 2x_{n-2} = 4n$.

Example 3 (continued).

Extra space.

Lecture 16: Divide and Conquer Algorithms and Recurrences

Please use the notes on Canvas not 10.6 Grimaldi.

What is the fastest algorithm for sorting an array of n numbers ?
What is the fastest algorithm to multiply two polynomials of degree n ?

Sorting Algorithms

Suppose we want to sort an A of n integers e.g.

$$A = \begin{array}{|c|c|c|c|c|c|c|} \hline 9 & 3 & 11 & 2 & 6 & 13 & 5 \\ \hline \end{array}$$

To compare sorting algorithms, by tradition, we count the number of comparisons they do. Bubblesort does exactly $n(n-1)/2$ comparisons. Mergesort does at most $n \log_2 n - n + 1$ comparisons. Below is a table for various values of n comparing the number of comparisons of these two algorithms.

	n	4	16	64	1024	10^6
Bubblesort	$n(n-1)/2$	6	120	2016	523776	approx 5×10^{11}
Mergesort	$n \log_2 n - n + 1$	5	49	321	9217	approx 20×10^6

For $n = 10^6$ Mergesort does a factor of over $25,000$ fewer comparisons!
Demo Mergesort

```c
void Merge( int A[], int n1, int B[], int n2, int C[] ) {
// Merge the sorted arrays A of length n1 and B of length n2 into C
   int i,j,k;
   i = j = k = 0;
   while( i<n1 && j<n2 )
      if( A[i]<B[j] ) { C[k] = A[i]; i++; k++; }
      else { C[k] = B[j]; j++; k++; }
   while( i<n1 ) { C[k] = A[i]; i++; k++; }
   while( j<n2 ) { C[k] = B[j]; j++; k++; }
   return;
}
```

Figure: C code for merging two sorted arrays A and B into the array C

The Mergesort Algorithm

```
void Mergesort( int A[], int n, int C[] ) {
// sort A[0],A[1],...,A[n] into ascending order
// C is an array of length n for working storage
   int n1,n2,*B;
   if( n<=1 ) return;
   n1 = n/2;
   n2 = n-n1;
   B = A + n1;
   Mergesort(A,n1,C); // sort the first half of A
   Mergesort(B,n2,C); // sort the second half of A
   Merge(A,n1,B,n2,C); // merge A and B into C
   for( i=0; i<n; i++ ) A[i] = C[i]; // copy C into A
   return;
}
```

Solving $C(n) \leq 2C(n/2) + n - 1$ with $C(1) = 0$.

Divide and Conquer Algorithms

Suppose we are given a problem of size n.

S1: Divide the problem into $a \geq 2$ subproblems of approximately the same size, say size b. Algorithm Mergesort divided A into $a = 2$ subproblems of size $n_1 = n/2$ and $n_2 = n - n_1$.

S2: Solve the subproblems recursively using the same "divide-and-conquer" approach.

S3: Combine the results from the subproblems to obtain the final solution. Algorithm Mergesort merges two sorted arrays of size n_1 and n_2 into one sorted array of size n.

Example. Adding an array of numbers.

```
double Add( double A[], int n ) {
// Add A[0]+A[1]+...+A[n-1]
   double s1,s2,*B;  int n1,n2;
   if( n==1 ) return A[0];
   n1 = n/2; n2 = n-n1;
   s1 = Add(A,n1); // s1 = A[0]+A[1]+...+A[n1-1]
   B = A + n1; // B is a subarray of A starting at n1
   s2 = Add(B,n2); // s2 = A[n1]+A[n2+1]+...+A[n-1]
   return s1+s2;
}
```

Solving recurrences using Maple's rsolve command.

A second order recurrence

```
> re := a(n) = 5*a(n-1) - 6*a(n-2);
```

$$re := a(n) = 5\,a(n-1) - 6\,a(n-2)$$

```
> rsolve( {re,a(0)=1,a(1)=4}, a(n) );
```

$$2\,3^n - 2^n$$

The mergesort recurrence

```
> re := c(n) = 2*c(n/2) + n-1;
```

$$re := c\,(n) = 2\,c\,(n/2) + n - 1$$

```
> expand( rsolve( {re, c(1)=0}, c(n) ) );
```

$$-n + \frac{\ln(n)\,n}{\ln(2)} + 1$$

Lecture 17 Generating Functions

Grimaldi Chapter 9 Generating Functions

A new powerful way of counting.

The binomial coefficient $\binom{n}{k}$ counts different objects:

$$\begin{aligned}\binom{n}{k} &= \text{the number of subsets of } \{1,2,\ldots,n\} \text{ of size } k\\ &= \text{the number of binary strings of length } n \text{ with } k \text{ 1's}\\ &= \text{the coefficient of } x^k y^{n-k} \text{ in the expansion of } (x+y)^n\end{aligned}$$

Example $(1+x)^3 =$

Definition (coefficient)

If $P(x)$ is a polynomial we denote by $[x^k]P(x)$ the coefficient of x^k in $P(x)$.

Example 1 How many integer solutions $a_1 + a_2 + a_3 = 7$ have if $0 \leq a_i \leq 3$?

Example 2 Suppose we roll two dice. If we add the values of the dice, how many ways can we get 6?

Example 3 How many integer solutions does

$$a_1 + a_2 + a_3 = 9$$

have if $2 \le a_1 \le 4, 1 \le a_2 \le 5, 3 \le a_3 \le 7$?

$[x^9]P(x)$ where $P(x) =$

Exercise: What if a_1 is odd, a_2 is even and $a_3 \in \{0, 3, 6\}$?

$[x^9]P(x)$ where $P(x) =$

Example 4. How many integer solutitions does

$$a_1 + a_2 + a_3 = n \quad \text{have if } a_i \geq 0 \text{ ?}$$

Definition

The **generating function** for an infinite sequence $a_0, a_1, a_2, \ldots$ is the series

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_i x^i.$$

We are interested in the coefficients of $A(x)$ not the values of $A(x)$.
Example 5. What is the generating function for $1, 1, 1, \ldots$?

Example 6. What is the generating function for the sequence $1, 2, 3, 4, 5, \ldots$?

Lecture 18 Calculating with Generating Functions

Grimaldi 9.2

Definition (Generating Function)

Let $a_0, a_1, a_2, a_3, \ldots$ be a sequence of real numbers (or integers). The function

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

is called the **generating function** for the sequence.

Note: we are interested in the coefficients of $A(x)$ not the values of $A(x)$.
All polynomials may be viewed as generating functions.

Example 1

Example 2. How many ways can we make 30 cents from nickels, dimes and quarters?

Example 3. How many integer solutions does $x_1 + x_2 + x_3 = n$ have if $x_i \geq 0$?

Definition (Arithmetic for Generating Functions)

Let $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$
and $B(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots$ and c be a constant. Then

(1) Sum: $A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots = \displaystyle\sum_{n=0}^{\infty}(a_n + b_n)x^n.$

(2) Scalar product: $cA(x) = ca_0 + ca_1x + \cdots = \displaystyle\sum_{n=0}^{\infty}(ca_n)x^n.$

(3) Product:

$$A(x) \cdot B(x) = (a_0b_0) + (a_0b_1 + a_1b_0)x + \cdots = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n.$$

(4) Derivative: $A'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots = \displaystyle\sum_{n=1}^{\infty}(na_n)x^{n-1}.$

Example. Let $A(x) = 1 + x + x^2 + x^3 + \dots$ and $B(x) = 2 + 2x + 2x^2 + 2x^3 + \dots$.

$2A(x) + B(x) =$

$A(x) \cdot B(x) =$

$A'(x) =$

What about inverses? Let x be a real number.
The number 1 has the property $1 \cdot x = x$ for all x. [identity]
If x is non-zero it has an inverse $1/x$ so that $x \cdot \frac{1}{x} = 1$. [inverses]

Example 1 Let $A(x) = 1 + x + x^2 + \ldots$ and $B(x) = 1 - x$.
Verify that $A(x) \cdot B(x) = 1$ and conclude that $A(x) = 1/B(x) = 1/(1 - x)$.

Example 2. Find the inverse of $(1-x)^k$.

Let a_n be the number of integer solutions of $x_1 + x_2 + \cdots + x_k = n$ where $x_i \geq 0$.

Example 3 Let $C(x) = 1 + 2x + 4x^2 + 8x^3 + \ldots$ and
$N(x) = 1 + x^5 + x^{10} + x^{15} + \ldots$ (the GF for nickels).
Using $2xC(x)$ and $x^5N(x)$ find the inverse of $C(x)$ and $N(x)$.

Express $C(x)$ and $N(x)$ in terms of $A(x) = 1 + x + x^2 + x^3 + \cdots = 1/(1-x)$.

Grimaldi 9.2

$$A(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$

$$A'(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

We have already seen that the generating function

$$A(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

has a compact represenation as the rational function $\frac{1}{1-x}$. Generating functions which can be compactly represented as rational functions will be our main subject.

Definition

A generating function $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ is **rational** if it can be expressed as

$$A(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

(1) Given a sequence of numbers express it as a rational GF ?

(2) Given a rational GF, find the associated sequence (coefficient extraction)

Two useful generating functions

$$A(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

$$A'(x) = 1 + 2x + 3x^2 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}.$$

Using just these two GF's with basic arithmetic operations gives us the ability to describe many other GF's.

Example 1. Determine the sequence for the GF

$$\frac{x^3-2}{1-x}$$

Example 2. Determine the sequence for the GF

$$\frac{2x^2+5}{(1-x)^2}+7x$$

Definition (Substitution)

Let $A(x) = a_0 + a_1 x + a_2 x^2 + \dots$ be a GF and c be a constant. Define

$$A(cx^m) = a_0 + a_1(cx^m) + a_2(cx^m)^2 + a_3(cx^m)^3 \cdots = \sum_{n=0}^{\infty} a_n c^n x^{mn}.$$

Example 1. The GF for nickels is $N(x) = 1 + x^5 + x^{10} + \cdots = \sum_{n-0}^{\infty} x^{5n}$. Express $N(x)$ as a rational function.

Example 2. What is the GF for $1, -1, 1, -1, \ldots$?

Example 3. Express $C(x) = 1 - 2x + 4x^2 - 8x^3 + 16x^4 - \ldots$ as a rational function.

Example 4. Find a rational GF for the sequence $1, -2, 3, -4, 5, -6, \ldots$?

Example 5. Express $D(x) = -x + 2x^2 - 3x^3 + 4x^4 - \ldots$ as a rational function.

Finding Coefficients

Using substitution and our two basic GF's $A(x) = 1 + x + x^2 + \ldots$ and $A'(x)$ we can now determine the coefficients for any GF that has the form

$$\frac{p(x)}{ax+b} \quad \text{or} \quad \frac{p(x)}{(ax+b)^2}$$

Problem 1. Find the coefficient of x^k in the GF

$$C(x) = \frac{x^2}{2x+3}$$

Problem 2. Find the coefficient of x^k in the GF

$$D(x) = \frac{x^2}{(x+2)^2}$$

Lecture 20 Rational Generating Functions continued.

Grimaldi 9.2 Calculation Techniques

We have been working with the two basic GF's

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n$$

We already proved the more general generating function:

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$$

Combining this formula with **substitution** allows us to determine the coefficients of any rational function of the form

$$\frac{p(x)}{(ax+b)^k}.$$

On page 422 the textbook uses a natural generalization of binomial coefficients, called the extended binomial theorem to get these coefficients. We will use substitutions instead.

Example 1. For $A(x) = \dfrac{x}{(1-x)^2}$ find $[x^n](A(x))$.

Example 2. Find the coefficient of x^5 of $A(x) = \dfrac{1}{(1-2x)^7}$

Partial Fractions

Question: How can we deterine the coefficients of GFs of the form

$$\frac{p(x)}{ax^2+bx+c} \quad \text{and} \quad \frac{p(x)}{ax^3+bx^2+cx+d} \quad \text{etc?}$$

If $ax^2+bx+c = a(x-\alpha)(x-\beta)$ and $\alpha \neq \beta$ solve

$$\frac{1}{(x-\alpha)(x-\beta)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta}$$

for A, B. If $ax^3+bx^2+cx+d = a(x-\alpha)(x-\beta)(x-\gamma)$ and $\alpha \neq \beta \neq \gamma$ solve

$$\frac{1}{(x-\alpha)(x-\beta)(x-\gamma)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma}$$

for A, B, C. If $ax^3+bx^2+cx+d = a(x-\alpha)(x-\beta)^2$ and $\alpha \neq \beta$ solve

$$\frac{1}{(x-\alpha)(x-\beta)^2} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{(x-\beta)^2}$$

for A, B, C. Then use the formula for $1/(1-x)^k$ with substitutions.

Example 1. Find the coefficient of x^n of $C(x) = \dfrac{3x}{x^2 - 3x + 2}$.

Example 2. Find values for A, B, C so that the expression below is true.

$$D(x) = \frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

Series Division

To find the series for the quotient

$$C(x) = \frac{A(x)}{B(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots}{b_0 + b_1 x + b_2 x^2 + \cdots}$$

let $C(x) = c_0 + c_1 x + c_2 x^2 + \cdots$ and write $A(x) = B(x)C(x)$ so that

$$(a_0 + a_1 x + a_2 x^2 + \cdots) = (b_0 + b_1 x + b_2 x^2 + \cdots)(c_0 + c_1 x + c_2 x^2 + \cdots)$$

In this equation the a_i and b_i are known coefficients, the c_i are unknown. Equating coefficients in x^i for $i = 0, 1, 2, \cdots$ and solving for c_i we obtain

$$\begin{array}{ll}
[x^0] & a_0 = b_0 c_0 \implies c_0 = a_0/b_0 \implies b_0 \neq 0 \\
[x^1] & a_1 = b_0 c_1 + b_1 c_0 \implies c_1 = (a_1 - b_1 c_0)/b_0 \\
[x^2] & a_2 = b_0 c_2 + b_1 c_1 + b_2 c_0 \implies c_2 = (a_2 - b_1 c_1 - b_2 c_0)/b_0 \\
\cdots & \\
[x^n] & a_n = b_0 c_n + b_1 c_{n-1} + \cdots + b_n c_0 \implies c_n = (a_n - b_1 c_{n-1} - \cdots - b_n c_0)/b_0
\end{array}$$

Example 1. Let $A(x) = (1+x)/(1-x)^2 = (1+x)/(1-2x+x^2)$. Find the coefficients of $A(x)$ up to x^3 using series division.

Example 2. Find a recurrence for the coefficients of $x/(1-x-x^2)$ using series division.

Exercise. Find the series for $(1+x)/(1-3x+3x^2-x^3)$ to x^4 using series division and determine a recurrence for the nth coefficient.

Lecture 21: Solving Recurrences using Generating Functions

Grimaldi 10.4

Given a_0, the recurrence $a_n = 3a_{n-1} + 1$ defines a sequence

$$a_0, a_1, a_2, \dots, a_n, \dots$$

which in turn defines the generating function

$$A(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

If we find a rational form for $A(x)$, that is

$$A(x) = \frac{p(x)}{q(x)}$$

for polynomials $p(x)$ and $q(x)$, then we can use partial fractions to get a formula for $a_n = [x^n]A(x)$.

Example 1. Solve $a_n - 3a_{n-1} = n$ for $n \geq 1$ and $a_0 = 1$.
The recurrence relation represents an infinite set of equations.

Multiply equation (k) be x^k we get

Adding all equations up gives

Now plug in a_0 and isolate $A(x)$.

Now we do a PDF and get a formula for a_n.

Example 2. Consider the sequence defined by $a_0 = 0, a_1 = 1$ and

$$a_n - 5a_{n-1} + 6a_{n-2} = 0 \text{ for } n \geq 2.$$

Find a rational expression for $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

Example 2 (cont.)

Method.

Let $a_0, a_1, a_2, \ldots$ be a sequence satisfying a recurrence

$$c_n a_n + c_{n-1} a_{n-1} + \cdots + c_k a_{n-k} = f(n).$$

Let $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

(1) Multiply the recurrence by $x^k, x^{k+1}, \ldots$ and sum both sides to infinity.

(2) Rewrite the infinite sums on the LHS in terms of $A(x)$ and the sum on the RHS as a rational function.

(3) Isolate $A(x)$ and use partial fractions to calculate $a_n = [x^n]A(x)$.

Problem. Consider the sequence defined by

$$a_0 = 2, a_1 = 3 \text{ and } a_n - 4a_{n-1} + 4a_{n-2} = 2^n \text{ for } n \geq 2.$$

Solve the recurrence using a generating function.

Problem (cont.)

Lecture 22: The Summation Operator

Grimaldi 9.5

Problem: $\displaystyle\sum_{i=1}^{n} i^2 = ? \quad \sum_{i=0}^{n} i^3 = ? \quad \sum_{i=1}^{n} i^4 = ?$ etc.

Let $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$. What is

$$A(x)\frac{1}{1-x} =$$

Example 1. For $A(x) = x + x^2$ find $\dfrac{A(x)}{(1-x)}$, $\dfrac{A(x)}{(1-x)^2}$ and $\dfrac{A(x)}{(1-x)^3}$.

Example 2. We will find a formula for

$$\sum_{k=0}^{n} k^2 = 0^2 + 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

Example 2 (cont.)

Example 2 (cont.)

Exercise 1. Use the summation operator to show $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Proofs by induction.

Example 1. Show that $\sum_{k=1}^{n} 2k - 1 = n^2$ for $n \geq 1$ by induction on n.

Example 1 continued.

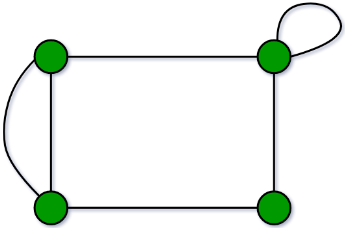
Exercise 2. Show that $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ for $n \geq 1$ by induction on n.

Example 2. Show that every integer $n \geq 2$ can be factored into a product of primes.

Example 2 continued.

Lecture 23 Graphs: Multigraphs

Grimaldi 11.1



Definition (multigraph)

A **multigraph** $G = (V, E)$ is a set V of vertices, and a multiset E of edges where each edge is in $V \times V$.

Example: $V = \{1, 2, 3, 4\}$, $E = \{(1, 2), (1, 2), (2, 3), (3, 3), (3, 4), (4, 1)\}$.

Simple graphs are multigraphs with no loops and no parallel edges.

Definition (walks in multigraphs)

Let x and y be two vertices in a multigraph $G = (V, E)$. A **walk** in G is a finite alternating sequence

$$x\, e_1\, x_1\, e_2\, x_3\, e_3 \ldots e_{n-1}\, x_{n-1}\, e_n\, y$$

of vertices $x_i \in V$ and edges $e_i \in E$ with $n \geq 0$ edges. The **length of the walk** is n, the number of edges. A walk from x to y is called a **closed walk** if $x = y$ and an **open walk** if $x \neq y$. Note, vertices and edges in walks need not be distinct.

Convention: Grimaldi allows walks to have length 0 which he calls **trivial walks**.

Examples

Definition (trails and circuits)

Let G be a multigraph and x and y be vertices in G.
A **trail** from x to y is an open walk in G that has no repeated edges.
A **circuit** from x to x is a closed walk in G that has no repeated edges.

Convention: Grimaldi says circuits must have at least 1 edge and cycles 3 edges.
We will allow both circuits and cycles to have 1 or more edges.

Examples

Theorem (trails and paths)

Let $G = (V, E)$ be a multigraph with vertices a and b. If there is a trail in G from a to b then there is a path in G from a to b.

Proof.

Definition (degree of a vertex in a multigraph)

If $G = (V, E)$ is a multigraph and $v \in V$, the **degree** of v, denoted $\deg(v)$, is the number of edges incident to v. Here a loop at v counts as two incident edges.

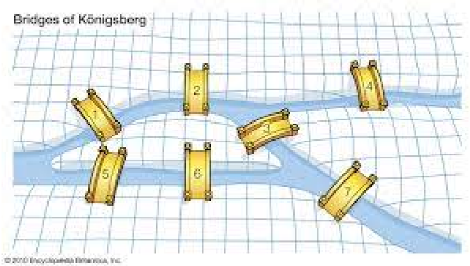
Example

Theorem

Every multigraph $G = (V, E)$ satisfies $\sum_{v \in V} \deg(v) = 2|E|$.

Poof.

The Bridges of Königsberg Problem



Question: Is it possible walk around the city, crossing each bridge exactly once, and end up where you started?

The definitions **subgraph**, **induced subgraph** and **spanning subgraph** that we made for simple graphs also work for multigraphs.

Definition (connected graph and connected components)

Let $G = (V, E)$ be a multigraph. We say G is **connected** if for all pairs $u, v \in V$ there is a path from u to v. The **connected components** of G are the maximal connected subgraphs of G.

Example

Definition (directed graphs)

A **directed graph** or **digraph** $G = (V, E)$ is a set V of vertices and a set E of edges where edges are ordered pairs of vertices. We draw arrows on edges to indicate direction. If a graph is not directed, we say it is an **undirected graph**.

Example. $V = \{1, 2, 3, 4, 5\}$, $E = \{(1, 2), (2, 3), (3, 1), (4, 1), (3, 5)\}$.

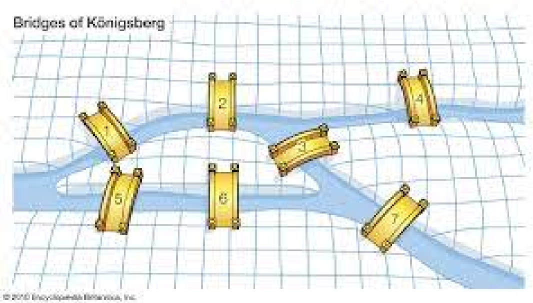
We will not study directed graphs in MACM 201.

Graph Terminology

- Graphs are classified as either directed graphs or undirected graphs.
- Simple graphs are graphs with no parallel edges and no loops.
- Adjacent vertices are also called neighbors.
- Graphs are also called networks. Usually a network refers to a real physical object whereas a graph could be abstract. Mathematicians and Computer Scientists usually use "graphs" whereas Engineers usually use "networks". Some terminology is different, for example

Graph Theory	Network Science
graph	network
vertex	node
edge	link

Grimaldi 11.3



Question: Is it possible walk around the city crossing each bridge exactly once, and end up where you started?

Definition (Eulerian circuit)

An **Euler ciruit** of a multi-graph $G = (V, E)$ is a circuit

$$W = v_1, e_1, v_2, e_2, \ldots, e_n, v_1$$

such that every edge in E appears once in W.

Examples.

Lemma

Let $G = (V, E)$ be a multigraph with $|E| \geq 1$.
If $\deg(v) \geq 2$ for all $v \in V$, then G contains a cycle of length ≥ 1.

Proof.

Proof (cont).

Theorem (Euler)

A connected multigraph $G = (V, E)$ which is not the singleton vertex, has an Euler circuit if and only if every vertex in V has even degree.

Proof.

Proof (cont.)

Proof (cont.)

The proof gives a recursive algorithm for finding an Euler circuit!

Definition

An **Euler trail** of a multi-graph $G = (V, E)$ is a trail

$$T = v_0, e_1, v_1, e_2, \ldots, e_n, v_n$$

such that every edge in E appears once in T.

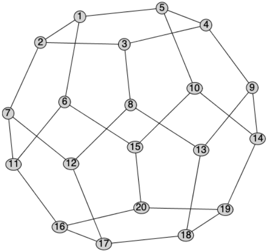
Example

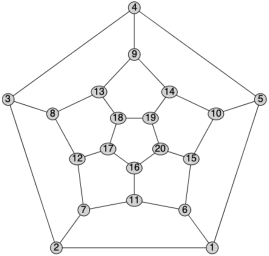
Corollary (of Euler's theorem)

A connected multigraph $G = (V, E)$ has an Euler trail if and only if there are exactly two vertices in G of odd degree.

Proof. Exercise.

Grimaldi 11.4





These are both drawings of the same graph. To see this locate the cycles $1-2-3-4-5-1$ and $16-17-18-19-20$ in both graphs.

Definition (planar graph)

A graph G is **planar** if G has a drawing (in the plane) so that the edges intersect only at the vertices of G. Such a drawing is called a **planar embedding** of G.

Examples

Observation: The graph $K_{3,3}$ is not planar.
Proof sketch (we will give a formal proof next day)

Observation: The graph K_5 is not planar.
Proof sketch (we will give a formal proof next day)

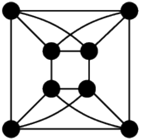
Definition (subdivision)

Let $G = (V, E)$ be a multigraph and let $e = \{u, v\}$ be an edge in E. To **subdivide** the edge e is to delete e and add a new vertex w and two new edges $e_1 = \{u, w\}$ and $e_2 = \{w, v\}$ to G. If the graph H is obtained from G by a sequence of subdivions, then H is called a subdivison of G.

Example

Observation. If H is a subdivision of G then H is planar if and only if G is planar. This means that every subdivision of $K_{3,3}$ and K_5 is nonplanar.

Example: Is this graph planar? I.e. can you find a planar embedding?



Exercise: find a subdivision of $K_{3,3}$ in the graph.

Question: Which graphs are planar ?

Definition

Let G and H be multigraphs. We say that G **contains a subdivision** of H if there is a subgraph of G isomorphic to some subdivision of H.

Theorem (Kuratowski-Wagner)

A multigraph G is planar if and only if G does not contain a subdivision of $K_{3,3}$ or a subdivision of K_5.

Notes.

Definition (Faces)

Let G be a planar graph embedded in the plane. The embedding partitions the plane into connected regions called **faces**. There is one unbounded region called the **infinite face**. All other faces are **internal faces**. If G is connected, every face has vertices and edges on its boundary. They form a closed walk called a **facial walk**

Example

Theorem (Euler's formula)

If $G = (V, E)$ is an connected multigraph embedded in the plane and F is the set of faces, then

$$|V| - |E| + |F| = 2.$$

This implies all embeddings of a planar graph have the same number of faces.

Example

Proof.

Proof (cont.)

Extra space.

Lecture 26: Planar Graphs continued

Grimaldi 11.4

Question: Can a given electronic circuit be layed out on an circuit board such that no wires cross each other?

Review:

A graph $G = (V, E)$ is **planar** if G has a drawing (in the plane) where the edges intersect only at the vertices of G. Such a drawing is called a **planar embedding** of G. The embedding partitions the plane into at set F of regions called **faces**. We proved Euler's formula $|V| - |E| + |F| = 2$.

Examples.

Definition (Face degrees)

Let $G = (V, E)$ be a connected multigraph embedded in the plane and let f be a face of this embedding. We define the **degree** of f, denoted $\deg(f)$, to be the number of edges in a facial walk of f.

Example

Theorem

If G has faces $f_1, f_2, \ldots, f_k$ then $\displaystyle\sum_{i=1}^{k} \deg(f_i) = 2|E|$.

Proof

What is the maximum number of edges a planar simple graph can have?

Theorem (Bound 1 for the number of edges)

If $G = (V, E)$ is a connected planar simple graph with $|V| \geq 3$ then

$$|E| \leq 3|V| - 6 \quad \text{and} \quad 2|E| \geq 3|F|.$$

Proof.

Corollary (to bound 1 for the number of edges)

The graph K_5 is not planar.

Proof.

Theorem (Bound 2 for the number of edges)

If $G = (V, E)$ is a connected planar simple graph with $|V| \geq 3$ and with no cycle of length 3 or less then

$$|E| \leq 2|V| - 4 \quad \text{and} \quad |E| \geq 2|F|.$$

Proof.

Corollary (to bound 2 for the number of edges)

The graph $K_{3,3}$ is not planar.

Proof.

Definition (Dual graphs)

Let $G = (V, E)$ be a connected multigraph embedded in the plane. The vertices of the **dual** multigraph G^* are the faces of G. If two faces f_i and f_j share an edge e then $e^* = \{f_i, f_j\}$ is an edge in G^*. This may be done so that e^* crosses e and G^* also ends up embedded in the plane.

Examples.

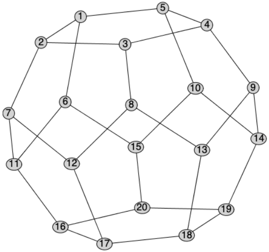
Features of duals

(1) Duals only exist for planar graphs
(2) If G^* is a dual of G then G is a dual of G^* !
(3) The degree of a vertex in G^* is the degree of the corresponding face of G.
(4) The dual of a simple graph may be a multigraph.

Examples

Grimaldi 11.5

In 1856 a mathematician William Hamilton invented a game in which the ojbect is to find a cycle along the edges of a dodecahedron.



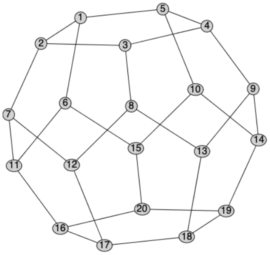
Problem: Can you find a cycle in the graph that includes all 20 vertices?

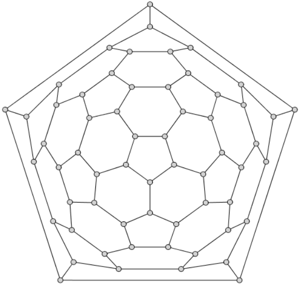
Definition

Let G be a graph. A path of G is a **Hamiltonian path** if it contains every vertex of G. A cycle of G is a **Hamiltonian cycle** if it contains every vertex of G.

Examples

Algorithm Exhaustive Search: try all possible paths.





Hamiltonian vs. Eulerian

The definition of Hamiltonian is very similar to Eulerian. In Hamiltonian each vertex appears exactly once. In Eulerian each edge appears exactly once. Although they look similar, having a Hamiltonian cycle and Having an Euler circuit is very different.

(1) There is a fast algorithm to test if a graph $G = (V, E)$ has an Euler circuit where the running time is a linear function of $|V| + |E|$, namely, test if G is connected and all vertices have even degree.

(2) No such fast test is known for a Hamiltonian cycle. The problem of deciding if a graph has a Hamiltonian path/cycle is **NP-complete**. So it is widely believed that there does not exist an algorithm which takes as input an arbitrary graph $G = (V, E)$ and determines if G has a Hamiltonian path/cycle where the running time is bounded by a polynomial function of $|V| + |E|$.

Definition (Necessary and sufficient conditions)

Let P be a property of graphs and C be a set of condtions.

(1) C is **necessary** for P if every graph satisfying P also satisfies C.

(2) C is **sufficient** for P if every graph satisfying C also satisfies P.

(3) If C is both **necessary and sufficient** for P, then a graph G satisfies P if and only if G satisfies C. We say C characterize P.

Examples

(1) It is necessary for a graph to be connected to have a H.P.

(2) Being a complete graph is a sufficient condition to have a H.P.

(3) $n > 1$ is odd is a necessary and sufficient condition for K_n to have an Euler circuit.

A sufficient condition for G to have an Hamiltonian path.

Theorem

Let $G = (V, E)$ *be a graph with* $|V| = n$*. If*

$$\deg(x) + \deg(y) \geq n - 1 \quad \textit{for all } x, y \in V \textit{ with } x \neq y$$

then G *has Hamiltonian path.*

Proof.

Proof (cont.)

Proof (cont.)

Corollary

If $G = (V, E)$ is a graph with $|V| = n$ and $\deg(v) \geq \frac{n-1}{2}$ holds for every $v \in V$, then G has a Hamiltonian path.

Proof.

Lecture 28: Trees

Grimaldi 12.1

Definition (tree, forest and leaf)

Let $G = (V, E)$ be a multigraph. G is a **tree** if G is connected and G does not contain a cycle. G is a **forest** if G does not contain a cycle. A vertex of degree 1 is called **leaf** or **pendant vertex**.

Examples

Since a tree cannot have loops or parallel edges, it is a simple graph.
We previously showed that every graph with all vertices of degree ≥ 2 must have a cycle. Therefore, every tree with ≥ 2 vertices must have a leaf. Later we will see that all trees with at least two vertices have at least two leaves.

Lemma

If $T = (V, E)$ is a tree with leaf v then $T - v$ is a tree.

Proof

This observation gives us a powerful tool for proving properties of trees. Try using induction on the number of vertices and, for the inductive step, deleting a leaf then applying the inductive hypothesis.

Theorem (unique paths)

If $T = (V, E)$ is a tree and $u, v \in V$ are distinct, there is a unique path in T with ends u, v.

Proof.

Theorem (main property of trees)

If $T = (V, E)$ is a tree then $|V| = |E| + 1$.
If $G = (V, E)$ is a forest with k trees then $|V| = |E| + k$.

Proof.

Proof (cont).

Lemma

If $G = (V, E)$ satisfies $|V| = |E| + 1$ then G must have a vertex of degree 0 or at least two of degree 1.

Proof.

Lemma

Every tree $T = (V, E)$ with $|V| \geq 2$ has at least two leaves.

Proof.

Exercise. Let T be a tree. Show that removing any edge from T disconnects T.

Lecture 29: Trees and Rooted Trees

Grimaldi 12.1, 12.2

Definition (tree)

A multigraph G is a **tree** if G is connected and G does not contain a cycle.

Theorem (main properties of trees)

If $T = (V, E)$ is a tree then $|V| = |E| + 1$ and secondly, there is a unique path in T between every pair of vertices.

Examples

Theorem (Characterization of Trees)

Let $G = (V, E)$ be a multigraph. The following statements are equivalent.

(1) *G is connected and has no cycle. (G is a tree)*

(2) *G is connected and $|V| = |E| + 1$.*

(3) *G has no cycle and $|V| = |E| + 1$.*

(4) *There is a unique path between every pair of vertices in G.*

Proof.

Proof (cont).

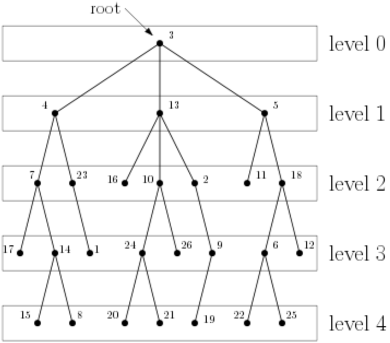
Definition (rooted tree)

A **rooted tree** $T = (V, E)$ is a tree with a distinguished vertex called the **root**. For every vertex $v \in V$ the **level** of v is the length of the path from v to the root. Note: the root is the unique vertex at level 0.

Example

Definition (rooted tree terminology)

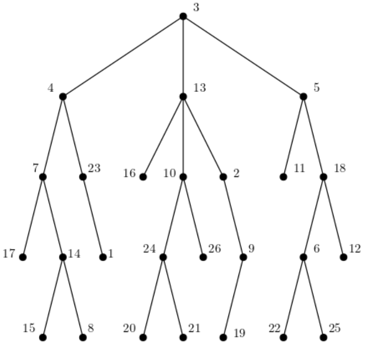
- The **height** of a rooted tree is the maximum level of a vertex. A rooted tree consisting of just a root vertex has height 0.
- Every non-root vertex v at level i is adjacent to exactly one vertex u at level $i-1$. We call u the **parent** of v and we say that v is a **child** of u.
- For every vertex v there is a walk "up the tree" to the root obtained by moving to the parent vertex at each step. If u is another vertex on this walk, we call u an **ancestor** of v and v a **descendant** of u.



We are frequently interested in working with rooted trees recursively. Therefore, it will be helpful to think of a rooted tree as composed out of smaller rooted trees.

Definition (subtree)

Let v be a vertex of a rooted tree T with level i. Define T' to be the subgraph of T induced by v together with its descendants. Then T' forms a new rooted tree with root vertex v. We say that T' with root v is the **subtree** of T at v.



Definition (isomorphism of rooted trees)

Let T_1, T_2 be rooted trees with $T_i = (V_i, E_i)$ for $i = 1, 2$. We say that T_1 and T_2 are **isomorphic** if there exists a bijection $f : V_1 \to V_2$ satisfying:

(1) $\{f(u), f(v)\} \in E_2 \Leftrightarrow \{u, v\} \in E_1$

(2) For every $v \in V_1$ the level of v and $f(v)$ is the same.
In particular, f sends the root of T_1 to the root of T_2.

Example.

Definition

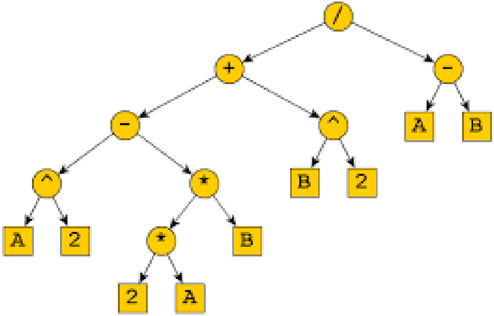
A rooted tree is m-**ary** if every internal node has at most m children. A 2-ary tree is called **binary** tree.

Exercise. Find all binary trees with height 0, 1, and 2 up to isomorphism.

Let b_n denote the number of binary trees of height at most n. Find b_0, b_1, b_2.

Use the recursive structure of rooted trees find a recurrence for b_n.

Grimaldi 12.2



What formula does this tree encode?

Ordered rooted trees

For some applications it is essential to have not just a rooted tree, but also an ordering of the children for each internal vertex.

Example

- Thesis
  - Ch1
    - S1.1
    - S1.2
  - Ch2
    - S2.1
    - S2.2
      - S2.2.1
    - S2.3
  - Ch3

How do we walk through and process a rooted tree?

Definition (preorder, postorder tree traversals)

A **preorder traversal** of a tree T first visits the root vertex then visits, in preorder, the vertices of the subtrees $T_1, T_2, \ldots, T_k$ of T.
A **postorder traversal** of a tree T visits, in postorder, the vertices of the subtrees $T_1, T_2, \ldots, T_k$ of T then visits the root.

Example

Exercise Draw the expression tree for $(3 \times 5) + ((7 - 4) \times 2)$ and give the postorder traversal.

Preorder is also called Polish notation and postorder is also called reverse Polish notation. HP calculators used postorder and a stack to evaluate expressions.

Definition (spanning tree)

Let G be a connected multigraph. A subgraph T of G is a **spanning tree** if T spans G (so T contains all vertices in G) and T is a tree.

Example

Question How many spanning trees does C_n have?

Theorem (existence of spanning trees)

Every connected multigraph $G = (V, E)$ has a spanning tree.
Here are three algorithms to construct a spanning tree in G:

(1) *Start from G. If there is a cycle C in G delete an edge from C. Repeat this until G has no cycles. Output G.*

(2) *Create the graph $H = (V, \phi)$. For each edge e in G add e to H if it does not make H have a cycle. Output H.*

(3) *The depth-first-search algorithm.*

Proof (1)

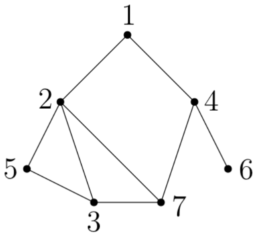
The Depth-First Search (DFS) algorithm

Input. A graph $G = (V, E)$.

Output. A set E_T of edges such that (V, E_T) is a spanning tree of G.

1. Let $v = 1$, $E_T = \phi$ and and mark vertex 1 as visited.
2. If all neighbors of v have been visited Then
   a) If $v = 1$ Then Return (V, E_T).
   b) Else (backtrack step) Let $v = parent(v)$ and Goto step 2.
3. Else
   a) Let i be the smallest neighbor of v that has not been visited.
   b) Mark i as visited.
   c) Add the edge $\{v, i\}$ to E_T and Let $parent(i) = v$.
   d) Let $v = i$ and Goto step 2.

Example



Extra space.

Lecture 31: Articulation Points and Biconnected Components

Grimaldi 12.5

An application of the depth-first search spanning tree.

Articulation Points

Definition (Articulation Point)

Let $G = (V, E)$ be a graph. A vertex v in G is an **articulation point** (AP) if removing v from G increases the number of connected components of G.

Lemma (12.3)

Let $G = (V, E)$ be a graph. A vertex $v \in V$ is an articulation point of G if and only if there are two vertices x and y in V such that $x \neq y \neq v$ and every path between x and y includes v.

Definition (Biconnected Component)

Let $G = (V, E)$ be a graph. A subgraph of G is **biconnected** if it is connected and has no articulation points. A maximal biconnected subgraph of G is called a **biconnected component** of G.

Lemma

Let $G = (V, E)$ be a graph. If G has a Hamiltonian cycle then G must have no APs, equivalently, G must be biconnected.

Exercise: Find a biconnected graph which does not have a Hamiltonian cycle.

How can we find the Articulation points in a **connected** graph G?

Algorithm 1.

> **set** AP $= \phi$.
> **for** each $v \in V$ do
>   **if** removing v from G disconnects G **then**
>   **set** AP $=$ AP $\cup\{v\}$.
>   **end if**
> **end for**
> **output** AP.

Algorithm 2

Step 1: Construct a DFS spanning tree T for G and number the edges in T in the order visited during the DFS.

Step 2: Traverse T in pre-order. If a vertex v has a backedge e_n from u to v, number all edges on the walk $v\, e_1\, x_1\, e_2 \ldots x_{n-2}\, e_{n-1}\, u\, e_n\, v$ from v down to u and back to v with the edge number on e_1.

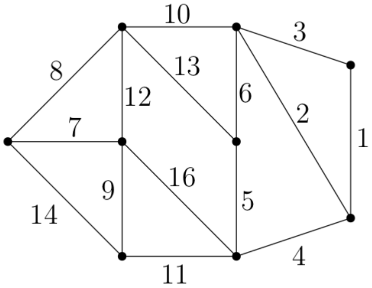
What are the articulation points?

What are the biconnected components?

Why is this algorithm better than Algorithm 1?
If implemented carefully, can be done in time proportional to $|V| + |E|$.

Lecture 32: Weighted Graphs and Minimum Spanning Trees

Grimaldi 13.2



Definition (Weighted Graph)

A **weighted graph** $G = (V, E)$ is a multigraph together with a function $w : E \to \mathbb{R}^+$ is called an **edge-weighting**.

Examples

Definition (Minimum Spanning Tree)

Let $G = (V, E)$ be a connected multigraph with edge-weighting w.
For any subgraph $H = (V', E')$ of G, the **weight** of H is

$$w(H) = \sum_{e \in E'} w(e).$$

A **minimum spanning tree** is a spanning tree of G of minimum weight.

Example.

Lemma (property of minimum spanning trees)

Let $G = (V, E)$ be a weighted connected graph. Let V_1 and V_2 be a partition of V. Amongst the edges in G with one vertex in V_1 and the other in V_2 let e one of minimum weight. There is a minimum spanning tree in G with e as one of it's edges.

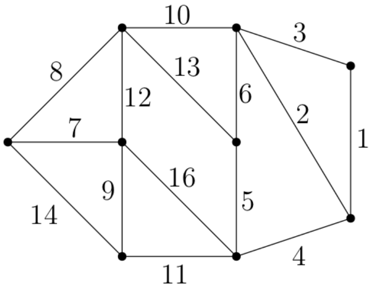
Proof.

Kruskal's algorithm to compute a minimum spanning tree

Input: a connected multigraph $G = (V, E)$ with an edge-weighting w.
Output: a mimimal spanning tree of G.

1. Set $E' = \phi$.
2. Sort the edges in E from least weight to highest weight.
3. While (V, E') is not connected do
   Let e be the next heaviest edge in E.
   If $(V, E' \cup \{e\})$ does not have a cycle set $E' = E' \cup \{e\}$.
4. Return the tree (V, E').

Example



Additional Space.

Additional Space.