Lecture 18 Calculating with Generating Functions
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# Lecture 18 Calculating with Generating Functions cont.



Grimaldi 9.2

Assignment #5 posted
due next Wednesday
Midterm #2 marks posted
Average is 60.2

Tutorial on recurrences this
Friday at 8pm for ~40 mins.
to prepare for final exam.

$$A(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$

$$A'(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$



We have already seen that the generating function

$$A(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

has a compact represenation as the rational function $\frac{1}{1-x}$. Generating functions which can be compactly represented as rational functions will be our main subject.

## Definition

A generating function $A(x)$ is called **rational** if it can be expressed as

$$A(x) = \frac{p(x)}{q(x)} = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

where $p(x)$ and $q(x)$ are polynomials. with $q(x) \neq 0$

Our main interests with rational GF's are:

(1) Given a sequence of numbers express it as a rational GF ?

(2) Given a rational GF, find the associated sequence (coefficient extraction)

## Two useful generating functions

$$A(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

$$A'(x) = 1 + 2x + 3x^2 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}.$$

Using just these two GF's with basic arithmetic operations gives us the ability to describe many other GF's.

Example 1. Determine the sequence for the GF

$$\frac{x^3-2}{1-x} = \frac{x^3 \cdot 1}{1-x} - \frac{2}{1-x} = x^3(1+x+x^2+\cdots) - 2(1+x+x^2+\cdots)$$

$$= x^3 + x^4 + \cdots \quad -2 - 2x - 2x^2 - 2x^3 - \cdots$$

$$= -2 - 2x - 2x^2 - 1x^3 - 1 \cdot x^4 - \cdots$$

So the sequence is $-2, -2, -2, -1, -1, -1, \cdots$



Example 2. Determine the sequence for the GF

$$\frac{2x^2+5}{(1-x)^2} + 7x = \frac{2x^2 \cdot 1}{(1-x)^2} + \frac{5}{(1-x)^2} + 7x$$

$$= 2x^2 \cdot (1 + 2x + 3x^2 + 4x^3 + \cdots) + 5(1 + 2x + 3x^2 + \cdots) + \underline{7x}$$

$$= 2(0 + 0x + \underline{x^2} + 2x^3 + 3x^4 + \cdots) + 5(1 + 2x + 3x^2 + \cdots) + \underline{7x}$$

$$= 5 + (10+7)x + (2+15)x^2 + \cdots + \underbrace{(2(n-1) + 5(n+1))}_{n \geq 2} \cdot x^n + \cdots$$

$$[x^n] = \begin{cases} 5 & \text{for } n=0 \\ 17 & \text{for } n=1 \\ 7n+3 & \text{for } n \geq 2. \end{cases}$$

## Definition ( Substitution )

Let $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ be a GF and $c$ be a constant. Define

$$A(cx^m) = a_0 + a_1(cx^m) + a_2(cx^m)^2 + a_3(cx^m)^3 \cdots = \sum_{n=0}^{\infty} a_n c^n x^{mn}.$$

1 nickel    2 nickels.

Example 1. The GF for nickels is $N(x) = \underline{1 + x^{5} + x^{10} + \cdots} = \sum_{n=0}^{\infty} x^{5n}$. Express $N(x)$ as a rational function.

$$A(x) = 1 + x + x^2 + \cdots = 1/(1-x).$$

$$A(1\cdot x^5) = \underbrace{1 + x^5 + x^{10} + \cdots}_{N(x)} = 1/(1-x^5).$$

So $N(x) = 1/(1-x^5)$.



Example 2. What is the GF for $1, -1, 1, -1, \ldots$ ?

$$\rightarrow B(x) = 1 - x + x^2 - x^3 + x^4 - \cdots =$$

$$A(-x) = 1 + (-x) + (-x)^2 + (-x)^3 + \cdots = B(x) = 1/(1-(-x)) = \underline{\frac{1}{1+x}}.$$

Example 3. Express $C(x) = 1 - 2x + 4x^2 - 8x^3 + 16x^4 - \ldots$ as a rational function.

$$C(x) = 1 + (-2x) + (2x)^2 + (-2x)^3 + (-2x)^4 + \cdots$$

$$C(x) = A(-2x) = 1/(1-(-2x)) = 1/(1+2x).$$

Exercise. Find a rational GF for the sequence $1, -2, 3, -4, 5, -6, \ldots$ ?

$$A(x) = 1 + x + x^2 + \cdots = 1/(1-x)$$
$$A'(x) = 1 + 2x + 3x^2 + \cdots = 1/(1-x)^2 \quad x \to -x$$
$$B(x) = 1 - 2x^1 + 3x^2 - 4x^3 + \cdots = \underline{A'(-x)} = 1/(1-(-x))^2 = 1/(1+x)^2$$

Example 4. Express $D(x) = -x^1 + 2x^2 - 3x^3 + 4x^4 - \ldots$ as a rational function.

$$D(x) = -x(1 - 2x + 3x^2 - 4x^3 + \cdots) = -x B(x) = \frac{-x}{(1+x)^2}$$



Using substitution and our two basic GF's $A(x) = 1 + x + x^2 + \ldots$ and $A'(x)$ we can now determine the coefficients for any GF that has the form

$$\frac{p(x)}{ax+b} \quad \text{or} \quad \frac{p(x)}{(ax+b)^2}$$

Problem 1. Find the coefficient of $x^k$ in the GF

$$C(x) = \frac{x^2/3}{2x+3/3} = \frac{x^2/3 \cdot 1}{1 + \frac{2}{3}x}$$

$$A\left(-\tfrac{2}{3}x\right) = \frac{1}{1+\frac{2}{3}x} = 1 - \frac{2}{3}x + \left(\frac{2}{3}\right)^2 x^2 - \left(\frac{2}{3}\right)^3 x^3 + \cdots + (-1)^n \left(\frac{2}{3}\right)^n x^n + \cdots$$

$$C(x) = \frac{1}{3}x^2\left[1 - \frac{2}{3}x + \frac{2^2}{3^2}x^2 - \cdots + \boxed{(-1)^n \cdot \left(\frac{2}{3}\right)^n x^n} + \cdots\right]$$

$$[x^k]\,C(x) = \frac{1}{3}(-1)^{k-2} \cdot \left(\frac{2}{3}\right)^{k-2} = \frac{1}{3} \cdot (-1)^k \left(\frac{2}{3}\right)^{k-2} \quad k \geq 2$$
$$= 0 \quad k = 0, 1.$$

Problem 2. Find the coefficient of $x^k$ in the GF

$$D(x) = \frac{x^2 /4}{(x+2)^2 /4} = \frac{x^2/4}{(1+x/2)^2} = \frac{1}{4}\cdot x^2 \cdot \frac{1}{(1+\frac{x}{2})^2}$$

$$A'(-x) = \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$$

$$A'(-\tfrac{x}{2}) = \frac{1}{(1+\frac{x}{2})^2} = 1 - x + 3\frac{x^2}{2^2} - 4\cdot\frac{x^3}{2^3} + \cdots + \boxed{\frac{(-1)^n\cdot(n+1)}{2^n}}\cdot x^n + \cdots$$

$$D(x) = \frac{1}{4}x^2 \, A'(\tfrac{-x}{2})$$

$$[x^k]\,D(x) = \frac{1}{4}(-1)^{k-2}\cdot\frac{(k-1)}{2^{k-2}} = \frac{(-1)^k}{2^k}(k-1) \cdot \text{ for } k \geq 2.$$

$$[x^k]\,D(x) = 0 \text{ for } k = 0, 1.$$

Check? $k=3$: $\frac{(-1)^3}{2^3}\cdot 2 = -\frac{1}{4}$.