Lecture 13 Second Order Recurrence Relations

Grimaldi 10.2

$$f_{n+1} = f_n + f_{n-1}$$

$$a_n = a_{n-1} + 2a_{n-2}$$

For constants a, b, c consider a recurrence relation of the form

$$a\,x_n + b\,x_{n-1} + c\,x_{n-2} = 0 \quad \text{for } n \geq 2. \tag{1}$$

Suppose that $x_n = r^n$ is a solution to equation (1). In this case we have

$$ar^n + br^{n-1} + cr^{n-2} = 0 \quad \text{for all } n \geq 2. \tag{2}$$

Observe that the $n \geq 2$ condition is redundant in equation (2). If this holds for $n = 2$, then it holds for all larger values (multiplying by powers of r gives the other equations). This reduces us to a familiar equation

$$ar^2 + br + c = 0$$

Conclusion: A number r satisfies $ar^2 + br + c = 0$ if and only if $x_n = r^n$ is a solution to our recurrence.

Definition

The homogeneous second order linear recurrence relation

$$ax_n + bx_{n-1} + cx_{n-2} = 0$$

has **characteristic equation**

$$ar^2 + br + c = 0.$$

The roots of $ar^2 + br + c$ are precisely those numbers r for which $x_n = r^n$ satisfies the above recurrence.

Exercise. Find all real numbers r so that $x_n = r^n$ is a solution to the recurrence

$$x_n - 5x_{n-1} + 6x_{n-2} = 0$$

Theorem (Linearity)

Both of the properties below hold for the recurrence relation

$$ax_n + bx_{n-1} + cx_{n-2} = 0 \tag{3}$$

(A) *If $x_n = r^n$ is a solution of (3) then Cr^n is a solution to (3) for any constant C.*

(B) *If $x_n = s^n$ and $x_n = t^n$ are solutions of (3) then $s^n + t^n$ is a solution.*

It follows from (A) and (B) that $Cs^n + Dt^n$ is a solution for any constants C, D.

Proof:

Exercise. The recurrence relation

$$x_n - 5x_{n-1} + 6x_{n-2} = 0$$

has the solutions $x_n = 3^n$ and $x_n = 2^n$. Check that $C2^n + D3^n$ is a solution.

How do we determine what C and D are? With two consecutive initial values. Find the solution with the initial values $x_0 = 6$ and $x_1 = 13$.

General solutions

Theorem

Let a, b, c be fixed constants with $a \neq 0$ and consider the recurrence

$$ax_n + bx_{n-1} + cx_{n-2} = 0. \tag{4}$$

If the characteristic equation,

$$ar^2 + br + c = 0$$

has two **distinct** *real roots, say r_1 and r_2, then every sequence satisfying this recurrence has the form*

$$x_n = Cr_1^n + Dr_2^n \tag{5}$$

where C and D are fixed constants. Accordingly, we will call equation (5) the ***general solution*** *to the recurrence.*

Example. The Fibonacci sequence $(f_1, f_2, f_3, \ldots)$ is generated by the recurrence

$$f_n = f_{n-1} + f_{n-2} \qquad \text{for } n \geq 2$$

together with the initial values $f_0 = 0$ and $f_1 = 1$.

(1) Find the general solution to the above recurrence.

(2) Find a closed form (a formula in n) for the Fibonacci sequence.

Solving $ar^2 + br + c = 0$ using the quadratic formula we get

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac = 0$ then we have two repeated real roots.
If $b^2 - 4ac < 0$ we have two complex roots.

Theorem (Repeated real roots case)

Let a, b, c be real constants with $a \neq 0, c \neq 0$ and consider the recurrence

$$a\,x_n + b\,x_{n-1} + c\,x_{n-2} = 0.$$

If the characteristic polynomial $ar^2 + br + c$ has a repeated root r then every sequence satisfying this recurrence has the form

$$x_n = Cr^n + D\,nr^n \tag{6}$$

where C and D are constants. Equation (6) is the **general solution** *to the recurrence.*

Example. Solve the following recurrence

$$x_n - 6x_{n-1} + 9x_{n-2} = 0 \quad \text{with} \quad x_0 = 2,\ x_1 = 3.$$