Lecture 14: Solving Non-Homogeneous Relations

Grimaldi 10.3

Definition

If $f(n) \neq 0$, a recurrence relation of the form

(1) $ax_n + bx_{n-1} = f(n) \qquad a \neq 0, b \neq 0$

(2) $ax_n + bx_{n-1} + cx_{n-2} = f(n) \qquad a \neq 0, c \neq 0$

is called a **non-homogeneneous** recurrence relation.
The **associated homogeneous relation** is obtained by setting f to be 0

(1) $ax_n + bx_{n-1} = 0$

(2) $ax_n + bx_{n-1} + cx_{n-2} = 0$

Definition

- A **particular solution** is a single sequence $x_n^{(p)}$ satisfying a recurrence without consideration of the initial condition.
- The **general solution** to a recurrence is the set of all sequences x_n satisfying it (without consideration of the initial condition)

Theorem

The general solution to a non-homogeneous recurrence is given by ***one particular solution****, $x_n^{(p)}$, plus the* ***general solution*** *to the associated homogeneous equation, $x_n^{(h)}$. That is, the solution has the form*

$$x_n = x_n^{(p)} + x_n^{(h)}.$$

Example 1 $x_n = 6x_{n-1} + 3^n$ for $n > 1$ and $x_0 = 7$.

Example 2 $x_n - 4x_{n-1} + 3x_{n-2} = \frac{2^n}{4}$ and $x_0 = 5, x_0 = 6.$

To find a particular solution to a non-homogeneous recurrence of the form

$$ax_n + bx_{n-1} = f(n) \qquad n \geq 1$$

$$ax_n + bx_{n-1} + cx_{n-2} = f(n) \qquad n \geq 2$$

(1) Exponential functions $f(n) = kr^n$
   (a) If r is not a root of the char. poly. of the homog. recurrence then look for a particular solution of the form $x_n^{(p)} = Cr^n$.
   (b) If r is a root of multiplicity m then look for a particular solution of the form $x_n^{(p)} = Cn^m r^n$

(2) Power functions $f(n) = kn^d$
   (a) Look for a solution of the form $x_n^{(p)} = a_d n^d + a_{d-1} n^{d-1} \ldots + a_1 n + a_0$
   (b) If n^t, for some $t \leq d$, is a solution to the homogeneous equation then multiply the trial solution $x_n^{(p)}$ by the smallest power of n, say n^s, for which no summand of $n^s f(n)$ is a solution of the homog. relation.

See Grimaldi page 479-481 for examples on how to determine the form of $x_n^{(p)}$.

Example 3. Find a particular solution to $x_n - 3x_{n-1} + 2x_{n-2} = 4n$.

Example 3 (continued).