Lecture 29: Trees and Rooted Trees

Grimaldi 12.1, 12.2

Definition (tree)

A multigraph G is a **tree** if G is connected and G does not contain a cycle.

Theorem (main properties of trees)

If $T = (V, E)$ is a tree then $|V| = |E| + 1$ and secondly, there is a unique path in T between every pair of vertices.

Examples

Theorem (Characterization of Trees)

Let $G = (V, E)$ be a multigraph. The following statements are equivalent.

(1) *G is connected and has no cycle. (G is a tree)*

(2) *G is connected and $|V| = |E| + 1$.*

(3) *G has no cycle and $|V| = |E| + 1$.*

(4) *There is a unique path between every pair of vertices in G.*

Proof.

Proof (cont).

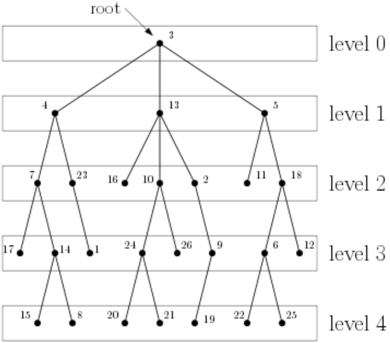
Definition (rooted tree)

A **rooted tree** $T = (V, E)$ is a tree with a distinguished vertex called the **root**.
For every vertex $v \in V$ the **level** of v is the length of the path from v to the root.
Note: the root is the unique vertex at level 0.

Example

Definition (rooted tree terminology)

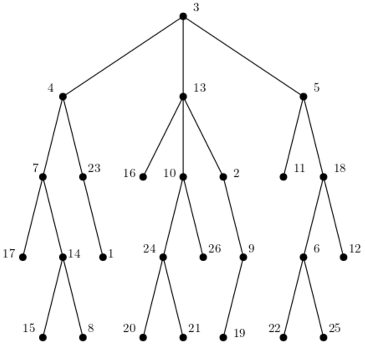
- The **height** of a rooted tree is the maximum level of a vertex. A rooted tree consisting of just a root vertex has height 0.
- Every non-root vertex v at level i is adjacent to exactly one vertex u at level $i-1$. We call u <u>the</u> **parent** of v and we say that v is <u>a</u> **child** of u.
- For every vertex v there is a walk "up the tree" to the root obtained by moving to the parent vertex at each step. If u is another vertex on this walk, we call u an **ancestor** of v and v a **descendant** of u.



We are frequently interested in working with rooted trees recursively. Therefore, it will be helpful to think of a rooted tree as composed out of smaller rooted trees.

Definition (subtree)

Let v be a vertex of a rooted tree T with level i. Define T' to be the subgraph of T induced by v together with its descendants. Then T' forms a new rooted tree with root vertex v. We say that T' with root v is the **subtree** of T at v.



Definition (isomorphism of rooted trees)

Let T_1, T_2 be rooted trees with $T_i = (V_i, E_i)$ for $i = 1, 2$. We say that T_1 and T_2 are **isomorphic** if there exists a bijection $f : V_1 \to V_2$ satisfying:

(1) $\{f(u), f(v)\} \in E_2 \Leftrightarrow \{u, v\} \in E_1$

(2) For every $v \in V_1$ the level of v and $f(v)$ is the same.
In particular, f sends the root of T_1 to the root of T_2.

Example.

Definition

A rooted tree is m-**ary** if every internal node has at most m children. A 2-ary tree is called **binary** tree.

Exercise. Find all binary trees with height 0, 1, and 2 up to isomorphism.

Let b_n denote the number of binary trees of height at most n. Find b_0, b_1, b_2.

Use the recursive structure of rooted trees find a recurrence for b_n.