Lecture 17 Generating Functions Michael Monagan

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Lecture 17 Generating Functions

Grimaldi Chapter 9 Generating Functions

Midterm #2 Monday March 8th.
Same procedure as Midterm #1.
1 Cheat sheet allowed.
Material : Lectures 9 to 16, assigns #3 and #4
Review problems posted tomorrow morning.

A new powerful way of counting.

The binomial coefficient $\binom{n}{k}$ counts different objects:

$$\begin{aligned}\binom{n}{k} &= \text{the number of subsets of } \{1,2,\ldots,n\} \text{ of size } k\\ &= \text{the number of binary strings of length } n \text{ with } k \text{ 1's}\\ &= \text{the coefficient of } x^k y^{n-k} \text{ in the expansion of } (x+y)^n\end{aligned}$$

Example $\boxed{(1+x)^3} = (x^0+x^1)^3 = (x^0+x^1)\overbrace{(x^0+x^1)(x^0+x^1)}$

$$\begin{aligned}&= x^0\cdot x^0\cdot x^0 + x^0\cdot x^0\cdot x^1 + x^0\cdot x^1\cdot x^1 + x^1 x^1 x^1 = \underset{\binom{3}{0}}{1}\cdot x^0 + \underset{\binom{3}{1}}{3}\cdot x^1 + \underset{\binom{3}{2}}{3}x^2 + \underset{\binom{3}{3}}{1}\cdot x^3\\ &\quad + x^0\cdot x^1\cdot x^0 + x^1\cdot x^0\cdot x^1\\ &\quad + x^1\cdot x^0\cdot x^0 + x^1\cdot x^1\cdot x^0\end{aligned}$$

Binary Strings

000

001
010
100

011
101
110

111

The coefficient of x^k in $(1+x)^3 = 1+3x+3x^2+x^3$
Counts the # of binary strings of length n with k 1 bits.

Definition (coefficient)

If $P(x)$ is a polynomial we denote by $[x^k]P(x)$ the coefficient of x^k in $P(x)$.

Example 1 How many integer solutions $a_1 + a_2 + a_3 = 7$ have if $0 \le a_i \le 3$?

$$2+2+3=7$$
$$1+3+3=7$$

Consider $P(x) = (1+x^1+x^2+x^3)(1+x^1+x^2+x^3)(1+x^1+x^2+x^3)$

a_1 a_2 a_3

Answer: $[x^7]P(x)$.
$= 6$.

$x^1 \cdot x^3 \cdot x^3 = x^7$ $\quad$ $x^2 \cdot x^2 \cdot x^3$

$x^3 \cdot x^1 \cdot x^3 = x^7$ $\quad$ $x^2 \cdot x^3 \cdot x^2$

$x^3 \cdot x^3 \cdot x^1 = x^7$ $\quad$ $x^3 \cdot x^2 \cdot x^2$

Wolfram α : Coefficient[(1+x+x^2+x^3)^3, x, 7] $\longrightarrow$ 6.

Maple : coeff((1+x+x^2+x^3)^3, x, 7) $\longrightarrow$ 6.

Example 2 Suppose we roll two dice. If we add the values of the dice, how many ways can we get 6? $1+5$, $5+1$, $2+4$, $4+2$, $3+3$ 5 ways.

Consider $r_1 + r_2 = 6$ where $1 \le r_1 \le 6$ and $1 \le r_2 \le 6$.

first die second die.

Consider $P(x) = (x^1+x^2+x^3+x^4+x^5+x^6)(x^1+x^2+x^3+x^4+x^5+x^6)$.

Answer $= [x^6]P(x)$.

$$P(x) = x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9 + 3x^{10}+2x^{11}+1\cdot x^{12}$$

The coefficients of polynomials can count something.

Example 3 How many integer solutions does

$$a_1 + a_2 + a_3 = 9 \qquad 3+2+4=9$$

have if $2 \le a_1 \le 4, 1 \le a_2 \le 5, 3 \le a_3 \le 7$?

$[x^9]P(x)$ where $P(x) = (\underbrace{x^2}_{a_1} + x^3 + x^4)(\underbrace{x^1 + x^2}_{a_2} + x^3 + x^4 + x^5)(x^3 + x^4 + \underbrace{x^5}_{a_3} + x^6 + x^7)$

$a_1 = 1, 3, 5, 7, 9.$

Exercise: What if a_1 is odd, a_2 is even and $a_3 \in \{0, 3, 6\}$?

$[x^9]P(x)$ where $P(x) = (x^1 + x^3 + x^5 + x^7 + x^9)(x^0 + x^2 + x^4 + x^6 + x^8)(x^0 + x^3 + x^6)$

$a_1 = 9, a_2 = 0 = a_3.$

Example 4. How many integer solutitions does

$$a_1 + a_2 + a_3 = n \quad \text{have if } a_i \ge 0 \text{ ?} \qquad (1)$$

$[x^n]\, P(x)$ where $P(x) = (x^0 + x^1 + \cdots + x^n)^3$

$[x^n]\, A(x)$ where $A(x) = (x^0 + x^1 + \cdots + x^n + x^{n+1} + x^{n+2} + \cdots)^3$

a series $= \left(\sum_{i=0}^{\infty} x^i\right)^3$

The generating function for the # of solutions to (1).

$[x^n]\, P(x)$
$[x^n]\, A(x)$ counts something.

Definition

The **generating function** for an infinite sequence $a_0, a_1, a_2, \ldots$ is the series

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_i x^i.$$

We are interested in the coefficients of $A(x)$ not the values of $A(x)$.
Example 5. What is the generating function for $1, 1, 1, \ldots$?

$$A(x) = \boxed{1 + 1\cdot x + 1\cdot x^2 + 1\cdot x^3 + 1\cdot x^4 + \cdots = \frac{1}{1-x}}$$ ✓ for $|x|<1$

$$(1-x)(1 + 1\cdot x + 1\cdot x^2 + 1\cdot x^3 + 1\cdot x^4 + \cdots) = 1 + 0\,x + 0\,x^2 + 0\cdot x^3 + \cdots$$
$$= 1$$

Question: What is the generating function for the sequence $1, 2, 3, 4, 5, \ldots$?

$$B(x) = 1 + 2\cdot x + 3\cdot x^2 + 4\cdot x^3 + 5x^4 + \cdots = ? = \frac{1}{(1-x)^2}$$

$$A(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \frac{1}{1-x} = (1-x)^{-1}$$

$$A'(x) = 0 + 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = -1(1-x)^{-2}(-1)$$
$$= \frac{1}{(1-x)^2}$$

is the G.F. for $1, 2, 3, 4, 5, \ldots$