Lecture 3: Combinations and the Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Grimaldi Section 1.3

The quantity $\binom{n}{k}$ is the number of ways of choosing a set of size k from a set of size n. We also saw that it is the number of binary strings of length n with k 1's so

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Theorem ($\binom{n}{k} = \binom{n}{n-k}$)

Proof.

Theorem ($\displaystyle\sum_{k=0}^{n}\binom{n}{k}=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$)

Expanding $(x + y)^n$

Theorem (The Binomial Theorem)

If n is a positive integer then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots + \binom{n}{n} x^n y^0.$$

Because of this theorem the numbers $\binom{n}{k}$ are called binomial coefficients
We now have three equivalent ways to think of $\binom{n}{k}$:

Using the Binomial Theorem

Exericse. Find the coefficient of $x^5 y^{95}$ in $(3x - y)^{100}$.

Theorem (The Multinomial Theorem)

If $x_1, x_2, \ldots, x_m$ are variables and n a positive integer, then,

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$$

Proof:

Example. What is the coefficient of xy^2z^2 in $(w + x + y + z)^5$?