Lecture 26: Planar Graphs continued

Grimaldi 11.4

Question: Can a given electronic circuit be layed out on an circuit board such that no wires cross each other?

Review:

A graph $G = (V, E)$ is **planar** if G has a drawing (in the plane) where the edges intersect only at the vertices of G. Such a drawing is called a **planar embedding** of G. The embedding partitions the plane into at set F of regions called **faces**. We proved Euler's formula $|V| - |E| + |F| = 2$.

Examples.

Definition (Face degrees)

Let $G = (V, E)$ be a connected multigraph embedded in the plane and let f be a face of this embedding. We define the **degree** of f, denoted $\deg(f)$, to be the number of edges in a facial walk of f.

Example

Theorem

If G has faces $f_1, f_2, \ldots, f_k$ then $\displaystyle\sum_{i=1}^{k} \deg(f_i) = 2|E|$.

Proof

What is the maximum number of edges a planar simple graph can have?

Theorem (Bound 1 for the number of edges)

If $G = (V, E)$ is a connected planar simple graph with $|V| \geq 3$ then

$$|E| \leq 3|V| - 6 \quad \text{and} \quad 2|E| \geq 3|F|.$$

Proof.

Corollary (to bound 1 for the number of edges)

The graph K_5 is not planar.

Proof.

Theorem (Bound 2 for the number of edges)

If $G = (V, E)$ is a connected planar simple graph with $|V| \geq 3$ and with no cycle of length 3 or less then

$$|E| \leq 2|V| - 4 \quad \text{and} \quad |E| \geq 2|F|.$$

Proof.

Corollary (to bound 2 for the number of edges)

The graph $K_{3,3}$ is not planar.

Proof.

Definition (Dual graphs)

Let $G = (V, E)$ be a connected multigraph embedded in the plane. The vertices of the **dual** multigraph G^* are the faces of G. If two faces f_i and f_j share an edge e then $e^* = \{f_i, f_j\}$ is an edge in G^*. This may be done so that e^* crosses e and G^* also ends up embedded in the plane.

Examples.

Features of duals

(1) Duals only exist for planar graphs
(2) If G^* is a dual of G then G is a dual of G^* !
(3) The degree of a vertex in G^* is the degree of the corresponding face of G.
(4) The dual of a simple graph may be a multigraph.

Examples