# Lecture 20 Generating Functions continued.



Grimaldi 9.2 Calculation Techniques

We have been working with the two basic GF's

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)x^n$$

We already proved the more general generating function:

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$$

Combining this formula with **substitution** allows us to determine the coefficients of any rational function of the form

$$\frac{p(x)}{(ax+b)^k}.$$

On page 422 the textbook uses a natural generalization of binomial coefficients, called the extended binomial theorem to get these coefficients. We will use substitutions instead.

Example 1. For $A(x) = \frac{x}{(1-x)^2}$ find $[x^n](A(x))$.

Example 2. Find the coefficient of $x^5$ of $A(x) = \frac{1}{(1-2x)^7}$

# Partial Fractions

Question: How can we deterine the coefficients of GFs of the form

$$\frac{p(x)}{ax^2+bx+c} \quad \text{and} \quad \frac{p(x)}{ax^3+bx^2+cx+d} \quad \text{etc?}$$

If $ax^2+bx+c = a(x-\alpha)(x-\beta)$ and $\alpha \neq \beta$ solve

$$\frac{1}{(x-\alpha)(x-\beta)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta}$$

for $A, B$. If $ax^3+bx^2+cx+d = a(x-\alpha)(x-\beta)(x-\gamma)$ and $\alpha \neq \beta \neq \gamma$ solve

$$\frac{1}{(x-\alpha)(x-\beta)(x-\gamma)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma}$$

for $A, B, C$. If $ax^3+bx^2+cx+d = a(x-\alpha)(x-\beta)^2$ and $\alpha \neq \beta$ solve

$$\frac{1}{(x-\alpha)(x-\beta)^2} = \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{(x-\beta)^2}$$

for $A, B, C$. Then use the formula for $1/(1-x)^k$ with substitutions.

Example 1. Find the coefficient of $x^n$ of $C(x) = \dfrac{3x}{x^2 - 3x + 2}$.

Example 2. Find values for $A, B, C$ so that the expression below is true.

$$D(x) = \frac{1}{(x-3)(x-2)^2} = \frac{A}{x-3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

# Series Division

To find the series for the quotient

$$C(x) = \frac{A(x)}{B(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots}{b_0 + b_1 x + b_2 x^2 + \cdots}$$

let $C(x) = c_0 + c_1 x + c_2 x^2 + \cdots$ and write $A(x) = B(x)C(x)$ so that

$$(a_0 + a_1 x + a_2 x^2 + \cdots) = (b_0 + b_1 x + b_2 x^2 + \cdots)(c_0 + c_1 x + c_2 x^2 + \cdots)$$

In this equation the $a_i$ and $b_i$ are known coefficients, the $c_i$ are unknown. Equating coefficients in $x^i$ for $i = 0, 1, 2, \cdots$ and solving for $c_i$ we obtain

$$\begin{array}{ll}
[x^0] & a_0 = b_0 c_0 \implies c_0 = a_0/b_0 \implies b_0 \neq 0 \\
[x^1] & a_1 = b_0 c_1 + b_1 c_0 \implies c_1 = (a_1 - b_1 c_0)/b_0 \\
[x^2] & a_2 = b_0 c_2 + b_1 c_1 + b_2 c_0 \implies c_2 = (a_2 - b_1 c_1 - b_2 c_0)/b_0 \\
\cdots & \\
[x^n] & a_n = b_0 c_n + b_1 c_{n-1} + \cdots + b_n c_0 \implies c_n = (a_n - b_1 c_{n-1} - \cdots - b_n c_0)/b_0
\end{array}$$

Example 1. Calculate $A(x) = (1 + x)/(1 - x)^2 = (1 + x)/(1 - 2x + x^2)$ to $x^3$.

Example 2. Find the series for $x/(1-x-x^2)$ using series division.

Exercise. Find the series for $(1+x)/(1-3x+3x^2-x^3)$ to $x^4$ using series division and determine a recurrence for the $n$th coefficient.