Lecture 25 Planar Graphs Michael Monagan

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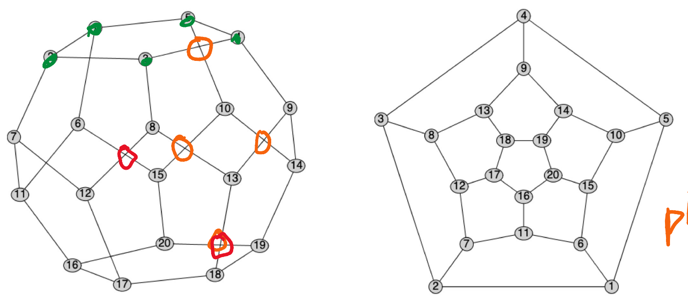
Lecture 25: Planar Graphs

Grimaldi 11.4

Assignment #6 is due tonight.
Midterm #3 next Monday.
Same rules & procedure as Midterm #2.

Dodecahedron.



planar embedding of the Dodecahedron.

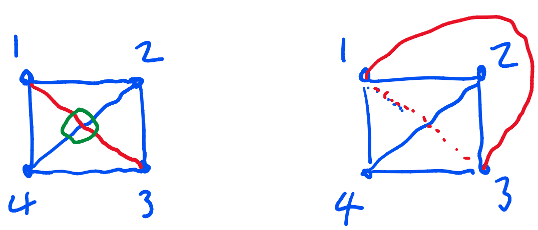
These are both drawings of the same graph. To see this locate the cycles $1-2-3-4-5-1$ and $16-17-18-19-20$ in both graphs.

Definition (planar graph)

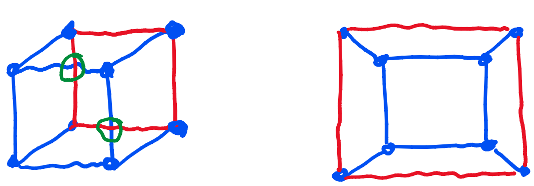
A graph G is **planar** if G has a drawing (in the plane) so that the edges intersect only at the vertices of G. Such a drawing is called a **planar embedding** of G.

Examples

K_4



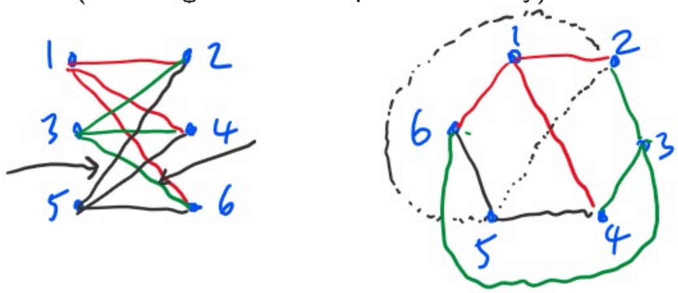
a planar embedding of K_4.
K_4 is planar.



Cube?

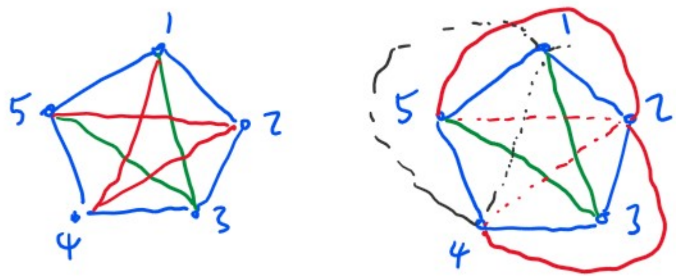
a planar embedding of the cube.

Observation: The graph $K_{3,3}$ is not planar.
Proof sketch (we will give a formal proof next day)



Get Stuck.

Observation: The graph K_5 is not planar.
Proof sketch (we will give a formal proof next day)

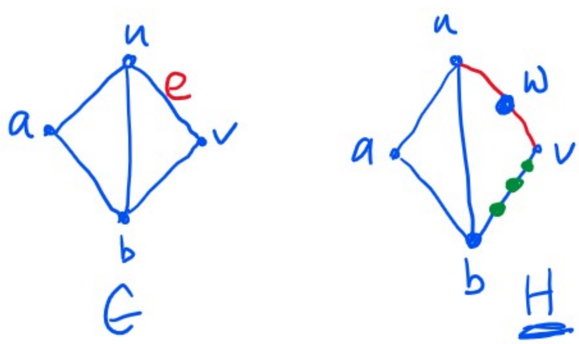


Get Stuck.

Definition (subdivision)

Let $G = (V, E)$ be a multigraph and let $e = \{u, v\}$ be an edge in E. To **subdivide** the edge e is to delete e and add a new vertex w and two new edges $e_1 = \{u, w\}$ and $e_2 = \{w, v\}$ to G. If the graph H is obtained from G by a sequence of subdivions, then H is called a subdivison of G.

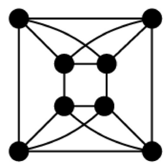
Example



Observation. If H is a subdivision of G then H is planar if and only if G is planar. This means that every subdivision of $K_{3,3}$ and K_5 is nonplanar.

Example: Is this graph planar? I.e. can you find a planar embedding?

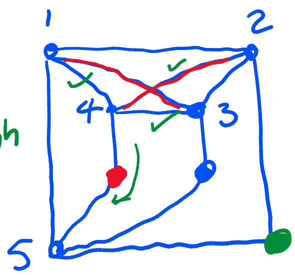
G

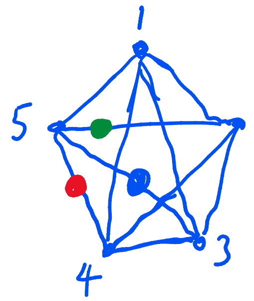


K_5?

If we think G is planar then we can try and find a planar embedding. If not we look for a subdivision of K_5 or $K_{3,3}$ in G.

a subgraph of G





a subdivision of K_5 that is a subgraph of G.

So G is not planar.

Exercise. Find a subdivision of $K_{3,3}$ in G.

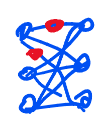
Question: Which graphs are planar ?

Definition

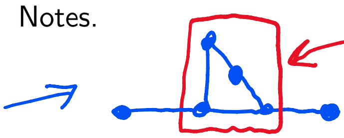
Let G and H be multigraphs. We say that G **contains a subdivision** of H if there is a subgraph of G isomorphic to some subdivision of H.

Theorem (Kuratowski-Wagner)

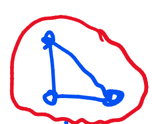
A multigraph G is planar if and only if G does not contain a subdivision of $K_{3,3}$ or a subdivision of K_5.



Notes.



G



K_3

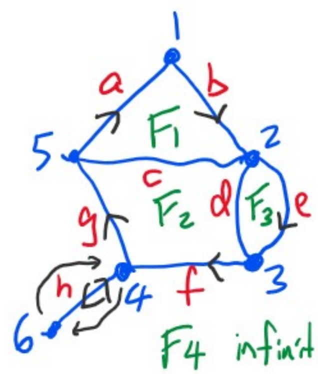
G contains a subdivision of K_3.

1. If G is planar then every subgraph of G is planar.
2. The Hopcroft–Tarjan planarity test (1974) takes linear time in $|V| + |E|$.

Definition (Faces)

Let G be a planar graph embedded in the plane. The embedding partitions the plane into connected regions called **faces**. There is one unbounded region called the **infinite face**. All other faces are **internal faces**. If G is connected, every face has vertices and edges on its boundary. They form a closed walk called a **facial walk**

Example



F4 infinite face

Facial walks

F_1 1 b 2 c 5 a 1 — 1, 2, 5 ; a, b, c

F_2 2 d 3 f 4 g 5 c 2.

F_3 2 d 3 e 2

F_4 1 b 2 e 3 f 4 h 6 h 4 g 5 a 1.

$|V| = 6,\ |E| = 8,\ |F| = 4 \qquad |V| - |E| + |F| = 6 - 8 + 4 = 2.$ ✓

Theorem (Euler's formula)

If $G = (V, E)$ is an connected multigraph embedded in the plane and F is the set of faces, then

$$|V| - |E| + |F| = 2. \Rightarrow |F| = |E| - |V| + 2$$

This implies all embeddings of a planar graph have the same number of faces.

Example — By induction on $|E|$ in G.

Proof.

Base $|E| = 0$. $G = \bullet\ F_1$ $\quad |V| - |E| + |F| = 1 - 0 + 1 = \underline{2.}$

Ind. Step. $|E| = n \geq 1$. Assume $|V| - |E| + |F| = 2$ holds for graphs with $|E| < n$. (Ind. Hyp.)

Case (i) G has a cycle. Let e be an edge on the cycle. Notice e separates two faces F_j and F_i.

Let $\underline{H} = G - e$. Then H has $\underline{|F|} - 1$ faces and $|E| - 1$ edges. but the same # of vertices.

Proof (cont.)

H is also connected. By the Ind. Hyp, H satisfies Euler's formula $|V| - (|E|-1) + (|F|-1) = 2$ ← Euler's theorem to H

$\Rightarrow$ $|V| - |E| + |F| = 2$ in G. G in G

Case (iii) If G has no cycle but is connected then G is a tree, and it has one face.

Let's delete a leaf vertex v from G.

Let $G' = G - v$. Observe G' is connected and

$|E'| = |E| - 1$, $|V'| = |V| - 1$, $|F'| = |F|$.

Applying the Ind. Hyp to G' we have

$|V'| - |E'| + |F'| = 2$

G' is a tree

$\Rightarrow$ $|V| - 1 - (|E| - 1) + |F| = 2$

$\Rightarrow$ $|V| - |E| + |F| = 2$ i.e. Eulers formula holds

By induction on $|E|$, $|V| - |E| + |F| = 2$ holds for all connected planar graphs.