Lecture 15, Second order non-homogenous Recurrences, Michael Monagan

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Lecture 15: Solving Non-Homogeneous Relations

Assignment #3 solutions are posted.
Assignment #4 due Monday March 1st.
Midterm #2 Monday March 8th.

Grimaldi 10.3

Definition

If $f(n) \neq 0$, a recurrence relation of the form

(1) $ax_n + bx_{n-1} = f(n)$ $\quad a \neq 0, b \neq 0$ first order.

(2) $ax_n + bx_{n-1} + cx_{n-2} = f(n)$ $\quad a \neq 0, c \neq 0$ second order.

is called a **non-homogeneneous** recurrence relation.
The **associated homogeneous relation** is obtained by setting f to be 0

(1) $ax_n + bx_{n-1} = 0$

(2) $ax_n + bx_{n-1} + cx_{n-2} = 0$

Definition

- A **particular solution** is a single sequence $x_n^{(p)}$ satisfying a recurrence without consideration of the initial condition.
- The **general solution** to a recurrence is the set of all sequences x_n satisfying it (without consideration of the initial condition)

Theorem

The general solution to a non-homogeneous recurrence is given by ***one particular solution***, $x_n^{(p)}$, *plus the* ***general solution*** *to the associated homogeneous equation,* $x_n^{(h)}$. *That is, the solution has the form*

$$x_n = x_n^{(p)} + x_n^{(h)}.$$

Example 1 $x_n = 6x_{n-1} + 3^n$ for $n > 1$ and $x_0 = 7$. first order

① Ass Hom RR: $x_n = 6x_{n-1}$ $\quad x_n = A6^n$ $\quad x_n^{(h)} = A6^n$

② $x_n = 6x_{n-1} + 3^n$

Try $x_n^{(p)} = B3^n$: $B\cdot 3^n = 6\cdot B3^{n-1} + 3^n$ $\quad \div 3^n$

$B\cdot 3^n = 2\cdot B3^n + 3^n \Rightarrow B = 2B+1$

$\Rightarrow -1 = 1\cdot B$

So $x_n^{(p)} = -3^n$ is a particular solution.

$x_n = x_n^{(h)} + x_n^{(p)} = A6^n - 3^n$.

③ $x_0 = 7 = A\cdot 6^0 - 3^0 = A - 1 \Rightarrow A = 8$.

$x_n = 8\,6^n - 3^n$.

Check: $x_n = 6x_{n-1} + 3^n$

$8\,6^n - 3^n$ $\qquad 6\cdot(8\cdot 6^{n-1} - 3^{n-1}) + 3^n = 8\cdot 6^n - 2\cdot 3^n + 3^n = 8\cdot 6^n - 3^n$

Example 2 $x_n - 4x_{n-1} + 3x_{n-2} = \frac{2^n}{4}$ and $x_0 = 5$ $x_0 = 6$. $x_1 = 6$

① $1\cdot x_n - 4x_{n-1} + 3x_{n-2} = 0$ $\quad 1\cdot r^2 - 4r + 3 = (r-3)(r-1) = 0$

$\Rightarrow r = 3, 1.$

$x_n^{(h)} = A1^n + B\cdot 3^n$

② $x_n - 4x_{n-1} + 3x_{n-2} = 2^n/4$

Try $x_n^{(p)} = C2^n$: $C2^n - 4\cdot C\cdot 2^{n-1} + 3\cdot C\cdot 2^{n-2} = 2^{n-2}$

$\div 2^{n-2} \Rightarrow 4C - 8C + 3C = 1 \Rightarrow -C = 1 \Rightarrow C = -1.$

So $x_n^{(p)} = -2^n$.

③ $x_n = x_n^{(h)} + x_n^{(p)} = A + B\cdot 3^n - 2^n$

$n=0$ $\quad x_0 = A + B - 1 = 5 \Rightarrow A + B = 6$
$n=1$ $\quad x_1 = A + 3B - 2 = 6 \Rightarrow A + 3B = 8$

$2B = 2 \Rightarrow B = 1$
$\Rightarrow A = 5$

$x_n = 5 + 1\cdot 3^n - 2^n$.

To find a particular solution to a non-homogeneous recurrence of the form

$$a x_n + b x_{n-1} = f(n) \qquad n \geq 1$$

$$a x_n + b x_{n-1} + c x_{n-2} = f(n) \qquad n \geq 2$$

(1) Exponential functions $f(n) = kr^n$ $\quad 2\cdot 3^n$

(a) If r is not a root of the char. poly. of the homog. recurrence then look for a particular solution of the form $x_n^{(p)} = Cr^n$. $\quad C$ is an unknown constant

(b) If r is a root of multiplicity m then look for a particular solution of the form $x_n^{(p)} = Cn^m r^n$

(2) Power functions $f(n) = kn^d$ $\quad f(n) = k_0 + k_1 n + \cdots + k_d \cdot n^d$ a polynomial of degree d.

(a) Look for a solution of the form $x_n^{(p)} = a_d n^d + a_{d-1} n^{d-1} \ldots + a_1 n + a_0$

(b) If n^t, for some $t \leq d$, is a solution to the homogeneous equation then multiply the trial solution $x_n^{(p)}$ by the smallest power of n, say n^s, for which no summand of $n^s f(n)$ is a solution of the homog. relation.

See Grimaldi page 479-481 for examples on how to determine the form of $x_n^{(p)}$.

Example 3. Find a particular solution to $x_n - 3x_{n-1} + 2x_{n-2} = 4n$.

Try $x_n^{(p)} = Cn + D$: $\quad Cn + D - 3(C(n-1)+D) + 2(C(n-2)+D) = 4n$

$\Rightarrow Cn + D - 3Cn + 3C - 3D + 2Cn - 4C + 2D = 4n$

$\Rightarrow n(\underbrace{C - 3C + 2C}_{=0}) + (D + 3C - 3D - 4C + 2D) = 4n.$

This has no solution

Try $x_n^{(p)} = n(Cn + D) = Cn^2 + Dn$

$Cn^2 + Dn - 3(C(n-1)^2 + D(n-1)) + 2(C(n-2)^2 + D(n-2)) = 4n$

$\Rightarrow Cn^2 + Dn - 3Cn^2 + 6Cn - 3C - 3Dn + 3D$
$+ 2Cn^2 - 8Cn + 8C + 2Dn - 4D \quad = 4n$

$\Rightarrow n^2(\underbrace{C - 3C + 2C}_{=0}) + n(\underbrace{D + 6C - 3D - 8C + 2D}_{=-2C}) + (\underbrace{-3C + 3D + 8C - 4D}_{=5C-D}) = 4n.$

so $-2C = 4 \Rightarrow C = -2$

$\Rightarrow -10 - D = 0$

$\Rightarrow D = -10.$

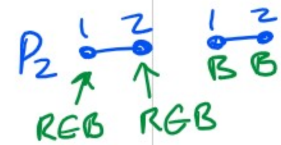
So $x_n^{(p)} = -2n^2 - 10n = -2n(n+5).$

Example 3 (continued).

Assignment #3 Question 15.

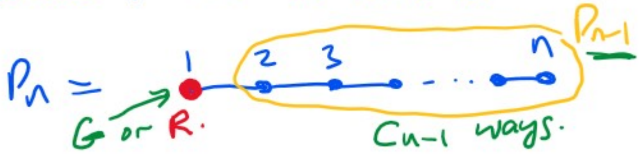
$P_n = $ 1 2 3 ... n Three colours R G B.

$C_n = \#$ ways to colour the vertices of P_n s.t. no two adjacent vertices can be Blue.



$C_2 = 9 - 1 = 8.$

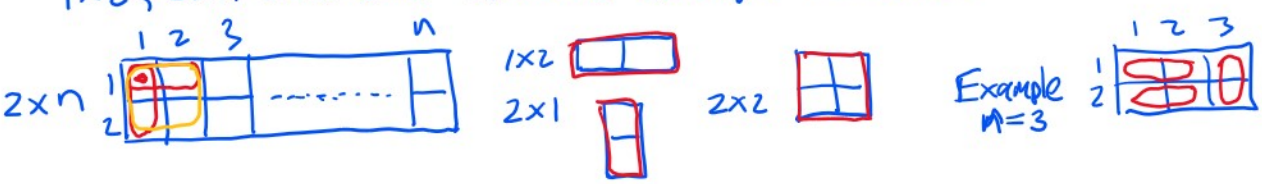
Vertex 1 must be R G B.

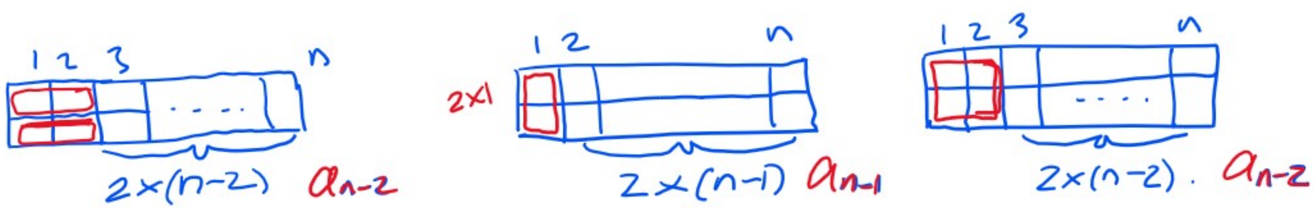


Therefore $C_n = \underset{\substack{\uparrow \\ \text{first vertex} \\ \text{is R or G}}}{2} \cdot C_{n-1} + \underset{\substack{\uparrow \\ \text{second vertex} \\ \text{is R or G.}}}{2} \cdot C_{n-2}.$ Second order.

Assignment #4 10.2 Exercise 24.

Let a_n be the # ways to "tile" a $2\times n$ chessboard with 1×2, 2×1, and 2×2 dominoes (tiles). Find and solve a RR for a_n





$a_n = a_{n-1} + 2a_{n-2}.$