Lec12 Solving First order Recurrences

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Lecture 12 Solving First Order Recurrence Relations

Grimaldi Chapter 10.1

Midterm average 68.6
median 71.3

$p_n = n\,p_{n-1}$

$c_n = (n-1) + c_{n-1}$

$h_n = 1 + 2h_{n-1}$

Assignment 3 posted.
Due Monday Feb 22nd @ 11pm.


```
 1: void Bubblesort( double A[], int n ) {
 2: // sort the array A of size n into ascending order
 3:    int i; double t;
 4:    if( n==1 ) return;
 5:    for( i=1; i<=n-1; i++ )
 6:       if( A[i-1] > A[i] ) {
 7:          t = A[i-1]; A[i-1] = A[i]; A[i] = t;
 8:       }
 9:    Bubblesort(A,n-1);
10:    return;
11: }
```

i++ $i = i+1$.

$C_1 = 0$.

line 6 is executed $n-1$ times.

recursive call to Bubblesort to sort the first $n-1$ entries of A. This does C_{n-1} comparisons.

What should we count to determine the cost of the Bubblesort algorithm?
We will count the number of comparisons between elements of A in line 6.
Let c_n be the number of comparisons.

line 6 line 9.

$n=1 \quad C_1 = 0$

$n>1 \quad C_n = n-1 + C_{n-1}$

This is the same RR as k_n the # of edges in K_n.

$k_n = k_{n-1} + n-1$ and $k_1 = 0$.

A First Look at Solving Recurrence Relations.

How can we solve RRs like

(1) $b_n = 2b_{n-1}$ for $n \geq 2$ and $b_1 = 2$.

(1) $c_n = c_{n-1} + (n-1)$ for $n \geq 2$ and $c_1 = 0$.

$b_n = 2^n$

$b_n = \#$ of binary strings of length n.

Example $b_n = 2b_{n-1}$

$n \to n-1 \Rightarrow b_n = 2^1 b_{n-1}$

$= 2(2b_{n-2}) = 2^2 b_{n-2}$

$= 2^2(2 \cdot b_{n-3}) = 2^3 b_{n-3}$

$= 2^3(2 \cdot b_{n-4}) = 2^4 b_{n-4}$

$\vdots$

$= 2^{n-1} \cdot b_1 = 2^{n-1} \cdot 2 = 2^n$

$1 + n - 1 = n$

$2 + n - 2 = n$

$3 + n - 3 = n$

$4 + n - 4 = n$

$\square + 1 = n$.

We can check that our solution $b_n = 2^n$ satisfies the RR by substituting it into the RR and IV:

$b_1 = 2^1$ $b_n = 2^n$ $b_{n-1} = 2^{n-1}$

$b_n = 2b_{n-1}$

$2^n = 2 \cdot 2^{n-1}$ ✓

$b_1 = 2$ ✓

Example $c_n = c_{n-1} + (n-1)$ $c_1 = 0$.

$n = 2$

$\cancel{c_n} = \cancel{c_{n-1}} + n - 1$

$\cancel{c_{n-1}} = \cancel{c_{n-2}} + n - 2$.

$\cancel{c_{n-2}} = \cancel{c_{n-3}} + n - 3$.

$\vdots$

$\cancel{c_2} = \cancel{c_1} + 1$

$\cancel{c_1} = 0$

Add these equations.

$c_n = n - 1 + n - 2 + n - 3 + \dots + 1 = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$n \to n-1$

$\sum_{i=1}^{n-1} i = \frac{(n-1)(n)}{2}$

Check $c_n = c_{n-1} + n - 1$

$c_n = \frac{n(n-1)}{2}$: $\frac{n(n-1)}{2} = \frac{(n-1)(n-2)}{2} + (n-1) = (n-1)\left[\frac{n-2}{2} + 1\right] = (n-1) \cdot \frac{n}{2}$

Theorem (A different way to solve first order RRs)

Every sequence $x_0, x_1, x_2, \ldots$ satisfying the recurrence $x_n = dx_{n-1}$ has the general solution $x_n = cd^n$ for some constant c. (The sequence is a geometric progression.)

$b_n = 2 \cdot b_{n-1}$

Proof (substitution)

$x_n = d \cdot x_{n-1}$

Sol: $x_n = c \cdot d^n$: $c \cdot d^n = d \cdot (c \cdot d^{n-1})$

$\Rightarrow x_{n-1} = c \cdot d^{n-1}$

This suggests the following general strategy for solving RRs:

(1) Find the **general solution** to the RR. This will have one or more constants. Note: a RR of order k will has k constants.

(2) Use the k initial values to determine the constants. This gives a **unique** solution.

Example. Suppose $x_n = 5x_{n-1}$ and $x_0 = 7$. First find the general solution then the unique solution satisfying $x_0 = 7$.

$x_n = d \cdot 5^n$ for some constand d.

$n=0$: $\left.\begin{array}{l} x_0 = d \cdot 5^0 = d. \\ x_0 = 7. \end{array}\right\} \Rightarrow d = 7.$

$x_n = 7 \cdot 5^n$

Exercise 1. Solve $p_n = np_{n-1}$ where $p_1 = 1$.

$p_n = n \cdot p_{n-1} = n(n-1)p_{n-2} = n(n-1)(n-2) \cdot p_{n-3}$

$= \cdots$

$= n(n-1)(n-2) \cdots 3p_2$

$= n(n-1) \cdots 3 \cdot 2 \cdot p_1$

$= n(n-1) \cdots 3 \cdot 2 \cdot 1$

$= n!$

$p_{n-1} = (n-1) \cdot p_{n-2}$

$p_{n-2} = (n-2) \cdot p_{n-3}$

Exercise 2. Solve $x_n = x_{n-1} + An + B$ for $x_1 = C$.

Q2(a). How many strings of length 4 over $\{A,B,C,D\}$ have BB in them?

any of A,B,C,D

B B _ _ $4 \cdot 4 = 16$ _ B B _ $4 \cdot 4 = 16$ _ _ B B $4 \cdot 4 = 16$

total $= 3 \cdot 16 = 48$.
This is incorrect.

any of A,C,D

B B _ _ $3 \cdot 3 = 9$ _ B B _ $3 \cdot 3 = 9$ _ _ B B $3 \cdot 3 = 9$ 27

A,C,D

B B B _ 3 + B _ B B 3 + _ B B B 3 + B B _ B 3 = 12.

B B B B 1 = 1

40.

Q2 (b) How many ways can three teams of 5 players be selected from 15 players?

$$\binom{15}{5} \cdot \binom{10}{5} \cdot \binom{5}{5} \Big/ 6 \quad (6 = 3!) \text{ ways.}$$

team1 team2 team3

one way	1,3,5,7,9	2,4,6,8,10	11,12,13,14,15
same teams.	2,4,6,8,10	1,3,5,7,9	11,12,…,15

There are 3! ways to permute the 3 teams so we need to divide by $3! = 6$.