Lec 2 Basic Counting Principles Michael Monagan

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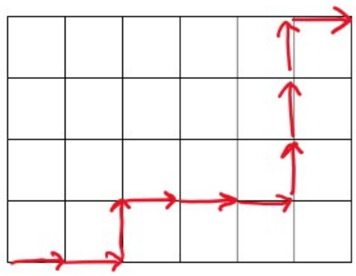
Lecture 2: Basic Counting Principles

I have an office hour tomorrow (Saturday) 8–9pm.

Reading: Grimaldi Sections 1.1, 1.2

Lattice paths arise in theoretical physics.



A lattice path.

How many lattice paths are there from $(0, 0)$ to $(6, 4)$ if we are restricted to North steps and East steps only?

Definition (Rule of Sum)

If there are m ways to perform task X and n ways to perform task Y, there are $m + n$ ways to perform **either** X or Y.

Definition (Rule of Product)

If there are m ways to perform task X and n ways to perform task Y, there are $m\,n$ ways to perform **both** X and Y.

Examples.

10 juices
11 Sodas
9 teas

If I order one drink so a juice or a soda or a tea then I have $10 + 11 + 9 = 30$ choices.

If I order one juice and one soda and one tea then I have $10 \cdot 11 \cdot 9 = 990$ choices.

Exercise. If there are 10 people at a party and all hug each other, how many hugs are there?

$\binom{10}{2} = \frac{10 \cdot 9}{2} = 45$ hugs.

By the rule of product:

There are $10 \cdot 9 = 90$ hugs.

$\boxed{\text{person 1}} \neq \boxed{\text{person 2}}$

10 choices　　9 choices.

This counts $\boxed{\begin{array}{l}\text{Dave hugs Mike} \\ \text{Mike hugs Dave}\end{array}}$ as two hugs.

So $10 \cdot 9 = 90$ counts each hug twice.

So there are $90/2 = 45$ hugs.

Theorem (Strings)

If Σ is an alphabet with k letters, the number of strings of length n over Σ is k^n.

Proof.

k choices　and　k choices　and　k choices　and.　k choices

letter　$\underline{\quad}_1$　$\underline{\quad}_2$　$\underline{\quad}_3$　. . . . .　$\underline{\quad}_n$

By the product rule we have $\underbrace{k \cdot k \cdot k \cdots k}_{n \text{ times}} = k^n$ choices.

Theorem (Permutations)

The number of permutations of a set of n distinct objects is $n!$.

Proof.

n choices and $n-1$ choices and $n-2$ choices ... 1 choices

letters $\underset{1}{\_\_}$ $\underset{2}{\_\_}$ $\underset{3}{\_\_}$ ..... $\underset{n}{\_\_}$

By the rule of product there are $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 = n!$

Definition (Permutations with Repetition)

Suppose there k_1 objects of type A, k_2 of type B, ..., and k_r of type R and let $n = k_1 + k_2 + \cdots + k_n$ r be the total number of objects. The number of distinct permutations is denoted by $\binom{n}{k_1, k_2, \ldots, k_r}$.

Example. Consider the letters M, E, E, N, N. How many permutations are there?

ME ENN
ME NEN
ME NNE

MN EEN
MN ENE
MN NEE

There are 6 permutations that begin with M.
Notice the M can go in positions 1, 2, 3, 4, 5.
There are $5 \cdot 6 = 30$ permutations.

I have enumerated all permutations.

1 M $k_1 = 1$
2 E $k_2 = 2$
2 N $k_3 = 2$
$n = 5$

$$\frac{n!}{k_1!\,k_2!\,k_3!} = \frac{5!}{1!\,2!\,2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2}{4} = 5 \cdot 3 \cdot 2 = 30.$$

Theorem (Permutations with Repetition)

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{k_1!, k_2!, \ldots, k_r!}.$$

Proof. If all n objects were distinct then $n!$ permutations.
Suppose there k_1 A's $A_1, A_2, \ldots, A_{k_1}$ Consider

letters $\underset{1}{\_}\ \underset{2}{A_1}\ \underset{3}{\_}\ \underset{4}{A_2}\ \underset{5}{A_3}\ \underset{6}{\_}\ \underset{7}{\_}\ \ldots\ A_{k_1}\ \_\ \underset{n}{\_}$

There are $k_1!$ permutations of the A's. Since the A's are the same $n!$ overcounts by a factor of $k_1!$ $\quad n!/k_1!$

Hence $n!$ overcounts by a factor of $k_1! \cdot k_2! \cdot k_3! \cdots k_r!$

Therefore the total # of permutations is $\dfrac{n!}{k_1!\, k_2! \cdots k_r!}$

overcounting A's $\longrightarrow k_1!$ $\quad$ $\longleftarrow$ overcounting of B's. ($k_2!$)

Exercise. How many binary strings of length 20 are there with exactly 13 1's?

$n = 20$ $\qquad \Sigma = \{0, 1\}.$

13 1 bits $\qquad$ 7 0 bits

$k_1 = 13$ $\qquad$ $k_2 = 7$

There are $\binom{n}{k_1, k_2} = \dfrac{n!}{k_1!\, k_2!} = \dfrac{20!}{13!\, 7!}$.

notation $\qquad$ formula.

Theorem (Subsets and Combinations)

If S is a set of size n, the number of subsets of size k $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Proof. Let $S = \{a_1, a_2, \ldots, a_n\}$

Each subset A of S corresponds to a binary string of length n $b_1 b_2 \cdots b_n$ where

$a_i \in A \iff b_i = 1$
$a_i \notin A \iff b_i = 0$

E.g. $S = \{a_1, a_2, a_3, a_4, a_5\}$ $n=5$

$A = \{a_2, a_3\}$ $\quad$ $A = \{a_1, a_3, a_4\}$

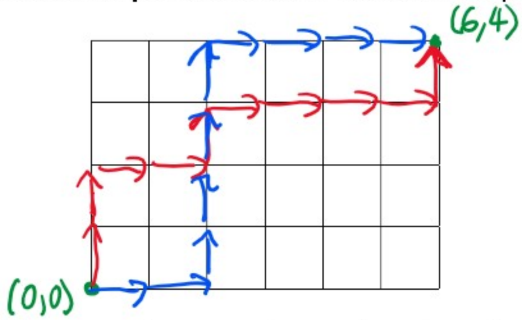
0 1 1 0 0 $\quad$ 1 0 1 1 0

So the # of subsets of of size k [which is denoted $\binom{n}{k}$] is the # of binary strings with k 1 bits. and $n-k$ 0 bits.

which is $\binom{n}{k_1, k_2} = \frac{n!}{k_1!\, k_2!}$

Lattice paths arise in theoretical physics.

This is a lattice path.



NNEENEEEEN $\leftarrow$ 2 letters 6 E's 4 N's.

1 1 0 0 1 0 0 0 0 1

How many lattice paths are there from $(0, 0)$ to $(6, 4)$ if we are restricted to North steps and East steps only?

There must be 6 E steps and 4 N steps for a total of $n = 10$ steps.

The # of lattice paths is $\binom{10}{6,4} = \frac{10!}{6!\,4!} = \frac{10 \cdot \cancel{9}^{3} \cdot \cancel{8} \cdot 7 \cdot \cancel{6!}}{\cancel{4 \cdot 3 \cdot 2} \cdot \cancel{6!}}$

midterm. $= 10 \cdot 3 \cdot 7 = 210.$