Lecture 17 Generating Functions

Grimaldi Chapter 9 Generating Functions

A new powerful way of counting.

The binomial coefficient $\binom{n}{k}$ counts different objects:

$$\begin{aligned}\binom{n}{k} &= \text{the number of subsets of } \{1,2,\ldots,n\} \text{ of size } k \\ &= \text{the number of binary strings of length } n \text{ with } k \text{ 1's} \\ &= \text{the coefficient of } x^k y^{n-k} \text{ in the expansion of } (x+y)^n\end{aligned}$$

Example $(1+x)^3 =$

Definition (coefficient)

If $P(x)$ is a polynomial we denote by $[x^k]P(x)$ the coefficient of x^k in $P(x)$.

Example 1 How many integer solutions $a_1 + a_2 + a_3 = 7$ have if $0 \leq a_i \leq 3$?

Example 2 Suppose we roll two dice. If we add the values of the dice, how many ways can we get 6?

Example 3 How many integer solutions does

$$a_1 + a_2 + a_3 = 9$$

have if $2 \leq a_1 \leq 4, 1 \leq a_2 \leq 5, 3 \leq a_3 \leq 7$?

$[x^9]P(x)$ where $P(x) =$

Exercise: What if a_1 is odd, a_2 is even and $a_3 \in \{0, 3, 6\}$?

$[x^9]P(x)$ where $P(x) =$

Example 4. How many integer solutitions does

$$a_1 + a_2 + a_3 = n \quad \text{have if } a_i \geq 0 \text{ ?}$$

Definition

The **generating function** for an infinite sequence $a_0, a_1, a_2, \ldots$ is the series

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_i x^i.$$

We are interested in the coefficients of $A(x)$ not the values of $A(x)$.
Example 5. What is the generating function for $1, 1, 1, \ldots$?

Example 6. What is the generating function for the sequence $1, 2, 3, 4, 5, \ldots$?