Lecture 22 Michael Monagan

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Lecture 22: The Summation Operator

Grimaldi 9.5

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Problem: $\sum_{i=1}^{n} i^2 = ?$ $\quad \sum_{i=0}^{n} i^3 = ?$ $\quad \sum_{i=1}^{n} i^4 = ?$ etc.

Let $A(x) = a_0 + a_1x + a_2x^2 + \dots$. What is

$$A(x)\frac{1}{1-x} = (a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots)$$

$$= a_0 + (a_0+a_1)x + (a_0+a_1+a_2)x^2 + (a_0+a_1+a_2+a_3)\cdot x^3 + \cdots$$

$$+ (a_0+a_1+\cdots+a_n)x^n + \cdots$$

So $A(x)\frac{1}{1-x}$ generates the sums $a_0, a_0+a_1, a_0+a_1+a_2, \cdots$

We call $\frac{1}{1-x}$ the summation operator.

Example 1. For $A(x) = x + x^2$ find $\frac{A(x)}{(1-x)}$, $\frac{A(x)}{(1-x)^2}$ and $\frac{A(x)}{(1-x)^3}$.

$A(x) = x + x^2$ generates $0, 1, 1, 0, 0, 0, \dots$

$A(x) \cdot \frac{1}{1-x}$ generates $0, 0+1, 0+1+1, 0+1+1+0, 0+1+1+0+0, \dots$

$= 0, 1, 2, 2, 2, 2, \dots$

$A(x) \cdot \frac{1}{1-x}$ generates $0, 0+1, 0+1+2, 0+1+2+2, 0+1+2+2+2, \dots$

$= 0, 1, 3, 5, 7, 9$

$A(x) \frac{1}{1-x}$ generates $0, 0+1, 0+1+3, 0+1+3+5, 0+1+3+5+7, \dots$

$= 0, 1, 4, 9, 16, \dots, n^2$

This says $1 + 3 + 5 + \cdots + \underline{2n-1} = n^2$ or $\sum_{i=1}^{n} (2i-1) = n^2$

Example 2. We will find a formula for

$$\sum_{k=0}^{n} k^2 = 0^2 + 1^2 + 2^2 + 3^2 + \cdots + n^2.$$

$$\frac{A(x)}{(1-x)^3} = \frac{x^2+x}{(1-x)^3} = 0^2 + 1^2 \cdot x + 2^2 \cdot x^2 + 3^2 \cdot x^3 + \cdots + n^2 \cdot x^n$$

$$\frac{A(x)}{(1-x^4)} = \frac{x^2+x}{(1-x)^4} = 0^2 + (0^2+1^2) \cdot x + (0^2+1^2+2^2)x^2 + \cdots + (0^2+1^2+2^2+\dots+n^2) \cdot x^n + \cdots$$

Thus $[x^n] \frac{x^2+x}{(1-x)^4} = 0^2 + 1^2 + 2^2 + \dots + n^2 = \sum_{k=0}^{n} k^2$

PF. $\frac{x^2+x}{(1-x)^4} = \frac{A}{(1-x)} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{(1-x^4)}$

$1 \cdot x^2 + x = \overset{0}{\overset{\shortparallel}{A}}(1-x)^3 + B(1-x)^2 + C(1-x) + D$

Example 2 (cont.)

$$[x^3] \quad 0 = -A \Rightarrow A = 0$$
$$[x^2] \quad 1 = B \Rightarrow B = 1$$
$$x^2 + x = 1\cdot(1-x)^2 + C(1-x) + D$$
$$x=1 \quad 2 = 0 + 0 + D \Rightarrow D = 2.$$
$$\cancel{x^2} + x = 1 - 2x + \cancel{x^2} + C - Cx + 2.$$
$$[x] \quad 1 = -2 - C \Rightarrow -C = 3 \Rightarrow C = -3.$$

Thus $\dfrac{x^2+x}{(1-x)^4} = \dfrac{1}{(1-x)^2} - \dfrac{3}{(1-x)^3} + \dfrac{2}{(1-x)^4}.$

Recall $[x^n]\dfrac{1}{(1-x)^k} = \binom{n+k-1}{n}$

$k=2$ $\quad k=3$ $\quad k=4$

Example 2 (cont.)

$$[x^n]\left(\frac{x^2+x}{(1-x)^4}\right) = \binom{n+1}{n} - 3\binom{n+2}{n} + 2\cdot\binom{n+3}{n} \qquad \binom{n+2}{n} = \frac{(n+2)!}{2!\,n!}$$
$$= n+1 - 3\,\frac{(n+2)\cdot(n+1)}{2} + 2\,\frac{(n+3)(n+2)(n+1)}{6=3!}$$
$$= (n+1)\left[1 - \frac{3(n+2)}{2} + \frac{1}{3}(n+3)(n+2)\right]$$
$$= \ldots\ldots$$
$$\sum_{k=1}^{n} k^2 = \quad = \frac{n(n+1)(2n+1)}{6}$$

Exercise: Use the summation operator to show $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$

Proofs by induction.

Example 1. Show that $\sum_{k=1}^{n} 2k-1 = n^2$ for $n \geq 1$ by induction on n.

$n \to n-1.$

Induction Base: $n=1$. LHS: $\sum_{k=1}^{1} 2k-1 = 2\cdot 1-1 = 1.$ RHS: $1^2 = 1.$

Induction Step $n>1$ Assume $\sum_{k=1}^{n-1} 2k-1 = (n-1)^2$ Induction Hypothesis.

$$\sum_{k=1}^{n} 2k-1 = 1+3+5+\cdots+2(n-1)-1 + 2n-1$$

$$= \sum_{k=1}^{n-1} 2k-1 + 2n-1$$

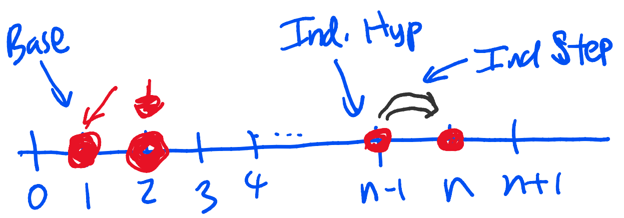
Using the I.H.

$$= (n-1)^2 + 2n-1$$

$$= n^2 \cancel{-2n} \cancel{+1} \cancel{+2n} \cancel{-1}$$

$$= n^2.$$

Example 1 continued.



Principle of math. induction.

By the principle of mathematical induction $\sum_{k=1}^{n} 2k-1 = n^2$ for all $n \geq 1$.

Exercise 1. Show that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ for $n \geq 1$ by induction on n.

Example 2. Show that every integer $n \geq 2$ can be factored into a product of primes.

$n = 12 = 3 \cdot 2 \cdot 2$

Base $n=2$ 2 is a product of primes.

Step $n>2$. Assume the statement is true for $2 \leq x < n$. Ind. Hyp.

Case 1: n is prime. n is a product of primes. e.g. $n=3$.

Case 2: n is not prime.

$\underline{12} = \underline{4} \cdot \underline{3}$ $n = x \cdot y$ where $2 \leq x < n$ and $2 \leq y < n$.

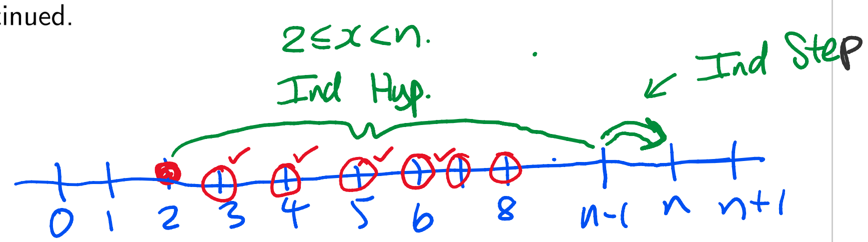
Ind Hyp. $x = p_1 \cdot p_2 \cdot p_3 \cdots p_r$ for some primes p_i

Ind Hyp $y = q_1 \cdot q_2 \cdot q_3 \cdots q_s$ for some primes q_i.

$n = x \cdot y = (p_1 \cdot p_2 \cdots p_r)(q_1 \, q_2 \cdots q_s)$

$=$ a product of primes !!

Example 2 continued.



$4 = 2 \cdot 2$ $6 = 3 \cdot 2$ $8 = 4 \cdot 2$

$3 \cdot 2$ $2 \cdot 2$ 2

By the principle of mathematical induction n factors into a product of primes for $n \geq 2$.

Exercise 2. Let $\alpha = (1+\sqrt{5})/2$ and f_n be the n'th Fibonacci number for $n \geq 0$. Show that $\alpha^2 = \alpha + 1$. Now prove $\alpha^n = \alpha f_n + f_{n-1}$ for $n \geq 1$ by induction on n.