Lec 18 Calculating with GFs, Michael Monagan

March 1, 2021 11:47 AM

Lecture 18 Calculating with Generating Functions

Grimaldi 9.2

Review 2 problems posted.
I'll post the solutions this afternoon.
Midterm 2 is on Monday March 8th.
Same camera position requirement:
must show us your hands and your desktop.

Definition (Generating Function)

Let $a_0, a_1, a_2, a_3, \ldots$ be a sequence of real numbers (or integers). The function

$$A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = \sum_{i=0}^{\infty} a_i x^i$$

is called the **generating function** for the sequence.

Note: we are interested in the coefficients of $A(x)$ not the values of $A(x)$.

All polynomials may be viewed as generating functions.
Example

$$(1+x)^3 = 1 + 3x + 3x^2 + 1x^3$$

$$A(x) = \underset{a_0}{1} + \underset{a_1}{3}\cdot x + \underset{a_2}{3}\cdot x^2 + \underset{a_3}{1}\cdot x^3 + \underset{a_4}{0}\cdot x^4 + 0\cdot x^5 + \cdots$$

Example $A(x) = 0 + 1\cdot x + 1\cdot x^2 + 2\cdot x^3 + 3\cdot x^4 + 5\cdot x^5 + 8\cdot x^6 + \cdots$
is a G.F. for the Fibonacci sequence $0, 1, 1, 2, 3, 5, \ldots$

Example 1. How many ways can we make 30 cents from nickels, dimes and quarters?

no nickels, 1 nickel, 2 nickels, no dimes, 1 dime, 1 quarter

$$A(x) = (1 + x^5 + x^{10} + x^{15} + \cdots)(1 + x^{10} + x^{20} + \cdots)(1 + x^{25} + x^{50} + x^{75} + \cdots)$$

The answer is the coefficient of x^{30} in $A(x) = [x^{30}]A(x)$.

The # of ways to get n cents is $[x^n]A(x)$.

$0+3+1=4$

$1+1+2=4$

Example 2. How many integer solutions does $x_1 + x_2 + x_3 = n$ have if ~~$x_i \geq 0$~~? $x_i \geq 0$?

$n=4$ $\quad x_1=1 \quad x_2=1 \quad x_3=2$

$$A(x) = (\underbrace{1 + x + x^2 + x^3 + \cdots}_{x_1})(\underbrace{1 + x + x^2 + x^3 + \cdots}_{x_2})(\underbrace{1 + x + x^2 + \cdots}_{x_3}).$$

$x_1=0 \quad x_2 \quad x_3$

The answer is $[x^n]A(x) = \binom{n+3-1}{n}$ from Lecture 4 Combos with repetition.

Definition (Arithmetic for Generating Functions)

Let $A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$
and $B(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \ldots$ and c be a constant. Then

$(a_0 b_2 + a_1 \cdot b_1 + a_2 \cdot b_0)x^2$

(1) Sum: $A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots = \sum_{n=0}^{\infty}(a_n + b_n)x^n$.

(2) Scalar product: $cA(x) = ca_0 + ca_1x + \cdots = \sum_{n=0}^{\infty}(ca_n)x^n$.

(3) Product: $A(x) \cdot B(x) = (a_0b_0) + (a_0b_1 + a_1b_0)x + \cdots = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right)x^n$.

(4) Derivative: $A'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots = \sum_{n=1}^{\infty}(na_n)x^{n-1}$.

Note $[x^n]\, A(x) \cdot B(x) = \underbrace{a_0 b_n}_{k=0} + \underbrace{a_1 \cdot b_{n-1}}_{k=1} + \underbrace{a_2 \cdot b_{n-2}}_{k=2} + \cdots + \underbrace{a_n \cdot b_0}_{k=n}$

Example. Let $A(x) = 1 + x + x^2 + x^3 + \ldots$ and $B(x) = 2 + 2x + 2x^2 + 2x^3 + \ldots$.

$2A(x) + B(x) = 2(1+x+x^2+\cdots) + (2+2x+2x^2+\cdots)$
$= (2+2x+2x^2+\cdots) + (2+2x+2x^2+\cdots)$
$= 4+4x+4x^2+\cdots = \sum_{n=0}^{\infty} 4\cdot x^n.$

$A(x)\cdot B(x) = (1+1\cdot x+1\cdot x^2+1\cdot x^3+\cdots)(2+2x+2x^2+2x^3+\cdots)$
$= 2+4x+6x^2+8x^3+\cdots = \sum_{n=0}^{\infty} 2(n+1)x^n.$

$A'(x) = 0+1+2x+3x^2+4\cdot x^3 =$
$= 1+2\cdot x+3\cdot x^2+4\cdot x^3$

$A'(x)$ is the G.F. for the natural numbers $1, 2, 3, 4, \ldots$

What about inverses? Let x be a real number. 1 is the multiplicative identity.
The number 1 has the property $1 \cdot x = x$ for all x. [identity]
If x is non-zero it has an inverse $1/x$ so that $x \cdot \frac{1}{x} = 1$. [inverses] x^{-1}

Let $A(x) = a_0 + a_1x + a_2x^2 + \cdots$

identity $\rightarrow 1\cdot A(x) = 1\cdot a_0 + 1\cdot a_1 x + 1\cdot a_2x^2 + \cdots = a_0 + a_1x + a_2x^2 + \cdots = A(x).$

If $A(x)\cdot B(x) = 1$ we say $B(x)$ is the inverse of $A(x)$
we write $B(x) = 1/A(x)$ or $B(x) = A(x)^{-1}$.

Example 1 Let $A(x) = 1 + x + x^2 + \ldots$ and $B(x) = 1 - x$.
Verify that $A(x) \cdot B(x) = 1$ and conclude that $A(x) = 1/B(x) = 1/(1 - x)$.

$A(x)\cdot B(x) = (1+x+x^2+\cdots)(1-x)$
$= 1 + 0\cdot x + 0\cdot x^2 + \cdots$
$= 1.$

$A(x)\cdot(1-x) = 1 \Rightarrow \boxed{\frac{1}{(1-x)} = (1+x+x^2+\cdots)} = A(x)$

Example 2. Find the inverse of $(1-x)^k$. Find $1/(1-x)^k = \cdots$
Let a_n be the number of solutions of $x_1 + x_2 + \cdots + x_k = n$ where $x_i \geq 0$.

From Lecture 4 $a_n = \binom{n+k-1}{n}$.

Let $A(x) = \frac{1}{(1-x)^k} = \frac{1}{(1-x)} \cdot \frac{1}{(1-x)} \cdot \cdots \cdot \frac{1}{(1-x)}$ [k times]

$= (1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdot\cdots\cdot(1+x+x^2+\cdots)$ k times.

The $[x^n]A(x) = \#$ of solutions to $x_1+x_2+\cdots+x_k = n$ where $x_i \geq 0$

$= a_n = \binom{n+k-1}{n}$.

So $A(x) = \frac{1}{(1-x)^k} = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \cdots = \sum_{n=0}^{\infty} \binom{n+k-1}{n} \cdot x^n$

E.g. $k=3$

$\frac{1}{(1-x)^3} = \underbrace{\binom{0+3-1}{0}}_{n=0} + \underbrace{\binom{1+2}{1} \cdot x}_{n=1} + \underbrace{\binom{2+2}{2} \cdot x^2}_{n=2} + \underbrace{\binom{3+2}{3} \cdot x^3}_{n=3} + \cdots$

$= \binom{2}{0} + \binom{3}{1} \cdot x + \binom{4}{2} \cdot x^2 + \binom{5}{3} \cdot x^3 + \cdots$

$= 1 + 3 \cdot x + 6 \cdot x^2 + 10 \cdot x^3 + \cdots$

The numbers 1, 3, 6, 10, 15, ... are called the triangle numbers.

Example 3 Let $C(x) = 1 + 2x + 4x^2 + 8x^3 + \ldots$ and
$N(x) = 1 + x^5 + x^{10} + x^{15} + \ldots$ (the GF for nickels).
Using $2xC(x)$ and $x^5N(x)$ and the inverse of $C(x)$ and $N(x)$.

$2x\,C(x) = 2x(1 + 2x + 4x^2 + 8x^3 + \cdots) = 2x + 4x^2 + 8x^3 + 16x^4 + \cdots = C(x) - 1$.

$2x\,C(x) = C(x) - 1 \Rightarrow (2x-1)C(x) = -1 \Rightarrow C(x) = \frac{-1}{2x-1} = \frac{1}{1-2x}$.

Therefore $\frac{1}{C(x)} = 1-2x$.

$x^5 N(x) = x^5(1 + x^5 + x^{10} + x^{15} + \cdots) = x^5 + x^{10} + x^{15} + x^{20} + \cdots = N(x) - 1$.

So $x^5 N(x) = N(x) - 1 \Rightarrow (x^5 - 1)N(x) = -1 \Rightarrow N(x) = \frac{1}{1-x^5}$

So the inverse of $N(x)$ is $1/N(x) = 1 - x^5$.

Express $C(x)$ and $N(x)$ in terms of $A(x) = 1 + x + x^2 + x^3 + \cdots = 1/(1-x)$.