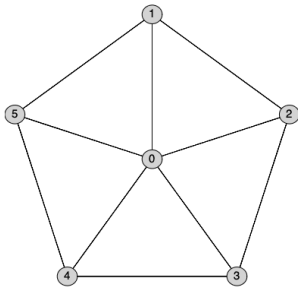# Lecture 5: Counting in Graphs

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Grimaldi 11.1, 11.4 (bipartite)

The Wheel graph W_5.

Problem: How many cycles does W_5 have?

Draw the graph $G = (V, E)$ where $V = \{1, 2, 3, 4, 5\}$ and
$E = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{2, 5\}\}$.

Definition (Bipartite graph)

A graph $G = (V, E)$ is **bipartite** if we can partition the vertices in V into two
non-empty sets V_1 and V_2 such that

(1) $V_1 \cap V_2 = \emptyset$

(2) $V_1 \cup V_2 = V$

(3)   every edge in E is incident with one vertex in V_1 and one vertex in V_2.

Definition ($K_{m,n}$)

For integers $n \geq 1$ and $m \geq 1$ we define the **complete bipartite graph** $K_{m,n}$ to be the bipartite graph with $|V_1| = n$, $|V_2| = m$ and

$$E = \{\{v_1, v_2\} \mid v_1 \in V_1 \text{ and } v_2 \in V_2\}.$$

Example $K_{2,3}$

Question 1: How many edges are in a path on n vertices?
Question 2: How many edges are in a cycle on n vertices?

Question 3: How many edges are in K_n ?
Question 4: How many edges are in $K_{m,n}$?

Question 5: How many graphs are there with n vertices?

Question 6: How many graphs have n vertices and m edges?

Let V_1, V_2 be disjoint sets with $|V_1| = n_1$ and $|V_2| = n_2$.

Question 7: How many graphs have bipartition (V_1, V_2)?

Question 8: How many graphs have bipartition (V_1, V_2) with m edges?

Definition (Subgraph)

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs.
G' is a **subgraph** of G if $V' \subseteq V$ and $E' \subseteq E$.
If $V' = V$ then we call G' a **spanning** subgraph of G.

Example.

Question 9: How many spanning subgraphs does K_{n_1, n_2} have?
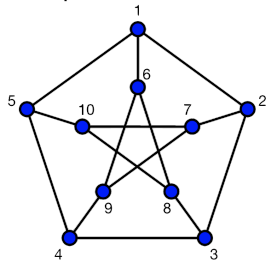Question 10: How many spanning subgraphs of K_{n_1, n_2} have exactly m edges?

Definition (Paths and Cycles)

If P is a subgraph of G that is a path we call P **a path of** G.
If C is a subgraph of G that is a cycle we call C **a cycle of** G.

Example.

Question 11: How many 4-vertex paths does the graph K_n have?

Definition (induced subgraph)

Let $G = (V, E)$ be a graph and let $V' \subseteq V$. The subgraph of G **induced** by V' is the graph $G' = (V', E')$ where

$$E' = \{\{x, y\} \mid x \in V', y \in V' \text{and } \{x, y\} \in E\}.$$

For the graph below determine the induced subgraph for the vertex sets $\{1, 3, 4\}$ and $\{1, 0, 3, 4\}$.