Lecture 6 Graph Isomorphism, Michael Monagan

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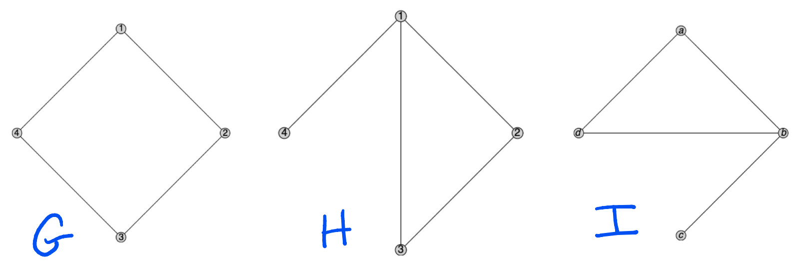
Lecture 6: Graph Isomorphism

Grimaldi 11.2
Which of the following graphs are the "same" ?



G is a cycle. H and I are not cycles.
H and I both have a triangle and one extra edge.
Ignoring vertex labels and how we draw H and I,
H and I are the same graphs.

Definition (isomorphic graphs)

Let $G = (V_1, E_1)$ and $H = (V_2, E_2)$ be two graphs. Then G is **isomorphic** to H (has the same structure as) if there is a bijection $f : V_1 \to V_2$ such that

$$\{u, v\} \in E_1 \iff \{f(u), f(v)\} \in E_2.$$

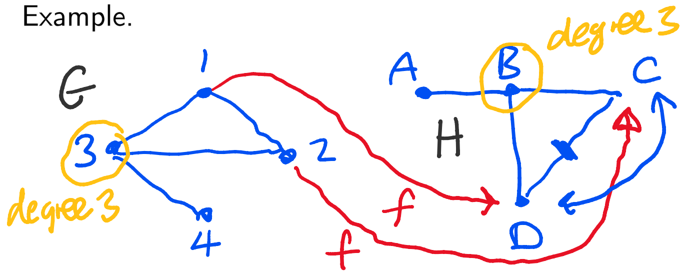
$\Rightarrow |V_1| = |V_2|$

$\Rightarrow |E_1| = |E_2|$

The function f is called an **isomorphism**.

iso = same
morphic = structure

Example.



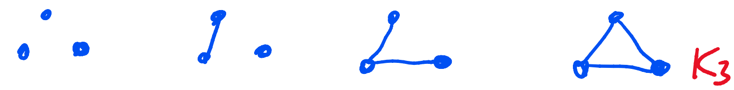
Check $\{1,2\}$
$\{f(1), f(2)\} = \{D, C\} \in H$ ✓

$f(1) = D$	$f(1) = C$
$f(2) = C$	$f(2) = D$
$f(3) = B$	$f(3) = B$
$f(4) = A$ ✓	$f(4) = A$ ✓

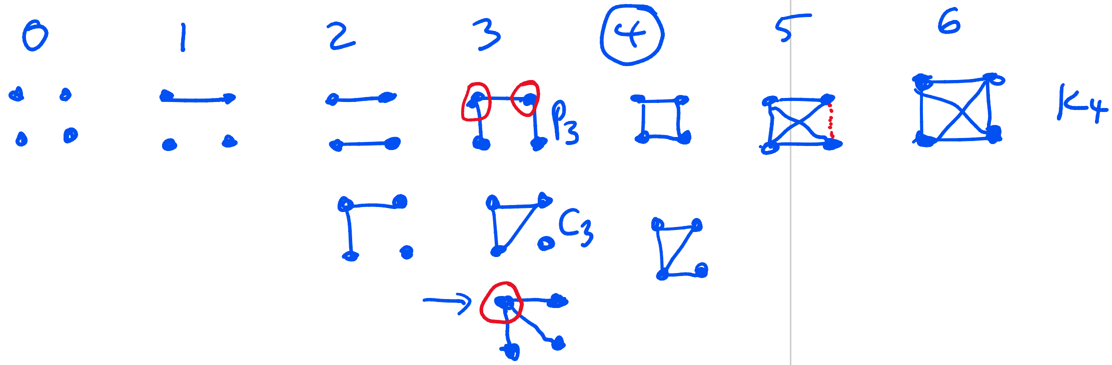
Vertexes 4 and A have degree 1.
Vertexes 3 and B have degree 3.
There are two isomorphisms.

Example. Draw all non-isomorphic graphs with $|V| = 3$ and $|V| = 4$.

$|V| = 3.$



$|V| = 4$



Exercise. Draw all non-isomorphic graphs with 5 vertices and ~~5~~ 4 edges. 6 graphs.

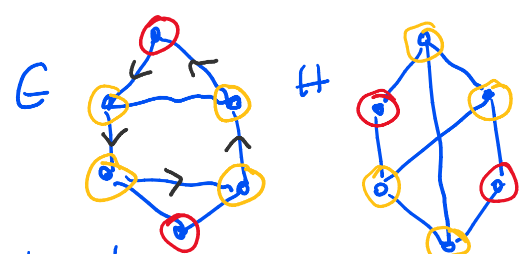
How can we test if two graphs G and H are isomorphic? $n!$

They must have the same properties.

(1) The same # of vertices and edges.

(2) The same vertex degrees

(3) The same # cycles of a given size.



vertices = 6
edges = 8

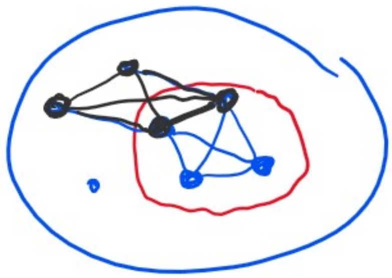
degrees 2, 2, 3, 3, 3, 3.

G has cycles of length 3, 3
H has no cycles of length 3.
G has two copies of C_5
H has none.
G is NOT isomorphic to H.

An "efficient" graph isomorphism algorithm is not known.

Example. For $n \geq t$, how many subgraphs of K_n are isomorphic to K_t ?

K_n
K_7



K_t $t=4$

6 edges.

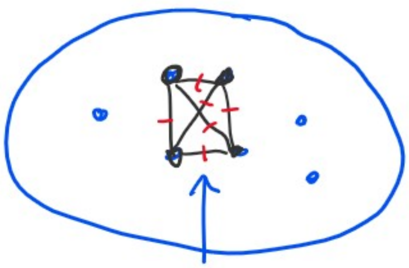
Each selection of $t=4$ vertices from K_n induces a subgraph of K_n which is isomorphic to (a copy of) K_t.

Answer $\binom{n}{t}$

Example. Let K_4^- be K_4 less one edge.
How many subgraphs of K_n are isomorphic to K_4^-?

K_4  K_4^- 

K_n



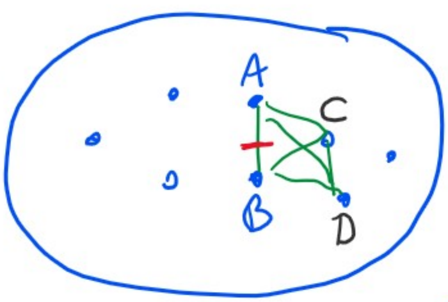
①

Pick any 4 vertices in K_n.
Delete any edge to get K_4^-.

choose 4 vertices   removed 1 edge.

$\boxed{\binom{n}{4} \cdot 6}$

K_n



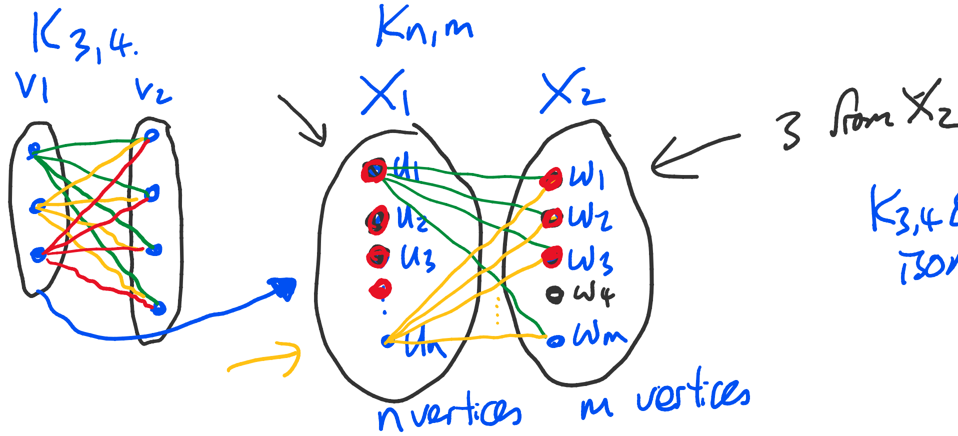
②

Choose A, B in K_n. $\binom{n}{2}$.
Choose C, D in K_n $\binom{n-2}{2}$
Delete edge $\{A, B\}$ to get K_4^-

choose 2   choose 2 more vertices   delete A, B.

This gives $\boxed{\binom{n}{2}\binom{n-2}{2} \cdot 1}$

Exercise. How many subgraphs of $K_{n,m}$ are isomorphic to $K_{3,4}$? $n \geq 3 \quad m \geq 4.$



$K_{3,4}$ & $K_{4,3}$ are isomorphic.

$$\binom{n}{3}\cdot\binom{m}{4} + \binom{n}{4}\cdot\binom{m}{3}$$

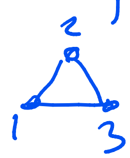
choosing 3 from X_1

choosing 4 vertices from X_2.

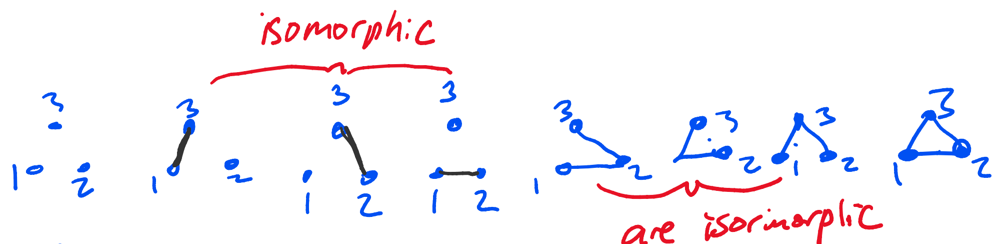
choosing 4 from X_1

choosing 3 from X_2.

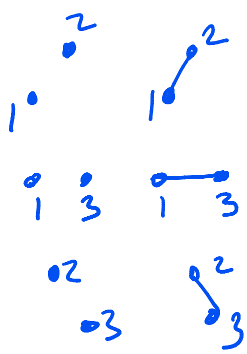
How many subgraphs does K_3 have?

$K_3 =$ 

8 subgraphs with 3 vertices



6 with 2 vertices



3 with 1 vertex