# Lecture 7 Basic Probability  Michael Monagan

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## Lecture 7: Basics of Discrete Probability

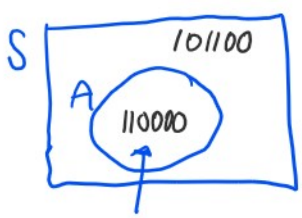
 1cm
Grimaldi 3.4 and 3.5

Suppose we pick a binary string $x$ of length 6 at **random**.
What is the **probability** that $x$ has two 1's in it?



Example 1. Suppose we pick a binary string $x$ of length 6 at **random**. 101100
What is the **probability** that $x$ has two 1's in it?

Let $S$ be the set of all bin. strs. of length 6. $|S| = 2^6 = 64$.
Let $A \subseteq S$ with two 1 bits in them. $|A| = \binom{6}{2} = \frac{6!}{2!4!} = \frac{6\cdot 5}{2} = 15$.




Suppose we pick $x \in S$ at random.
The probability that $x \in A$ is

$$Pr(A) = \frac{|A|}{|S|} = \frac{15}{64}.$$

S is the sample space (set of all possible outcomes).
A is the set of considered outcomes.

## Definition (Probability of an Event)

**Hypothesis:**
$S$ is a **set** of possible outcomes called the **sample space**, all having equal likelihood. Each subset $A \subseteq S$ is called an **event**, i.e. a set of considered outcomes. Each element of $S$ determines an **outcome**.

**Experiment:**
We generate an event by "drawing" at random an outcome $x$ from $S$.
Note: Other words used for "drawing" are "choosing", "selecting" and "picking".

**Event:** Let $Pr(A)$ denote the probability that $x \in A$.

**Question:** What is $Pr(A)$ ?
**Answer:** If each outcome is equally likely and $|S|$ is finite

$$Pr(A) = \frac{|A|}{|S|}$$

Algorithm
⋮
Pick $x \in S$ at random.
⋮

Fundamental Principle. Calculating $Pr(A)$ requires defining the two sets $S$ and $A$. If all outcomes are equally likely, we just need to calculate $|S|$ and $|A|$.



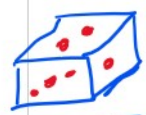
Example 2. What is the probability that a random binary string of size $n \geq 2$ starts with 11 ?

$S$ = set of binary strings of length $n$. $|S| = 2^n$

$A$ = set of bin. strs. that start with 11

$= \{ 1\ 1\ \underbrace{\text{anything}}_{n-2 \text{ bits}} \}$ $|A| = 2^{n-2}$

$Pr(A) = |A|/|S| = 2^{n-2}/2^n = 1/4.$



Example 3. What is the probability that the sum of two rolls of a dice is 7 ?

$S = \{(1,1),(1,2),(2,1),\ldots,(6,6)\}$ $|S| = 36$

$A = \{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}$

$Pr(A) = \frac{|A|}{|S|} = \frac{6}{36} = \frac{1}{6}.$

$\Sigma = \{1,2,3,4,5,6\}$

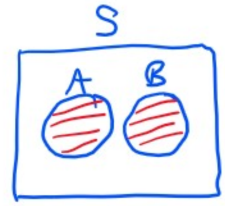
$\underline{\ \ }\ \underline{\ \ }$
6 choices 6 choices.

## Definition (Axioms of Probability)

Let $S$ be a sample space and let $A$ and $B$ be subsets of $S$.

1. $0 \leq Pr(A) \leq 1$
2. $Pr(S) = 1$
3. If $A \cap B = \phi$ then $Pr(A \cup B) = Pr(A) + Pr(B)$.

$A \subseteq S \quad 0 \leq |A| \leq |S| \qquad Pr(A) = \dfrac{|A|}{|S|}$




$Pr(A \cup B) = \dfrac{|A|}{|S|} + \dfrac{|B|}{|S|}$

**Note:** These axioms hold whether the outcomes of $S$ have equal likelihood or not.

## Theorem ( the rule of complement )

*Let $\bar{A} = S - A$ be the* **complement** *of $A$. Then $Pr(\bar{A}) = 1 - Pr(A)$.*

**Proof:**

$1 \overset{(2)}{=} Pr(S) = Pr(A \cup \bar{A}) \overset{(3)}{=} \underline{Pr(A)} + \underline{Pr(\bar{A})}$

$1 = Pr(A) + Pr(\bar{A})$

$\Rightarrow Pr(\bar{A}) = 1 - Pr(A).$


$A \cup \bar{A} = S$

$A \cap \bar{A} = \emptyset$

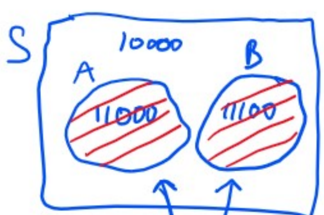


Example 3 (illustrating the third axiom ) What is the probability that a random binary string of size $n \geq 3$ has exactly two 1's or exactly three 1's ?

$S$ = Set of all bin. strs. of length $n \geq 3$. $|S| = 2^n$

Let $A \subset S$ be the bin strs with 2 1 bits. $|A| = \binom{n}{2}$

Let $B \subset S$ " " " " " 3 1 bits. $|B| = \binom{n}{3}$.




$n = 5$ $A \cap B = \emptyset$

$Pr(A \text{ or } B) = Pr(A \cup B)$

because $A \cap B = \emptyset$

$= Pr(A) + Pr(B)$

$= \dfrac{|A|}{|S|} + \dfrac{|B|}{|S|}$

$= \dfrac{\binom{n}{2}}{2^n} + \dfrac{\binom{n}{3}}{2^n}.$

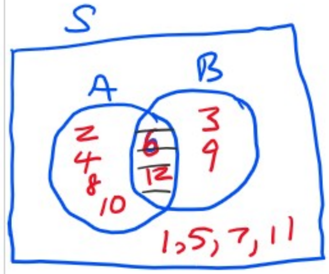
Example 4. Let $S = \{1, 2, 3, \ldots, 12\}$. If $x$ is chosen from $S$ at random what is the probability that $x$ is divisible by 2 OR 3 ?

$S = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \}$

$A = \{ 2, 3, 4, 6, 8, 9, 10, 12 \}$ $\quad Pr(A) = |A|/|S| = 8/12 = \frac{2}{3}.$

Let $A = \{2, 4, 6, 8, 10, 12\}$. $\quad |A| = 12/2 = 6.$

Let $B = \{3, 6, 9, 12\}$ $\quad |B| = 12/3 = 4.$




$$Pr(A \cup B) = \underbrace{Pr(A) + Pr(B)}_{\text{overcounts } A \cap B \text{ twice.}} - \underbrace{Pr(A \cap B)}_{\text{corrects for overcounting.}}$$

$$= \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|}$$

$$= \frac{6}{12} + \frac{4}{12} - \frac{2}{12} = \frac{8}{12} = \frac{2}{3}.$$



## Theorem ( the additive rule )

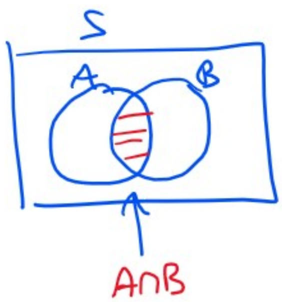
*Let $S$ be a sample space and $A, B \subseteq S$ be two events from $S$. Then*

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B).$$

**Proof:** (Assuming all outcomes in $S$ are equally likely).

$$Pr(A \cup B) = \frac{|A \cup B|}{|S|} = \frac{|A| + |B| - |A \cap B|}{|S|}$$

to avoid overcounting

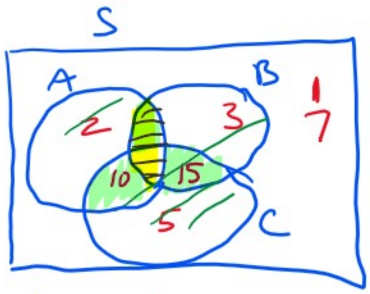



$$= \frac{|A|}{|S|} + \frac{|B|}{|S|} - \frac{|A \cap B|}{|S|}$$

$$= Pr(A) + Pr(B) - Pr(A \cap B)$$

Exercise. Let $S = \{1, 2, 3, \ldots, 60\}$. If $x$ is chosen from $S$ at random what is the probability that $x$ is divisible by 2 **or** divisible by 3 **or** divisible **or** 5 ?
Generalize the additive rule to three subsets $A, B, C$.

Let $A \subset S$ which are divisible by 2. $|A| = 60/2 = 30$
Let $B \subset S$ " " " " 3. $|B| = 60/3 = 20$
Let $C \subset S$ " " " " 5 $|C| = 60/5 = 12$.




$$Pr(A \cup B \cup C) = Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr(A \cap C) - Pr(B \cap C)$$

Almost right.

$$= \frac{|A|}{|S|} + \frac{|B|}{|S|} + \frac{|C|}{|S|} - \frac{|A \cap B|}{|S|}$$

$$= \frac{30}{60}$$

$|A \cap B| =$ divisible by 2 and 3 = divisible by $LCM(2,3) = 6$.