Lecture 9, Random Variables and Expectation

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Lecture 9: Discrete Random Variables

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Grimaldi 3.7 (we will not cover variance)

Assignment #2 due tonight.
Midterm #1 next Monday

Definition (Random Variable)

Let S be a sample space. A **random variable** X on S is a function $X : S \to \mathbb{R}$ that associates a numerical value to each possible outcome.
The **range** $r(X)$ of X is the set of all values it can take.

Example 1. If S is the set of all binary sequences of size $n = 4$. The function that counts the number of 1's is a random variable.

$S = \{0000, 1111, \ldots\}$ $|S| = 2^4$

$X(1011) = 3$ $r(X) = \{0,1,2,3,4\}$

Example 2. If S is the set of all rolls of two dice.
The function that adds the values of the dice is a random variable.

$X(\quad) = 5$ $r(X) = \{2, 3, \ldots, 12\}.$

Example 3. Suppose we throw m balls into n bins randomly.
Let X be the number of empty bins. $n = 5$ bins $m = 5$ balls.

$X(\quad) = 1$ $r(X) = ?$

Definition

Let S be a sample space and X a random variable on S. Let x be a value from the range of X. The probability of x, denoted by

$$Pr(X = x)$$

is the sum of the probabilities of all outcomes s of S such that $X(s) = x$.

Example 1 (cont.)
Let $X(s)$ be the number of 1 bits in a binary string with $n = 4$ bits.
Here $r(X) = \{0, 1, 2, 3, 4\}$

$S = \{0000, \ldots, 1111\}$.

$|S| = 2^4 = 16$

Pr(X = 0) = $1/16$ 0000

Pr(X = 1) = $4/16$ 1000 0100 0010 0001

Pr(X = 2) = $6/16$ 1100 0110 ...... $\binom{4}{2} = 6$.

Pr(X = 3) = $4/16$ 1110 1101 1011 0111 $\binom{4}{3}$

Pr(X = 4) = $1/16$ 1111

Pr(X = k) = $\binom{n}{k}/2^n$

Definition

The **expected value** of a random variable X on a sample space S is defined by

$$E(X) = \sum_{x \in r(X)} x Pr(X = x) = \sum_{s \in S} X(s) Pr(s).$$

$X(1011) = 3$

How many 1 bits do we expect to get?

Example 1 (cont.)

x	0	1	2	3	4 $= n$
$Pr(X = x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$E[X] = \sum_{x \in r(X)} x Pr(X = x) = 0 \cdot \frac{1}{16} + 1 \cdot \frac{4}{14} + 2 \cdot \frac{6}{16} + 3 \cdot \frac{4}{16} + 4 \cdot \frac{1}{16} = \frac{0+4+12+12+4}{16 = |S|} = \frac{32}{16} = 2 = \frac{4}{2}$

$Pr(X = k) = \binom{n}{k}/2^n = |S|$

$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} / 2^n$

$= \frac{1}{2^n} \sum_{k=0}^{n} k \binom{n}{k} = \frac{1}{2^n} n \cdot 2^{n-1} = \boxed{\frac{n}{2}}$

$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

$\frac{d}{dx}$ $n(x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k \cdot x^{k-1} y^{n-k}$.

$x = 1$
$y = 1$ $n \cdot 2^{n-1} = \sum_{k=0}^{n} \binom{n}{k} \cdot k \cdot 1 \cdot 1$

The Geometric Distribution

$1, 2, 3, 4, 5, 6$

$1, 3, 5, 6.$

Reference Example 9.18 on page 428 of Grimaldi $\quad p = \frac{1}{6}$

Example 4. On average, how many times must we roll a fair die before we get a 6?

Let r_i be the i'th roll of the die.

Let $p = Pr(r_i = 6) = 1/6$ and let $q = Pr(r_i \neq 6) = 5/6 = 1 - p.$

Let T be the # rolls till we get a 6.

$Pr(T=1) = 1/6 = p.$

$Pr(T=2) = Pr(r_1 \neq 6 \text{ and } r_2 = 6) =$

If A and B are independent events then

$Pr(A \text{ and } B) = Pr(A \cap B) = Pr(A) \cdot Pr(B)$

$= Pr(r_1 \neq 6) \cdot Pr(r_2 = 6) = 5/6 \cdot 1/6 = \frac{5}{36} = p \cdot q$

$Pr(T=3) = Pr(r_1 \neq 6 \text{ and } r_2 \neq 6 \text{ and } r_3 = 6) =$

$= Pr(r_1 \neq 6) \cdot Pr(r_2 \neq 6) \cdot Pr(r_3 = 6) = q \cdot q \cdot p = pq^2$

Example 4 cont.

$Pr(T=k) = Pr(r_1 \neq 6 \text{ and } \cdots \text{ and } r_{k-1} \neq 6 \text{ and } r_k = 6) = q^{k-1} \cdot p.$

$$E[T] = \sum_{k=1}^{\boxed{\infty}} k \cdot p \cdot q^{k-1} = p \sum_{k=1}^{\infty} k \cdot q^{k-1}$$

Let $A(q) = \sum_{k=1}^{\infty} k q^{k-1} = 1 + 2q + 3q^2 + 4q^3 + \cdots$ where $q = 1 - p.$

$$= \sum_{k=1}^{\infty} \frac{d}{dq} q^k \underset{\text{sum rule}}{=} \frac{d}{dq} \sum_{k=1}^{\infty} q^k = \frac{d}{dq}\left(\frac{1}{1-q} - 1\right) = \frac{1}{(1-q)^2}$$

Recall $\sum_{k=0}^{\infty} q^k = 1 + q + q^2 + \cdots = \frac{1}{1-q}$ for $|q| < 1$. from Math 152.

$$E[T] = p \cdot \frac{1}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$p = \frac{1}{6}$ So $\frac{1}{1/p} = 6.$

Example 4 cont. Alternative method to find $A(q)$

$$A(q) = 1 + 2q + 3q^2 + 4q^3 + \cdots$$
$$qA(q) = 0 + q + 2q^2 + 3q^3 + 4q^4 + \cdots$$
$$\underline{A(q) - qA(q)} = 1 + 1\cdot q + 1q^2 + 1\cdot q^3 + \cdots = \underline{\frac{1}{1-q}}$$
$$A(q)(1-q) = \frac{1}{1-q} \Rightarrow A(q) = \frac{1}{(1-q)^2}$$

Summary: We say T is geometrically distributed with parameter p and $\Pr(T = k) = p(1-p)^k$ for $k \geq 1$ and $E(T) = 1/p$.

The Binomial Distribution

Example 5. Suppose we toss a biased coin n times. $n=100$
Let the probability of getting heads be $p = 0.7$ and tails be $q = 0.3$.
Let H be the number of heads. What is $\Pr(H = k)$ and $E(H)$?

random variable H

$$\Pr(H=0) = q^n$$

T T T . . . T   0.3 · 3 · 3 · · · · · 3

n-1 tails

$$\Pr(H=1) = n\, p \cdot q^{n-1}$$

H T T . . . T

H H H

each toss could be a head   one head

$$\Pr(H=2) = \binom{n}{2} \cdot p^2 \cdot q^{n-2}$$

H H   n-2 tails $\binom{n}{2}$ ways of getting 2 heads.

$$\Pr(H=k) = \binom{n}{k} \cdot p^k \cdot q^{n-k}$$

Th 3.11

$$\underline{E(H)} = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k q^{n-k} = \cdots\cdots = \underline{n \cdot p}$$

Summary: We say X is binomially distributed with parameters p and n and $\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $0 \leq k \leq n$ and $E(X) = np$.

$p = .7,\ n = 100 \quad np = 70.$