Lecture 32 Labelled Trees and Prufer Codes

December 4, 2020 11:04 AM

Lecture 32 Labelled Trees and Prüfer ~~Sequences~~ Codes.

Assignment #8 due Monday.

Grimalid 12.1 Exercise 21 → Worth 3% + 10% of your final grade.

Final exam is next Friday @ 8:30am

Some review problems for assignments 7 and 8 are posted.

Question: How many trees with labels $1, 2, 3, \ldots, n$ are there?
Equivalently, how many spanning trees are there in K_n?

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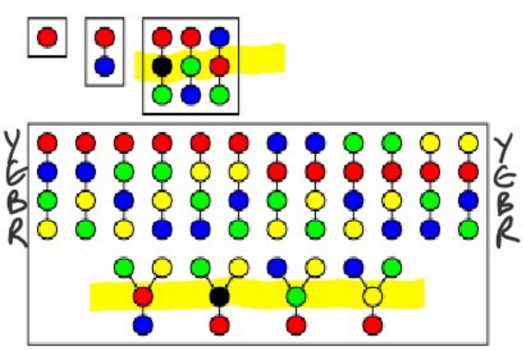
Equivalently, how many spanning trees are there in K_n?
Let T_n be the set of such trees and let $t_n = |T_n|$.

Unlabelled trees.

$n=3$ $n=4$

Labelled trees.



T_1, T_2, T_3, T_4 using colors for labels.
We have $t_1 = 1, t_2 = 1, t_3 = 3, t_4 = 16$.

$n=3$ 180°

1 2 3 = 3 2 1

2 1 3 = 3 1 2

1 3 2 = 2 3 1

$3! = 6$ permutations of 1, 2, 3.

So $t_3 = 3$.

$n=4$ label 1, 2, 3, 4

1 2 3 4

4 3 2 1 180°

$4!/2 = 12$ non-isomorphic unlabelled trees

So $t_4 = 4 + 12 = 16$.

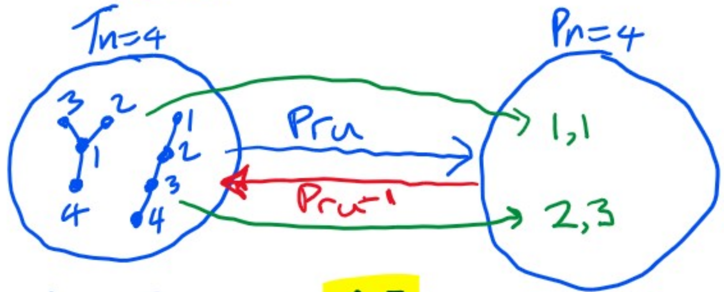
What is t_n?

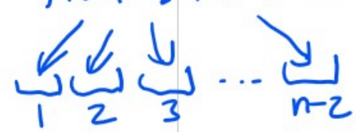
Theorem (Cayley's formula for the number of labelled trees)

The number of spanning trees of K_n is $t_n = n^{n-2}$ for $n \geq 2$.

Proof – Heinz Prüfer, 1918. $n=4$ $t_4 = 4^2 = 16.$

Let T_n be the set of labelled trees on n vertices.
Let P_n be the set of sequences in $V = \{1, 2, \ldots, n\}$ of length $n-2$.
The Prüfer code is a function $Pru : T_n \to P_n$.
We will show that Pru is a bijection hence $|T_n| = |P_n| = n^{n-2}$.





Sequences of length $n-2$ from $\Sigma = \{1,2,\ldots,n\}$

$|P_n| = n^{n-2}.$

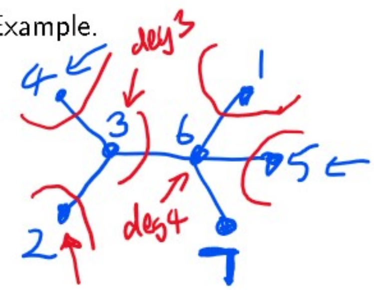
$t_n = |T_n| = |P_n| = n^{n-2}$

See the youtube video on Cayley's Formula by Sarda Herke.

Algorithm $Pru(T)$
Input: A tree T on n vertices.
Output: A Prüfer code $x \in P_n$ of length $n-2$.

1. For $i = 1, 2, \ldots, n-2$ do
   Let u be the leaf in T with smallest vertex label.
   Set x_i to be the unique neighbor of u in T.
   Remove the vertex u and edge $\{u, x_i\}$ from T.
2. Return $(x_1, x_2, \ldots, x_{n-2})$.

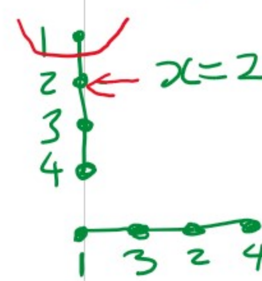
Example.



$n=7.$

$i=1$ $u=1$ $x_1=6$
$i=2$ $u=2$ $x_2=3$
$i=3$ $u=4$ $x_3=3$
$i=4$ $u=3$ $x_4=6$
$i=5$ $u=5$ $x_5=6$
$i=6$ DONE $x = \underline{63366}$

Exercise



$x=2,3.$

$x=3,2$

Note leaf vertices are not in $Pru(T)$.

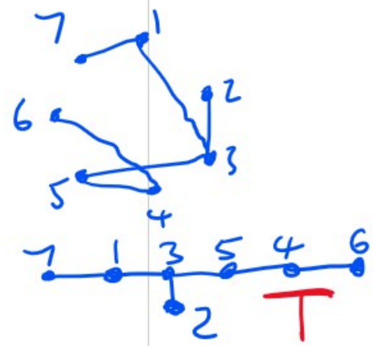
Notice that the number of times a vertex v appears in $Pru(T)$ is $\deg(v) - 1$.

Algorithm Tree($\mathbf{x}$).

Input $V = v_1, v_2, \ldots, v_n$ and a Prüfer code x of length $n-2$ on V.

Output a tree with vertices V

1. Set $L = V$ and $E = \phi$.
2. For $i = 1, 2, \ldots, n-2$ do
   Let y be the first element in L that is not in $x[i..n-2]$.
   Set $E = E \cup \{x_i, y\}$ and remove the vertex y from L.
3. Set $E = E \cup L$.
4. Return the tree (V, E). $V = \{1, 2, \ldots, 7\}$



Example. Determine the tree for the Prüfer code 3, 4, 5, 3, 1.

$x = \cancel{3}\,\cancel{4}\,\cancel{5}\,\cancel{3}\,\cancel{1}$

$L = \{1, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{5}, \cancel{6}, 7\}$

$E = \{\{3,2\}, \{4,6\}, \{5,4\}, \{3,5\}, \{1,3\}, \{1,7\}\}$

$n=7$

$i=1 \quad y=2$

$i=2 \quad y=6$

$i=3 \quad y=4$

$i=4 \quad y=5$

$i=5 \quad y=3$

$i=6 \quad$ DONE

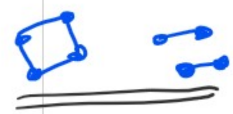
Additional Space.

To finish the proof, we could show $Tree(Pru(T)) = T$. for all labelled trees.

The converse of a statement.

Let G be a graph. ← preamble.

If G has a H.C. then G is connected. ← A necessary condition for G to have a HC.



converse?

Let G be a graph.

If G is connected then G has a H.C. true?

A counter example : $G =$  is connected

but G does not have a H.C.