Lec 4 Combinations with Repetition Michael Monagan

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Lecture 4: Combinations with repetition: Grimaldi 1.4

How many combinations of size 3 are there from $S = \{a, b, c\}$ if repetitions are allowed?

abc } one of each.

aab bba cca
aac bbc ccb } two of one + one

aaa bbb ccc } three of each

10 possibilities

Theorem (combinations with repetitions)

Let S be a set with n elements. The number of ways to select k objects from S, with repetition allowed, is

$$\binom{n+k-1}{k} = \frac{(n+k-1)!}{k!(n-1)!}.$$

Proof with binary strings.

Ex. $n=3$ $S=\{a,b,c\}$ bijection.
$k=7$.

4 a's 2 b's 1 c

aaaa bb c $\longleftrightarrow$ 0000 1 00 1 0 invertible mapping.

aa ccccc $\longleftrightarrow$ 00 1 1 00000

2 a's 0 b's 5 c's.

There is a one-to-one correspondance between the words of length 7 and the binary strings of length 7. The binary strings have k 0 bits and $n-1$ 1 bits. These are bin. strs. of length $k+n-1$ with k 0 bits. The # of them is $\binom{k+n-1}{k} = \binom{n+k-1}{k}$.

Example. How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 10$ with $x_i \geq 0$?

$2+2+1+1+4=10$ $\quad$ $3+4+1+1+1=10$

aa bb c d eeee $\quad$ aaa bbbb c d e

I.e. # of solutions to the equation is the same as choosing $k=10$ letters from $S=\{a,b,c,d,e\}$ of size $n=5$.

The # of solutions is $\binom{n+k-1}{k} = \binom{5+10-1}{10} = \binom{14}{10}$. *

Trap. it's not $\binom{n+k-1}{n} = \binom{5+10-1}{5} = \binom{14}{5} = \binom{14}{9}$

Example. How many integer solutions are there to $x_1 + x_2 \leq 7$ with $x_1 \geq 0$ and $x_2 \geq 0$?

E.g. $2+3 \leq 7$ corresponds to $2+3+2=7$.

Each solution to $x_1+x_2 \leq 7$ is a solution of $\underbrace{x_1+x_2+x_3}_{n=3} = \underbrace{7}_{k}$ where $x_3 = 7 - x_1 - x_2$.

slack Variable.

$n=3$, $k=7$ $\quad \binom{n+k-1}{k} = \binom{3+7-1}{7} = \binom{9}{7}$.

Example. How many ways are their to distribute 5 apples, 4 oranges and 3 pears to three people?

Alice $\quad$ Bob $\quad$ Chris.

Apples.

$n=3$ people $\quad a_1 + a_2 + a_3 = 5$ ← k.

ways $= \binom{n+k-1}{k} = \binom{3+5-1}{5} = \binom{7}{5}$.

Oranges

$k=4$, $n=3$ $\quad \binom{3+4-1}{4} = \binom{6}{4}$

Pears.

$k=3$, $n=3$ $\quad \binom{5}{3}$.

By the rule of product there are $\binom{7}{5} \cdot \binom{6}{4} \cdot \binom{5}{3}$.

Example. Consider the following code segments.
What is the value of counter after the loops have executed ?

```
counter = 0;
for( i=1; i<=20; i++ )
   for( j=1; j<=20; j++ )
      for( k=1; k<=20; k++ )
         counter = counter + 1;
```

$1 \le i \le 20$ 20
$1 \le j \le 20$ $\times 20$
$1 \le k \le 20$ $\times 20$

← how many times is this executed ?? $20^3 = 8000$.

```
counter = 0;
for( i=1; i<=20; i++ )
   for( j=i; j<=20; j++ )
      for( k=j; k<=20; k++ )
         counter = counter + 1;
```

$j \ge i$

$j \le k \le 20$

$\boxed{1 \le i \le j \le k \le 20}$

4 6 9
4 4 9
4 4 4

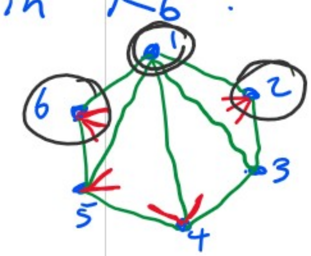
$i \le j \le k$ sorted
can have repetitions.

Counter = the # ways of choosing $k=3$ from $n=20$ objects $\{1,2,\dots,20\}$ with repetition allowed $= \binom{n+k-1}{k} = \binom{20+3-1}{3} = \binom{22}{3} = 1540.$

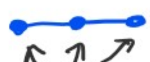
~~Combinatorial arguments~~

~~Example. Let $G = (V, E)$ be a graph with n vertices.~~
~~Two proofs that G can have at most $n(n-1)/2$ edges.~~

How many paths of length 2 edges are in K_6 ?

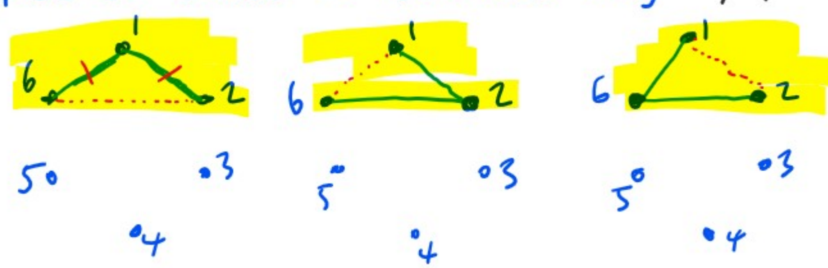


P_3



3 distinct vertices.

Suppose we choose 3 vertices say 1,2,6.



For each choice of 3 vertices in K_6 there are 3 paths of length 2.

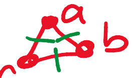
The # paths of length 2 in K_6 is $\binom{6}{3}\cdot 3$ ← # of paths on 3 vertices.

↑ # ways to choose 3 vertices

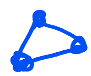
$$\binom{6}{3}\cdot 3 = \frac{6!}{3!\cdot 3!}\cdot 3 = \frac{\cancel{6}\cdot 5\cdot 4\cdot \cancel{3!}}{\cancel{3!}\cdot\cancel{3!}}\,3 = \underline{\underline{60}}$$

$$\binom{6}{3}\cdot 3 = \binom{6}{3}\binom{3}{1}$$

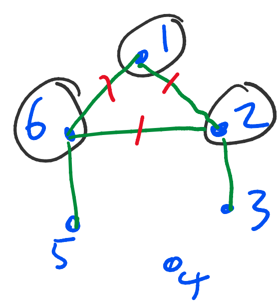
↗ # ways of choosing 3 vertices $\{a,b,c\}$ from 6.

↖ # ways of removing one edge from 

How many cycles of length 3 edges are in K_6?

C_3 

↑ 3 vertices



choose 3 vertices

↓

$$\binom{6}{3}\cdot\binom{3}{3}$$

↑ choose all 3 edges