Lec 19 Generating Functions

March 5, 2021 12:15 PM

Lecture 19 Calculating with Generating Functions cont.

Grimaldi 9.2

$$A(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$$

$$A'(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

We have already seen that the generating function

$$A(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

has a compact represenation as the rational function $\frac{1}{1-x}$. Generating functions which can be compactly represented as rational functions will be our main subject.

Definition

A generating function $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ is **rational** if it can be expressed as

$$\frac{1+x}{1-2x^2} \qquad A(x) = \frac{p(x)}{q(x)} = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$.

(1) Given a sequence of numbers express it as a rational GF ?

(2) Given a rational GF, find the associated sequence (coefficient extraction)

Two useful generating functions

$$A(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

$$A'(x) = 1 + 2x + 3x^2 + \ldots = \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}.$$

Using just these two GF's with basic arithmetic operations gives us the ability to describe many other GF's.

Example 1. Determine the sequence for the GF

$$\frac{x^3-2}{1-x} = \frac{x^3\cdot 1}{1-x} - \frac{2}{1-x} = x^3(1+x+x^2+\cdots) - 2(1+x+x^2+\cdots)$$

$$= x^3 + x^4 + x^5 + \cdots - (2 + 2x + 2x^2 + 2x^3 + \cdots)$$

$$= -2 - 2x - 2x^2 - x^3 - x^4 - \cdots$$

So the sequence is $-2, -2, -2, -1, -1, -1, \ldots$

Example 2. Determine the sequence for the GF

$$\frac{2x^2+5}{(1-x)^2} + 7x = 2x^2 \cdot \frac{1}{(1-x)^2} + 5 \cdot \frac{1}{(1-x)^2} + 7x$$

$$= 2x^2(1 + 2x + 3x^2 + 4x^3 + \cdots) + 5(1 + 2x + 3x^2 + 4x^3 + \cdots) + 7x$$

$$= 5 + (10+7)x + (2 + 5\cdot 3)x^2 + \cdots + \underbrace{(2(n-1) + 5(n+1))}_{7n+3} \cdot x^n + \cdots$$

$$[x^n] = \begin{cases} 5 & \text{for } n = 0 \\ 17 & \text{for } n = 1 \\ 7n+3 & \text{for } n \geq 2. \end{cases}$$

Definition (Substitution)

Let $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$ be a GF and c be a constant. Define

$$A(cx^m) = a_0 + a_1(cx^m) + a_2(cx^m)^2 + a_3(cx^m)^3 \cdots = \sum_{n=0}^{\infty} a_n c^n x^{mn}.$$

Example 1. The GF for nickels is $N(x) = 1 + x^5 + x^{10} + \cdots = \sum_{n=0}^{\infty} x^{5n}$. $= \frac{P(x)}{Q(x)}$.
Express $N(x)$ as a rational function.

no nickels 1nickel 2 nickels.

$A(x) = 1 + x + x^2 + x^3 + \cdots = 1/(1-x)$.
$N(x) = 1 + x^5 + x^{10} + x^{15} + \cdots = 1/(1-x^5)$
$N(x) = A(x^5)$

Example 2. What is the GF for $1, -1, 1, -1, \ldots$?

$B(x) = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$
$A(x) = 1 + x + x^2 + x^3 + \cdots = 1/(1-x)$
$B(x) = A(-x) = 1 - x + (-x)^2 + (-x)^3 = 1/(1-(-x)) = \frac{1}{1+x}$.

Example 3. Express $C(x) = 1 - 2x + 4x^2 - 8x^3 + 16x^4 - \ldots$ as a rational function.

$A(x) = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$.
$A(-2x) = 1 - 2x + (-2x)^2 + (-2x)^3 + (-2x^4) + \cdots$
$= 1 - 2x + 4x^2 - 8x^3 + 16x^4 - \cdots = C(x)$.
$C(x) = A(-2x) = \frac{1}{1-(-2x)} = \frac{1}{1+2x}$.

Exercise. Find a rational GF for the sequence $1, -2, 3, -4, 5, -6, \ldots$?

$A(x) = 1 + x + x^2 + x^3 + \cdots = 1/(1-x)$

$A'(x) = 1 + 2x + 3x^2 + \cdots = 1/(1-x)^2$.

$B(x) = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots = A'(-x)$

$= 1/(1-(-x))^2 = \dfrac{1}{(1+x)^2}$

Example 4. Express $D(x) = -x + 2x^2 - 3x^3 + 4x^4 - \ldots$ as a rational function.

$D(x) = -x\,[1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots]$

$= -x \cdot B(x)$

$= \dfrac{-x}{(1+x)^2}$

Using substitution and our two basic GF's $A(x) = 1 + x + x^2 + \ldots$ and $A'(x)$ ~~we~~ $= 1 + 2x + 3x^2 + \cdots$ can now determine the coefficients for any GF that has the form $\dfrac{1}{(1-x)}$ $= 1/(1-x)^2$.

$$\frac{p(x)}{ax+b} \text{ or } \frac{p(x)}{(ax+b)^2}$$

Problem 1. Find the coefficient of x^k in the GF

$C(x) = \dfrac{x^2/3}{(2x+3)/3} = \dfrac{x^2}{3} \cdot \left(\dfrac{1}{1+\frac{2}{3}x}\right)$

$A(-\tfrac{2}{3}x) = 1 - \tfrac{2}{3}x + (\tfrac{2}{3})^2 x^2 - \cdots (-1)^n (\tfrac{2}{3})^n x^n + \cdots = \dfrac{1}{1+\frac{2}{3}x}$

$C(x) = \dfrac{x^2}{3} \cdot [1 - \tfrac{2}{3}x + \tfrac{2^2}{3^2}x^2 - \cdots (-1)^n (\tfrac{2}{3})^n x^n + \cdots]$

$[x^k]\,C(x) = \tfrac{1}{3} \cdot (-1)^{k-2} (\tfrac{2}{3})^{k-2} = \begin{cases} \tfrac{1}{3}(-1)^k (\tfrac{2}{3})^{k-2} & \text{for } k \geq 2 \\ 0 & \text{for } k = 0, 1 \end{cases}$

Problem 2. Find the coefficient of x^k in the GF

$$D(x) = \frac{x^2}{(x+2)^2} = x^2 \cdot \frac{1/4}{(x+2)^2/4} = \frac{x^2}{4} \cdot \frac{1}{(1+\frac{1}{2}x)^2}$$

$$A'(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

$$A'(-\tfrac{x}{2}) = 1 - 1 \cdot x + \frac{3}{2^2} \cdot x^2 - 4\frac{x^3}{2^3} + \cdots + (-1)^n \frac{(n+1)}{2^n} x^n + \cdots$$

$$D(x) = \frac{x^2}{4} A'(-\tfrac{x}{2}) = \frac{x^2}{4} - \frac{x^3}{4} + \frac{1}{4} x^4 \frac{3}{2^2} - \cdots + \frac{(-1)^{n-2}(n-1)}{4 \cdot 2^{n-2}} x^n + \cdots$$

$$[x^k]\,D(x) = \begin{cases} \dfrac{(-1)^k (k-1)}{2^k} & \text{for } k \geq 2. \\ 0 & \text{for } k = 0, 1. \end{cases}$$

Check. $k=3$. $\frac{(-1)^3 \cdot 2}{2^3} = -\frac{1}{4}.$