Lecture 28 Trees and Rooted Trees
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Lecture 28 Trees and Rooted Trees

Grimaldi 12.1, 12.2

Assignment # 7 due tonight.

Definition (tree)

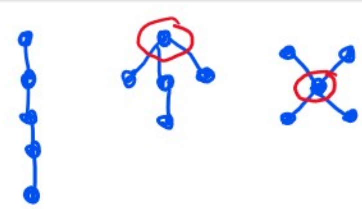
A multigraph G is a **tree** if G is connected and G does not contain a cycle.

Theorem (main properties of trees)

If $T = (V, E)$ is a tree then $|V| = |E| + 1$ and secondly, there is a unique path in T between every pair of vertices.

Examples

Trees with 5 vertices



$|E| = 4$

Removing an edge disconnects a tree.
Adding an edge creates a cycle.

Theorem (Characterization of Trees)

Let $G = (V, E)$ be a multigraph. The following statements are equivalent.

(1) *G is connected and has no cycle. (G is a tree)*

(2) *G is connected and $|V| = |E| + 1$.*

(3) *G has no cycle and $|V| = |E| + 1$.*

(4) *There is a unique path between every pair of vertices in G.*

Proof. We will prove $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$

We have proven $(1) \Rightarrow (4)$. Exercise: $(4) \Rightarrow (1)$.

$(1) \Rightarrow (2)$. Given G is connected and G has no cycle.

To prove: G is connected ✓ and $|V| = |E| + 1$ ✓ (last day).

$(2) \Rightarrow (3)$ Given G is connected and $|V| = |E| + 1$.

To prove: G has no cycle and $|V| = |E| + 1$ ✓

Let's use induction on $n = |V|$.

Base $|V| = 1$. Applying $|V| = |E| + 1 \Rightarrow |E| = 0$.

This means $G = \bullet$ which has no cycle.

Proof (cont). Ind. Step $n > 1$. Ind. Hyp. Assume $(2) \Rightarrow (3)$ for G with $|V| < n$.

If $|V| = |E| + 1$ a Lemma from last day Said G must have a vertex of degree 0 or ≥ 2 vertexes of degree 1.

But $|V| > 1$ and G is connected $\Rightarrow$ G has a leaf vertex v.

Observe $G - v$ is connected (because G is connected) and $G - v$ satisfies ($|V| = |E| + 1$ in G) $|V| - 1 = |E| - 1 + 1$. By the Ind. Hypothesis $G - v$ has no cycle. This implies G has no cycle since v is a leaf.

$(3) \Rightarrow (1)$ Given G has no cycle and $|V| = |E| + 1$.

To prove: G is connected and G has no cycle ✓

Since G has no cycle it is a forest with $k \geq 1$ trees.

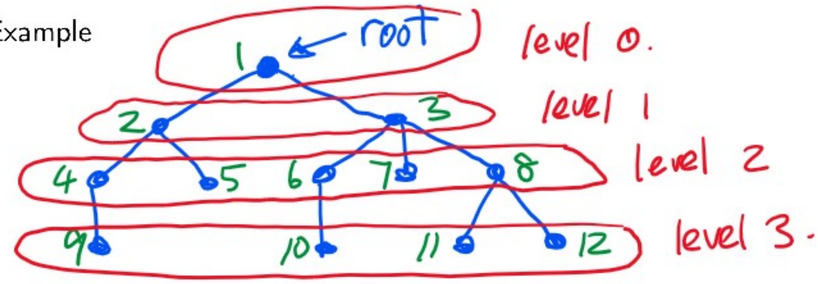
Thus $|V| = |E| + k$ by another Lemma from last day.

Also $|V| = |E| + 1$. $\Rightarrow k = 1 \Rightarrow G$ has 1 connected component. $\Rightarrow G$ is connected.

Definition (rooted tree) 12.2

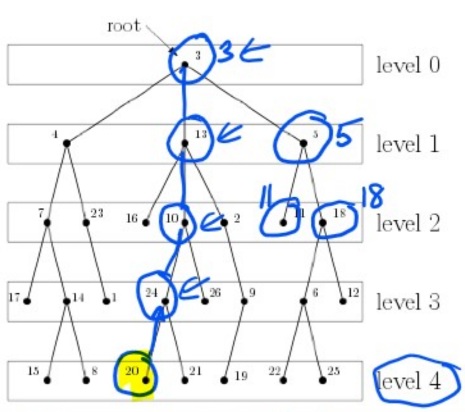
A **rooted tree** $T = (V, E)$ is a tree with a distinguished vertex called the **root**.
For every vertex $v \in V$ the **level** of v is the length of the path from v to the root.
Note: the root is the unique vertex at level 0.

Example



Definition (rooted tree terminology)

- The **height** of a rooted tree is the maximum level of a vertex.
  A rooted tree consisting of just a root vertex has height 0.
- Every non-root vertex v at level i is adjacent to exactly one vertex u at level $i-1$. We call u the **parent** of v and we say that v is a **child** of u.
- For every vertex v there is a walk "up the tree" to the root obtained by moving to the parent vertex at each step. If u is another vertex on this walk, we call u an **ancestor** of v and v a **descendant** of u.



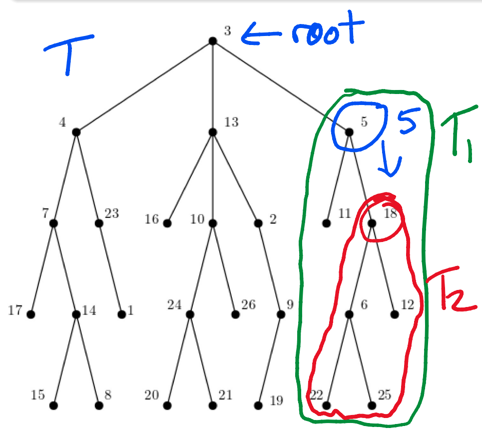
Height = 4

5 is the parent of vertex 18
18 is a child of vertex 5

We are frequently interested in working with rooted trees recursively. Therefore, it will be helpful to think of a rooted tree as composed out of smaller rooted trees.

Definition (subtree)

Let v be a vertex of a rooted tree T with level i. Define T' to be the subgraph of T induced by v together with its descendants. Then T' forms a new rooted tree with root vertex v. We say that T' with root v is the **subtree** of T at v.



T_1 is a subtree of T
T_2 is a subtree of T_1.

Definition (isomorphism of rooted trees)

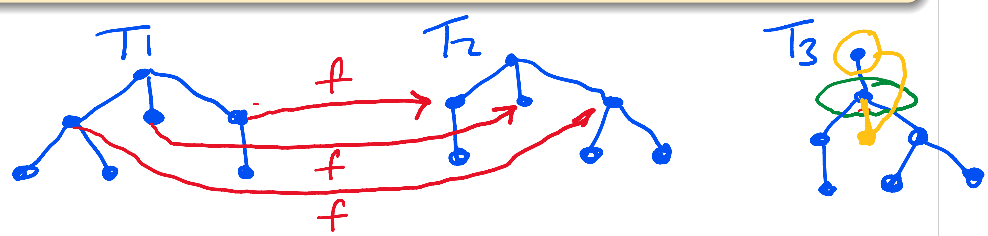
Let T_1, T_2 be rooted trees with $T_i = (V_i, E_i)$ for $i = 1, 2$. We say that T_1 and T_2 are **isomorphic** if there exists a bijection $f : V_1 \to V_2$ satisfying:

(1) $\{f(u), f(v)\} \in E_2 \Leftrightarrow \{u, v\} \in E_1$

(2) For every $v \in V_1$ the level of v and $f(v)$ is the same.
In particular, f sends the root of T_1 to the root of T_2.

root is at level 0.

Example.



$|V| = 7$
$|E| = 6$.

T_1 is isomorphic to T_2 as a rooted tree but not to T_3 because the degree of the root vertexes differ.
Notice T_1, T_2, T_3 are isomorphic as trees.

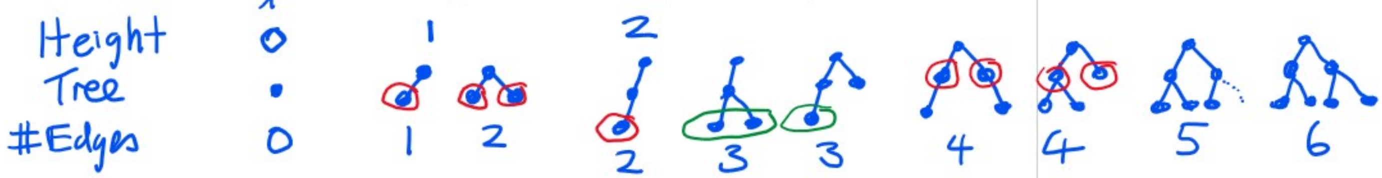
Definition

non-leaf vertex

A rooted tree is m-**ary** if every internal node has at most m children. A 2-ary tree is called **binary** tree.

rooted

Exercise. Find all binary trees with height 0, 1, and 2 up to isomorphism.



Let b_n denote the number of binary trees of height at most n. Find b_0, b_1, b_2.

$b_0 = 1 \quad b_1 = b_0 + 2 = 3 \quad b_2 = b_1 + 7 = 3 + 7 = 10.$

Use the recursive structure of rooted trees find a recurrence for b_n. (height at most n)

We will remove the root vertex which leaves at most two binary trees of less height.

Case 0 children 1 child 2 children

In Case 1 there are b_{n-1} binary trees for C

In Case 2 there are $\binom{b_{n-1}}{2}$ ways to choose two distinct binary trees for A and B and b_{n-1} ways to chose the same one.

Hence $b_n = 1 + b_{n-1} + \left(\binom{b_{n-1}}{2} + b_{n-1}\right).$

Check: $b_2 = 1 + b_1 + \binom{b_1}{2} + b_1$
$= 1 + 3 + \binom{3}{2} + 3 = 1 + 3 + 3 + 3 = 10.$ ✓