Lecture 9: Discrete Random Variables

Grimaldi 3.7 (we will not cover variance)

Definition (Random Variable)

Let S be a sample space. A **random variable** X on S is a function $X : S \to \mathbb{R}$ that associates a numerical value to each possible outcome.
The **range** $r(X)$ of X is the set of all values it can take.

Example 1. If S is the set of all binary sequences of size $n = 4$.
The function that counts the number of 1's is a random variable.

Example 2. If S is the set of all rolls of two dice.
The function that adds the values of the dice is a random variable.

Example 3. Suppose we throw m balls into n bins randomly.
Let X be the number of empty bins.

Definition

Let S be a sample space and X a random variable on S. Let x be a value from the range of X. The probability of x, denoted by

$$Pr(X = x)$$

is the sum of the probabilities of all outcomes s of S such that $X(s) = x$.

Example 1 (cont.)
Let $X(s)$ be the number of 1 bits in a binary string with $n = 4$ bits.
Here $r(X) = \{0, 1, 2, 3, 4\}$

Pr(X = 0) =

Pr(X = 1) =

Pr(X = 2) =

Pr(X = 3) =

Pr(X = 4) =

Pr(X = k) =

Definition

The **expected value** of a random variable X on a sample space S is defined by

$$E(X) = \sum_{x \in r(X)} xPr(X = x) = \sum_{s \in S} X(s)Pr(s).$$

Example 1 (cont.)

x	0	1	2	3	4
$Pr(X = x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$$E[X] = \sum_{x \in r(X)} xPr(X = x) =$$

The Geometric Distribution

Reference Example 9.18 on page 428 of Grimaldi

Example 4. On average, how many times must we roll a fair die before we get a 6?

Example 4 cont.

Example 4 cont.

Summary: We say T is geometrically distributed with parameter p and $\Pr(T = k) = p(1-p)^k$ for $k \geq 1$ and $E(T) = 1/p$.

The Binomial Distribution

Example 5. Suppose we toss a biased coin n times.
Let the probability of getting heads be $p = 0.7$ and tails be $q = 0.3$.
Let H be the number of heads. What is $\Pr(H = k)$ and $E(H)$?

Summary: We say X is binomially distributed with parameters p and n and $\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $0 \le k \le n$ and $E(X) = np$.