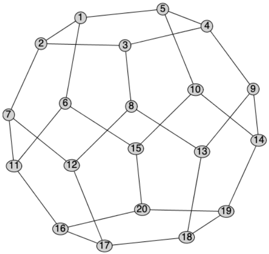
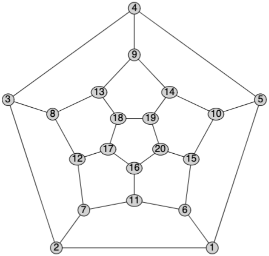
Lecture 25: Planar Graphs

Grimaldi 11.4





These are both drawings of the same graph. To see this locate the cycles $1-2-3-4-5-1$ and $16-17-18-19-20$ in both graphs.

Definition (planar graph)

A graph G is **planar** if G has a drawing (in the plane) so that the edges intersect only at the vertices of G. Such a drawing is called a **planar embedding** of G.

Examples

Observation: The graph $K_{3,3}$ is not planar.
Proof sketch (we will give a formal proof next day)

Observation: The graph K_5 is not planar.
Proof sketch (we will give a formal proof next day)

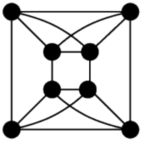
Definition (subdivision)

Let $G = (V, E)$ be a multigraph and let $e = \{u, v\}$ be an edge in E. To **subdivide** the edge e is to delete e and add a new vertex w and two new edges $e_1 = \{u, w\}$ and $e_2 = \{w, v\}$ to G. If the graph H is obtained from G by a sequence of subdivions, then H is called a subdivison of G.

Example

Observation. If H is a subdivision of G then H is planar if and only if G is planar. This means that every subdivision of $K_{3,3}$ and K_5 is nonplanar.

Example: Is this graph planar? I.e. can you find a planar embedding?



Exercise: find a subdivision of $K_{3,3}$ in the graph.

Question: Which graphs are planar ?

Definition

Let G and H be multigraphs. We say that G **contains a subdivision** of H if there is a subgraph of G isomorphic to some subdivision of H.

Theorem (Kuratowski-Wagner)

A multigraph G is planar if and only if G does not contain a subdivision of $K_{3,3}$ or a subdivision of K_5.

Notes.

Definition (Faces)

Let G be a planar graph embedded in the plane. The embedding partitions the plane into connected regions called **faces**. There is one unbounded region called the **infinite face**. All other faces are **internal faces**. If G is connected, every face has vertices and edges on its boundary. They form a closed walk called a **facial walk**

Example

Theorem (Euler's formula)

If $G = (V, E)$ is an connected multigraph embedded in the plane and F is the set of faces, then

$$|V| - |E| + |F| = 2.$$

This implies all embeddings of a planar graph have the same number of faces.

Example

Proof.

Proof (cont.)

Extra space.