Lec 28 Trees Michael Monagan

March 31, 2021 11:36 AM

Lecture 28: Trees

Grimaldi 12.1

Friday is a holiday (Good Friday)
Monday is a holiday. (Easter Monday)

Assignment #7 due Tuesday @ 11pm.

Final Exam — 40% of grade.
Half on A7 and A8 — 20% of grade.
Half on A1 — A6.

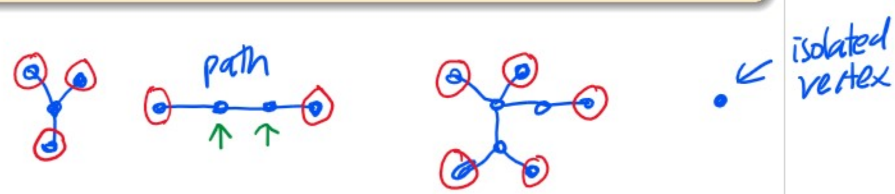
Midterm #3 Average 61.5
Median 62.5

Definition (tree, forest and leaf)

Let $G = (V, E)$ be a multigraph. G is a **tree** if G is connected and G does not contain a cycle. G is a **forest** if G does not contain a cycle. A vertex of degree 1 is called **leaf** or **pendant vertex**.

Examples

Trees



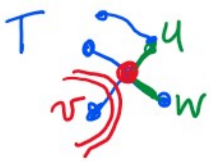
Forrests

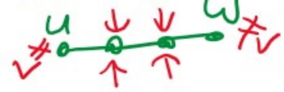
Since a tree cannot have loops or parallel edges, it is a simple graph.
We previously showed that every graph with all vertices of degree ≥ 2 must have a cycle. Therefor, evey tree with ≥ 2 vertices must contain a leaf. Later we will see that trees have ≥ 2 leaves.

Lemma

If $T = (V, E)$ is a tree with leaf v then $T - v$ is a tree.

Proof We must show $T-v$ is connected and acyclic.
First observe if $T-v$ has a cycle then T has a cycle which contradicts T is a tree.
So $T-v$ cannot have a cycle.



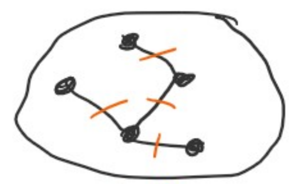


Let $u, w \in V$ with $\boxed{u \neq w \neq v.}$ There is a path in T from u to w as T is connected. The only two vertices of degree 1 on that path are u and w. All other vertices on the path have degree ≥ 2 in T. So v (has degree 1) is not on the path from u to w. Therefore in $T-v$ there is a path from u to w. So $T-v$ is connected.

This observation gives us a powerful tool for proving properties of trees. Try using induction on the number of vertices and, for the inductive step, deleting a leaf then applying the inductive hypothesis. to $T-v$.

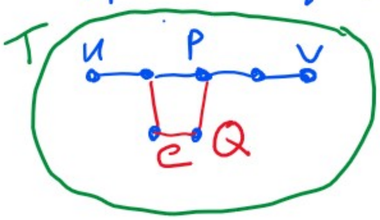
Theorem (unique paths)

If $T = (V, E)$ is a tree and $u, v \in V$ are distinct, there is a unique path in T with ends u, v. exactly one path.



Proof. (is a path from u to v) By definition a tree is connected so there is a path from u to v.

(path is unique) Let P be a path from u to v. Towards a contradiction suppose there is another path Q from u to v. Since $Q \neq P$ there must be at least one edge $e \in E$ that is on Q but not on P. Notice $T-e$ is connected. Therefore e is in a cycle in T. This contradicts "T is a tree". Therefore there is at most one path from u to v in T.



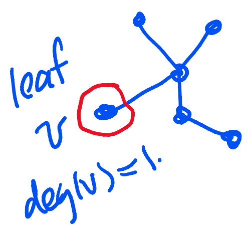
Theorem (main property of trees)

If $T = (V, E)$ *is a tree then* $|V| = |E| + 1$. $|E| = |V| - 1$.
If $G = (V, E)$ *is a forest with* k *trees then* $|V| = |E| + k$.

Proof. By induction on $|V|$. Let $n = |V|$ in T.

Base: $n = 1$ T is the singleton vertex •
Here $|V| = 1$, $|E| = 0$ and $|V| = |E| + 1$?

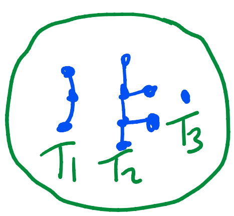
Step $n \geq 2$: Ind. Hypothesis. Assume $|V| = |E| + 1$ holds for any tree with $|V| < n$ vertices. Let v be a leaf vertex in T. Then $T - v$ is a tree with $|V| - 1$ vertices and $|E| - 1$ edges. By the Ind. Hyp. the tree



$T - v$ satisfies $(|V| - 1) = (|E| - 1) + 1$
$\Rightarrow \quad |V| = |E| + 1.$

Proof (cont). By induction on n, $|V| = |E| + 1$ for all trees with $n \geq 1$ vertices.

Forests

$G =$ 

Let $T_1, T_2, \ldots, T_k$ be the trees in G where $T_i = (V_i, E_i)$.
By the first part of the theorem.

$$|V_i| = |E_i| + 1 \quad \text{for } 1 \leq i \leq k.$$

Adding these k equations gives

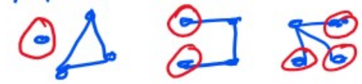
$$\underbrace{|V_1| + |V_2| + \cdots + |V_k|}_{= |V|} = \underbrace{(|E_1| + 1) + (|E_2| + 1) + \cdots + (|E_k| + 1)}_{= |E| + k}.$$

$\Rightarrow \quad |V| = |E| + k$

Lemma

If $G=(V,E)$ satisfies $|V|=|E|+1$ then G must have a vertex of degree 0 or at least two of degree 1.

$|V|=4$ $|E|=3$



Proof. Let $n=|V|$ and let k_0 be the # of vertices of degree 0
and k_1 " " " " " " " 1.

We want to show $k_0 \geq 1$ or $k_1 \geq 2$.

Consider

$$2|E| = \sum_{v \in V} \deg(v)$$

vertices have degree ≥ 2

$$2|E| = 2(|V|-1) \geq k_0 \cdot 0 + k_1 \cdot 1 + 2(\underbrace{n-k_0-k_1}_{\#\text{vertices of degree} \geq 2})$$

$$\Rightarrow \quad 2(\cancel{n}-1) \geq k_1 + \cancel{2n} - 2k_0 - 2k_1$$

$$\Rightarrow \quad -2 \geq -2k_0 - k_1$$

$$\Rightarrow \quad \underbrace{2k_0}_{\geq 0} + \underbrace{k_1}_{\geq 0} \geq 2. \quad \Rightarrow k_0 \geq 1 \text{ or } k_1 \geq 2$$

$\Rightarrow$ # singletons ≥ 1 OR #leaves ≥ 2.

Lemma

Every tree $T=(V,E)$ with $|V| \geq 2$ has at least two leaves.

Proof. T is a tree $\Rightarrow$ T is connected
and $T \neq \bullet$ $\Rightarrow$ every vertex has degree ≥ 1.
$\Rightarrow$ no vertices have degree 0
$\Rightarrow$ there are ≥ 2 vertices of degree 1 by previous Lemma.
$\Rightarrow$ T has ≥ 2 leaf vertices.

Exercise. Let $T=(V,E)$ be a tree with $|V| \geq 2$. So $T \neq \bullet$
Show that removing any edge disconnects T.