Lecture 21: Solving Recurrences using Generating Functions

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Given a_0, the recurrence $a_n = 3a_{n-1} + 1$ defines a sequence

$$a_0, a_1, a_2, \ldots, a_n, \ldots$$

which in turn defines the generating function

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \ldots$$

If we find a rational form for $A(x)$, that is

$$A(x) = \frac{p(x)}{q(x)}$$

for polynomials $p(x)$ and $q(x)$, then we can use partial fractions to get a formula for $a_n = [x^n]A(x)$.

Example 1. Solve $a_n - 3a_{n-1} = n$ for $n \geq 1$ and $a_0 = 1$.
The recurrence relation represents an infinite set of equations.

Multiply equation (k) be x^k we get

Adding all equations up gives

Now plug in a_0 and isolate $A(x)$.

Now we do a PDF and get a formula for a_n.

Example 2. Consider the sequence defined by $a_0 = 0, a_1 = 1$ and

$$a_n - 5a_{n-1} + 6a_{n-2} = 0 \text{ for } n \geq 2.$$

Find a rational expression for $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

Example 2 (cont.)

Method.

Let $a_0, a_1, a_2, \dots$ be a sequence satisfying a recurrence

$$c_n a_n + c_{n-1} a_{n-1} + \cdots + c_k a_{n-k} = f(n).$$

Let $A(x) = a_0 + a_1 x + a_2 x^2 + \dots$

(1) Multiply the recurrence by $x^k, x^{k+1}, \dots$ and sum both sides to infinity.

(2) Rewrite the infinite sums on the LHS in terms of $A(x)$ and the sum on the RHS as a rational function.

(3) Isolate $A(x)$ and use partial fractions to calculate $a_n = [x^n]A(x)$.

Problem. Consider the sequence defined by

$$a_0 = 2, a_1 = 3 \text{ and } a_n - 4a_{n-1} + 4a_{n-2} = 2^n \text{ for } n \geq 2.$$

Solve the recurrence using a generating function.

Problem (cont.)