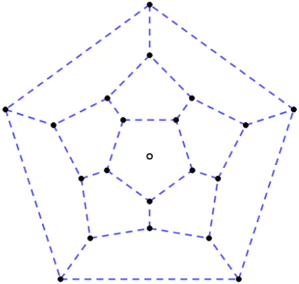
Lecture 27: Hamiltonian Paths and Cycles

Grimaldi 11.5

In 1856 a mathematician William Hamilton invented a game in which the ojbect is to find a cycle along the edges of a dodecahedron.



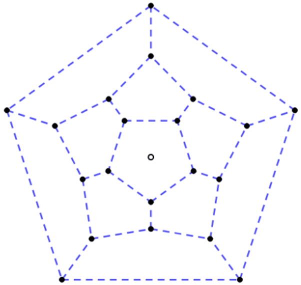
Problem: Can you find a cycle in the graph that includes all 20 vertices?

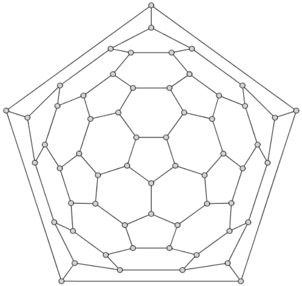
Definition

Let G be a graph. A path of G is a **Hamiltonian path** if it contains every vertex of G. A cycle of G is a **Hamiltonian cycle** if it contains every vertex of G.

Examples

Algorithm Exhaustive Search: try all possible paths.





Hamiltonian vs. Eulerian

The definition of Hamiltonian is very similar to Eulerian. In Hamiltonian each vertex appears exactly once. In Eulerian each edge appears exactly once. Although they look similar, having a Hamiltonian cycle and Having an Euler circuit is very different.

(1) There is a fast algorithm to test if a graph $G = (V, E)$ has an Euler circuit where the running time is a linear function of $|V| + |E|$, namely, test if G is connected and all vertices have even degree.

(2) No such fast test is known for a Hamiltonian cycle. The problem of deciding if a graph has a Hamiltonian path/cycle is **NP-complete**. So it is widely believed that there does not exist an algorithm which takes as input an arbitrary graph $G = (V, E)$ and determines if G has a Hamiltonian path/cycle where the running time is bounded by a polynomial function of $|V| + |E|$.

Definition (Necessary and sufficient conditions)

Let P be a property of graphs and C be a set of condtions.

(1) C is **necessary** for P if every graph satisfying P also satisfies C.

(2) C is **sufficient** for P if every graph satisfying C also satisfies P.

(3) If C is both **necessary and sufficient** for P, then a graph G satisfies P if and only if G satisfies C. We say C characterize P.

Examples

(1) It is necessary for a graph to be connected to have a H.P.

(2) Being a complete graph is a sufficient condition to have a H.P.

(3) $n > 1$ is odd is a necessary and sufficient condition for K_n to have an Euler circuit.

A sufficient condition for G to have an Hamiltonian path.

Theorem

Let $G = (V, E)$ *be a graph with* $|V| = n$. *If*

$$\deg(x) + \deg(y) \geq n - 1 \quad \textit{for all } x, y \in V \textit{ with } x \neq y$$

then G *has Hamiltonian path.*

Proof.

Proof (cont.)

Proof (cont.)

Corollary

If $G = (V, E)$ is a graph with $|V| = n$ and $\deg(v) \geq \frac{n-1}{2}$ holds for every $v \in V$, then G has a Hamiltonian path.

Proof.