Lecture 21 Michael Monagan

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Lecture 21: Solving Recurrences using Generating Functions

Assignment #5 due Monday.

Grimaldi 10.4

Given a_0, the recurrence $a_n = 3a_{n-1} + 1$ defines a sequence

$$a_0, a_1, a_2, \ldots, a_n, \ldots$$

which in turn defines the generating function

$$A(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \ldots$$

If we find a rational form for $A(x)$, that is

$$A(x) = \frac{p(x)}{q(x)} = \frac{A}{(x-\alpha)} + \frac{B}{(x-\beta)}$$

for polynomials $p(x)$ and $q(x)$, then we can use partial fractions to get a formula for $a_n = [x^n]A(x)$.

Example 1. Solve $a_n - 3a_{n-1} = n$ for $n \geq 1$ and $a_0 = 1$.
The recurrence relation represents an infinite set of equations.

$n=1$ $a_1 - 3a_0 = 1$ (1)

$n=2$ $a_2 - 3a_1 = 2$ (2)

$n=3$ $a_3 - 3a_2 = 3$

$\vdots$

Multiply equation (~~k~~) be x^k we get

$n=1$ $a_1 \cdot x^1 - 3a_0 x = 1 \cdot x$

$n=2$ $a_2 x^2 - 3a_1 x^2 = 2 \cdot x^2$

$n=3$ $a_3 x^3 - 3a_2 x^2 = 3 \cdot x^3$

$\vdots$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$\times x \qquad \times x$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

Adding all equations up gives

$$\sum_{n=1}^{\infty} a_n x^n - 3x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n x^n$$

$$[A(x) - a_0] - 3x\,A(x) = \frac{x}{(1-x)^2}$$

$a_0 = 1$

Let $A(x) = a_0 + a_1 x + a_2 x^2 + \cdots$

Now plug in a_0 and isolate $A(x)$.

$$\Rightarrow A(x)(1-3x) = \frac{x}{(1-x)^2} + 1 = \frac{x+(1-x)^2}{(1-x)^2}$$

$$\Rightarrow A(x) = \frac{x+(1-x)^2}{(1-3x)(1-x)^2}$$

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+\cdots$$

$$\frac{1}{(1-x)} = 1+x+x^2+\cdots$$

$x \to 3x$ $\quad$ $x \to 3x$

$$\frac{1}{1-3x} = 1+3x+3^2x^2+\cdots + 3^n x^n + \cdots$$

Now we do a PDF and get a formula for a_n.

$$A(x) = \frac{x+(1-x)^2}{(1-3x)(1-x)^2} = \frac{B}{(1-3x)} + \frac{C}{(1-x)} + \frac{D}{(1-x)^2}$$

$B = \frac{7}{4}$, $C = -\frac{1}{4}$, $D = -\frac{1}{2}$

$$[x^n]A(x) = \frac{7}{4}\cdot 3^n - \frac{1}{4}\cdot 1 - \frac{1}{2}\cdot(n+1).$$

$$\Rightarrow a_n = \frac{7}{4}3^n - \frac{3}{4} - \frac{1}{2}n.$$

Example 2. Consider the sequence defined by $a_0 = 0, a_1 = 1$ and

$a_n - 5a_{n-1} + 6a_{n-2} = 0$ for $n \geq 2$.

Find a rational expression for $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

$$a_2 - 5a_1 + 6a_0 = 0$$
$$a_2 = 5\cdot 1 - 6\cdot 0 = 5$$

$n=2$ $\quad a_2x^2 - 5a_1 x^2 + 6a_0x^2 = 0\cdot x^2$

$n=3$ $\quad a_3x^3 - 5a_2x^3 + 6a_1x^3 = 0$

$n=4$ $\quad a_4x^4 - 5a_3x^4 + 6a_2x^4 = 0$

$$\sum_{n=2}^{\infty} a_n x^n - 5x\sum_{n=1}^{\infty} a_n x^n + 6x^2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$A(x) - \underset{0}{a_0} - \underset{1}{a_1}x - 5x(A(x) - \underset{0}{a_0}) + 6x^2 A(x) = 0$$

$$A(x)\left[1 - 5x + 6x^2\right] - x = 0$$

$$A(x) = a_0 + a_1x + a_2x^2$$

Example 2 (cont.)

$$\Rightarrow A(x) = \frac{x}{1-5x+6x^2} = \frac{x}{(1-3x)(1-2x)}$$

$$\Rightarrow \frac{x}{(1-3x)(1-2x)} = \frac{B}{(1-3x)} + \frac{C}{(1-2x)}$$ for some constants C, D.

$$\Rightarrow x = (1-2x)B + (1-3x)\cdot C.$$

$x=\frac{1}{3}$: $\frac{1}{3} = \frac{1}{3}B + 0\cdot C \Rightarrow B = 1.$

$x=\frac{1}{2}$: $\frac{1}{2} = 0\cdot B + -\frac{1}{2}C \Rightarrow C = -1$

$$A(x) = \frac{1}{(1-3x)} - \frac{1}{(1-2x)}$$

$$a_n = [x^n]A(x) = 3^n - 2^n$$

$$a_2 = 3^2 - 2^2 = 9 - 4 = 5$$

Method.

Let $a_0, a_1, a_2, \ldots$ be a sequence satisfying a recurrence

$$c_n a_n + c_{n-1} a_{n-1} + \cdots + c_k a_{n-k} = f(n).$$

Let $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

(1) Multiply the recurrence by $x^k, x^{k+1}, \ldots$ and sum both sides to infinity.

(2) Rewrite the infinite sums on the LHS in terms of $A(x)$ and the sum on the RHS as a rational function.

(3) Isolate $A(x)$ and use partial fractions to calculate $a_n = [x^n]A(x)$.

Problem. Consider the sequence defined by

$a_0 = 2, a_1 = 3$ and $a_n - 4a_{n-1} + 4a_{n-2} = 2^n$ for $n \geq 2$.

Solve the recurrence using a generating function.

$n=2 \qquad a_2x^2 - 4a_1x^2 + 4a_0x^2 = 2^2x^2$

$n=3 \qquad a_3x^3 - 4a_2x^3 \ 4a_1x^3 = 2^3x^3$

$n=4 \qquad a_4x^4 - 4a_3x^4 + 4a_2x^4 = 2^4x^4$

$\vdots$

$$\frac{1}{1-2x} = 1 + 2x + 2^2x^2 + \cdots$$

$$\frac{4}{1-2x} = 4 + 2^3x + 2^4x^2 + \cdots$$

$$\frac{4x^2}{1-2x} = 4x^2 + 8x^3 + \cdots$$

$$[A(x) - \underbrace{a_0}_{=2} - \underbrace{a_1}_{=3}x] - 4x[A(x) - \underbrace{a_0}_{=2}] + 4x^2A(x) = \frac{4x^2}{1-2x}$$

$$A(x)[1 - 4x + 4x^2] - 2 - 3x + 8x = 4x^2/(1-2x)$$

$$A(x) = \left(\frac{4x^2}{(1-2x)} + 2 - 5x\right)/(1-2x)^2 = \frac{4x^2}{(1-2x)^3} + \frac{2}{(1-2x)^2} + \frac{-5}{(1-2x)^2}$$

$$A(x) = \frac{1}{(1-2x)^2}$$

$k=3 \qquad k=2$

Problem (cont.)

Recall $\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} x^n$

$x \to 2x$

$$\frac{1}{(1-2x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{n} 2^n \cdot x^n$$

$$a_n = [x^n]A(x) = 4\binom{n-2+3-1}{n-2} \cdot 2^{n-2} + 2\binom{n+2-1}{n} \cdot 2^n - 5\binom{n-1+2-1}{n-1} \cdot 2^{n-1}$$

$$= 4\binom{n}{n-2} \cdot 2^{n-2} + 2\binom{n+1}{n} \cdot 2^n - 5\binom{n}{n-1} \cdot 2^{n-1}$$

$$= \binom{n}{n-2} \cdot 2^n + (n+1) \cdot 2^{n+1} - 5n \cdot 2^{n-1}$$

$$= \binom{n}{2} = \frac{n(n-1)}{2}$$

$$= n(n-1) \cdot 2^{n-1} + (n+1) \cdot 2^{n+1} - 5n \cdot 2^{n-1}$$

$$= 2^{n-1}[n^2 - n + 4n - 5n + 4]$$

$$= 2^{n-1}[n^2 - 2n + 4].$$

$a_0 = \quad a_1 =$

$r^2 - 3r - 2 \leftarrow a_n = 3a_{n-1} + 2a_{n-2}$

factor

$a_n = \bigcirc$

$a_0, a_1, \ldots$

series ÷

$\frac{p(x)}{q(x)} \xrightarrow{\text{P.F.}} a_n = \bigcirc$

factor

$a_0, a_1, \ldots$

$a_n =$

$a_0, a_1, \ldots$

series
$\div$

factor

$a_0, a_1, \ldots$