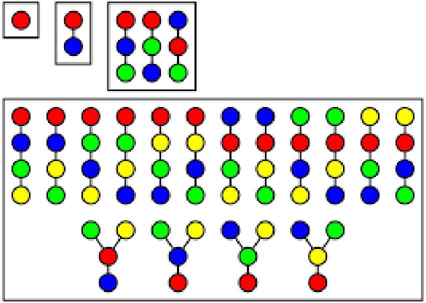
Lecture 32 Labelled Trees and Prüfer Sequences

Grimalid 12.1 Exercise 21

Question: How many trees with labels $1, 2, 3, \ldots, n$ are there?
Equivalently, how many spanning trees are there in K_n?

Question: How many trees with labels $1, 2, 3, \ldots, n$ are there?
Equivalently, how many spanning trees are there in K_n?
Let T_n be the set of such trees and let $t_n = |T_n|$.



T_1, T_2, T_3, T_4 using colors for labels.
We have $t_1 = 1, t_2 = 1, t_3 = 3, t_4 = 16$.

What is t_n?

Theorem (Cayley's formula for the number of labelled trees)

The number of spanning trees of K_n is $t_n = n^{n-2}$ for $n \geq 2$.

Proof – Heinz Prüfer, 1918.

Let T_n be the set of labelled trees on n vertices.
Let P_n be the set of sequences in $V = \{1, 2, \ldots, n\}$ of length $n - 2$.
The Prüfer code is a function $Pru : T_n \to P_n$.
We will show that Pru is a bijection hence $|T_n| = |P_n| = n^{n-2}$.

See the youtube video on Cayley's Formula by Sarda Herke.

Algorithm $Pru(T)$
Input: A tree T on n vertices.
Output: A Prüfer code $x \in P_n$ of length $n-2$.

1. For $i = 1, 2, \ldots, n-2$ do
   Let u be the leaf in T with smallest vertex label.
   Set x_i to be the unique neighbor of u in T.
   Remove the vertex u and edge $\{u, x_i\}$ from T.
2. Return $(x_1, x_2, \ldots, x_{n-2})$.

Example.

Notice that the number of times a vertex v appears in $Pru(T)$ is $\deg(v) - 1$.

Algorithm Tree($\mathbf{x}$).
Input $V = v_1, v_2, \ldots, v_n$ and a Prüfer code x of length $n-2$ on V.
Output a tree with vertices V

1. Set $L = V$ and $E = \phi$.
2. For $i = 1, 2, \ldots, n-2$ do
   Let y be the first element in L that is not in $x[i..n-2]$.
   Set $E = E \cup \{x_i, y\}$ and remove the vertex y from L.
3. Set $E = E \cup L$.
4. Return the tree (V, E).

Example. Determine the tree for the Prüfer code $3, 4, 5, 3, 1$.

Additional Space.

Additional Space.