How fast can we multiply and divide polynomials?

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Joint work with Roman Pearce.

The Mathematics Of Computer Algebra and Analysis project.

> factor$(2\,x^3 + x^2y^2 - x^3y + x^2 - 5\,yx - 3\,y^3 + 3\,y^2x - 3\,y - 1)$;

> solve$(\{x^2 + y^2 + z^2 - 4\,,\; x\,y\,z + 2\,,\; x\,y + z^3 - 1\})$;

> Determinant$\left(\begin{bmatrix} t & 1-2\,t & 1 \\ t^2 & t & 1 \\ 1+t & 1 & 1+t+t^2 \end{bmatrix}\right)$;

> $\int x^2\,\ln(x)\mathrm{e}^{-x} + (1-x)\,\ln(x)\mathrm{e}^{-x} - 2x\mathrm{e}^{-x}\,dx$;

The Mathematics Of Computer Algebra and Analysis project.

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> $\text{solve}(\{x^2 + y^2 + z^2 - 4\ ,\ x\,y\,z + 2\ ,\ x\,y + z^3 - 1\})$;

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Risch

$\mathrm{e}^{-x} \rightarrow \theta_1$
$\ln x \rightarrow \theta_2$

$$\int \overbrace{x^2\theta_2\theta_1 + (1-x)\theta_2\theta_1 - 2x\theta_1}^{\text{a polynomial}}\,\mathrm{d}x \quad \text{where} \quad \begin{array}{l} \theta_1' = -\theta_1 \\ \theta_2' = 1/x. \end{array}$$

Polynomials are the key!

Talk Outline:

- How do CAS represent polynomials?
- How do CAS multiply and divide polynomials?
- Our representation and algorithms.
- How fast we compared with other CAS?
- Immediate Monomial Project (for Maple 15)
- Parallel Multiplication (for Maple 15)

How do CAS *represent* polynomials?

Recursive and distributed polynomial representations.

The distributed represention: monomials $x^i y^j z^k$ are sorted in *lexicographical order* (Magma, Mathematica):

$$f = -6x^3 + 9xy^3z - 8xy^2z + 7y^2z^2 + 5$$

or *graded lex order* (Singular, Maple 15):

$$f = 9xy^3z - 8xy^2z + 7y^2z^2 - 6x^3 + 5.$$

Key property: if X, Y, Z are monomials then $Y > Z \implies XY > XZ$.

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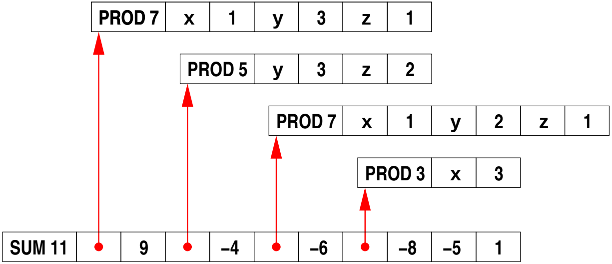
The recursive representation (Macsyma, REDUCE, Derive, Pari):

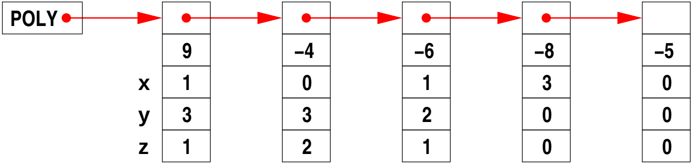
$$f = (-6)x^3 + ((9z)y^3 + (-8z)y^2)x^1 + ((7z^2)y^2 + 5y^0)x^0.$$

Maple's sum of products representation.

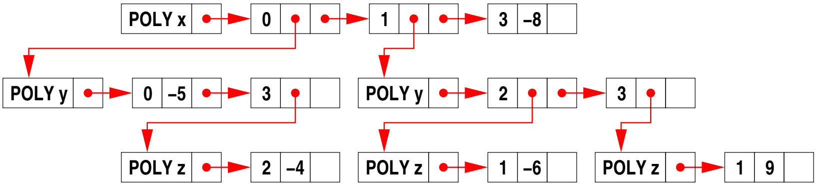


$$9xy^3z - 4y^3z^2 - 6xy^2z - 8x^3 - 5$$

Singular's distributed representation.

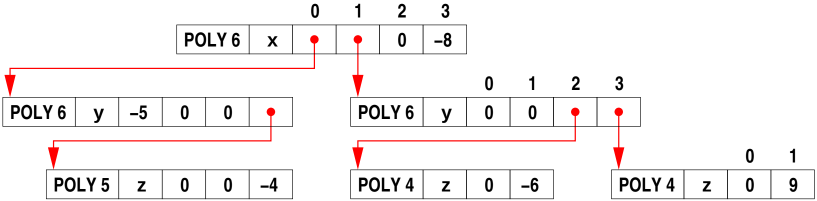


Trip's recursive sparse representation.



$$(-5y - 4z^2y^3) + (-6zy^2 + 9zy^3)x - 8x^3$$

Pari's recursive dense representation.



Our representation uses packed monomials.

Packing for $x^i y^j z^k$ in graded lex order with $x > y > z$:

$$\text{One 64 bit word}: \quad \underbrace{\boxed{i+j+k \;|\; i \;|\; j \;|\; k}}_{(i+j+k)2^{48}+2^{32}i+2^{16}j+k.}.$$

Why?

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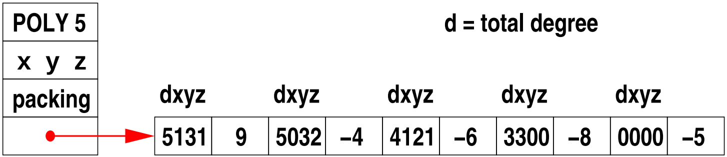
Our packed array for $9xy^3z - 4y^3z^2 - 6xy^2z - 8x^3 - 5$.



Why graded lex order?

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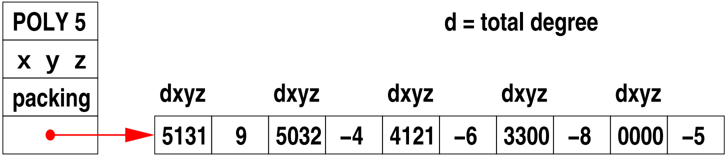

Why graded lex order? No exponent overflow in division.

How do CAS multiply and divide polynomials?

Let $f = f_1 + f_2 + \cdots + f_n$ and $g = g_1 + g_2 + \cdots + g_m$ where $f_1 > f_2 > \cdots > f_n$ and $g_1 > g_2 > \cdots > g_m$.

Using

$$h = f \times g = ((f_1 g + f_2 g) + f_3 g) + \cdots + f_n g \quad \text{and}$$
$$h \div g = f \;\; : \;\; (((h - f_1 g) - f_2 g) - f_3 g) - \cdots - f_n g$$

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takes $O(n^2 m)$ comparisons of monomials
and $O(nm)$ multiplications of coefficient and monomials.

Example:
$f = x^n + x^{n-1} + \ldots + x$ and $g = y^n + y^{n-1} + \ldots + y$.

Our algorithms for multiplication and division use heaps.

Heaps

A binary heap H with n entries is a partially ordered array satisfying

$$H_i \geq H_{2i} \quad \text{and} \quad H_i \geq H_{2i+1}.$$

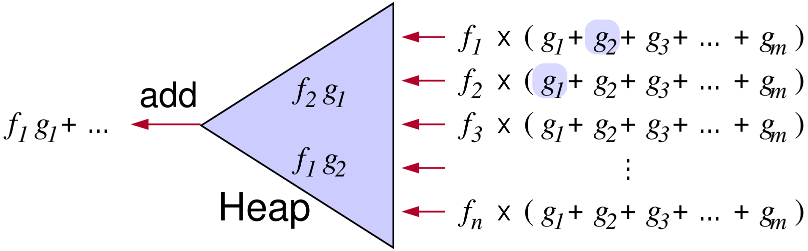
H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
10	9	8	3	5	4	7	–

- H_1 is the biggest entry in a heap.
- We can extract the maximum entry in $O(\log_2 n)$ comparisons.
- We can insert a new entry in $O(\log_2 n)$ comparisons.

Multiplication using a binary heap.

Johnson, 1974, a simultaneous n-ary merge:

$$\begin{aligned} f &= a_1X_1 + a_2X_2 + \cdots + a_nX_n \\ g &= b_1Y_1 + b_2Y_2 + \cdots + b_mY_m \end{aligned} \quad \text{(sorted)}$$



- $O(nm \log n)$ comparisons.
- Space for $\leq n$ monomials in the heap.
- Can pick $n \leq m$.

Division using a heap.

Johnson's quotient heap algorithm.

Dividing $f \div g = q$ compute $$f - \sum_{i=1}^{\#q} q_i \times g$$

- $O(\#f + \#q\#g \log \#q)$ comparisons
- $O(\#q)$ working memory

Our divisor heap algorithm.

Dividing $f \div g = q$ compute $$f - \sum_{i=2}^{\#g} g_i \times q$$

- $O(\#f + \#q\#g \log \#g)$ comparisons
- $O(\#g)$ working memory

Minimal heap division (Monagan & Pearce, 2008)

Problem: we don't know if $\#q > \#g$ when starting a division.
E.g. $(x^7 - y^7) \div (x - y) = x^6 + yx^5 + y^2x^4 + \cdots + y^6$.

Start with quotient heap, switch to divisor heap when $\#q = \#g$.

$$f \quad - \underbrace{\sum_{i=1}^{\min(\#q,\#g)} q_i \times g}_{\text{quotient heap}} \quad - \underbrace{\sum_{i=2}^{\#g} g_i \times (q_{\#g+1} + \cdots)}_{\text{divisor heap}}$$

- $O(\#f + \#q\#g \log \min(\#q, \#g))$ comparisons
- $O(\min(\#q, \#g))$ working memory

Which CAS is fastest?

Benchmark 1: A dense Fateman problem.

$$f = (1 + x + y + z + t)^{20} \quad g = f + 1$$

- f and g have 39 bit coefficients and 10,626 terms
- $h = f \cdot g$ has 83 bit coefficients and 135,751 terms

Intel Core2 3.0 GHz	multiply $p = f \times g$	divide $q = p/f$
Maple 12	289.23 s	187.72 s
Maple 13	187.35 s	159.12 s
Singular 3-0-4	62.00 s	20.00 s
Magma V2.14-7	23.02 s	22.76 s
Pari 2.3.3 (w/ GMP)	32.43 s	14.76 s
Trip v0.99	5.93 s	-
sdmp (unpacked)	5.15 s	5.44 s
sdmp (packed)	2.26 s	2.77 s
Maple 14	3.33 s	4.46s

Benchmark 2: A sparse 10 variable problem.

$$\begin{aligned} f &= (x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_6 + x_6x_7 + x_7x_8 + x_8x_9 + x_9x_{10} + x_1x_{10} \\ &\quad + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + 1)^4 \\ g &= (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2 + x_{10}^2 \\ &\quad + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + 1)^4 \end{aligned}$$

$6,746 \times 8,361 =$ 3,157,883 terms	multiply $p = f \times g$ seconds	divide $q = p/f$ secs
Maple 12	305.76s	280.65s
Maple 13	293.74s	312.29s
Singular 3-0-4	31.00s	18.00s
Magma V2.14-7	17.43s	197.72s
Pari 2.3.3 (w/ GMP)	7.06s	7.05s
Trip v0.99 (rationals)	8.13s	–
sdmp (unpacked)	11.12s	10.37s
sdmp (packed)	2.46s	2.61s
Maple 14	11.74s	14.45s

Benchmark 3: Factorization speedup in Maple 14.

In Maple 13,

```
> h := expand(f*g);
> divide(h,f,'q');
```

call `` `expand/bigprod` ``(f,g) and `` `expand/bigdiv` ``(h,f,q) for large inputs.

In Maple 14, we reprogrammed `` `expand/bigprod` `` and `` `expand/bigdiv` `` to convert to SDMP, multiply (divide) in SDMP, then convert back to Maple.

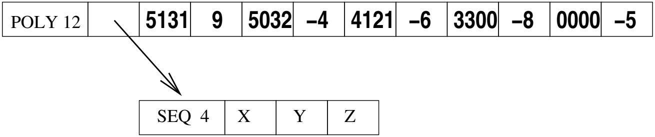
Benchmark	factor(h) where $h=(a+1)(a+2)$			
$a=$	Maple13	Magma	Maple14	Speedup
$(x+y+z)^{30}$	368.88	4.47	18.61	19.8 x
$(1+x+y+z)^{20}$	38.38	10.95	4.01	9.6 x
$(1+x+y+z)^{30}$	679.01	400.4	23.38	29.0 x
$(1+x+y+z+t)^{20}$	5390.32	1286.8	99.00	54.4 x

Table: Factorization Benchmark Timings (in CPU seconds)

The Immediate Monomial Project.

A joint MITACS project with Maplesoft.

A new data structure being implemented by Paul de Marco.



How will we pack monomials? E.g. $x^i y^j z^k$ on a 64 bit computer.

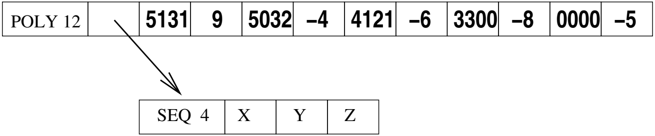
Always try to pack all monomials into one word.

If $i+j+k < 2^{16}$ pack | $i+j+k$ | i | j | k | in one word.

If $i+j+k \geq 2^{16}$ use Maple's existing representation.

So the number of variables determines the packing.

The Immediate Monomial Project



Let $f(x_1, x_2, \cdots, x_n) = f_1 + f_2 + \cdots + f_m$.

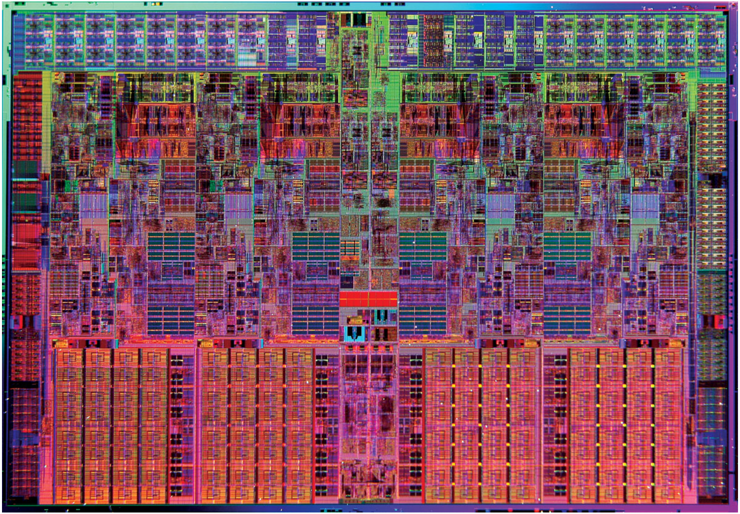
$O(nm) \Rightarrow O(1)$: `lcoeff(f);  degree(f);  indets(f);`

$O(nm) \Rightarrow O(n)$: `f;  has(f,x);  type(f,polynom(integer));`

$O(nm) \Rightarrow O(m)$: `degree(f,x);  diff(f,x);  coeffs(f,x);`

A $10 - 20\%$ gain in overall efficiency gain for Maple 14 ?

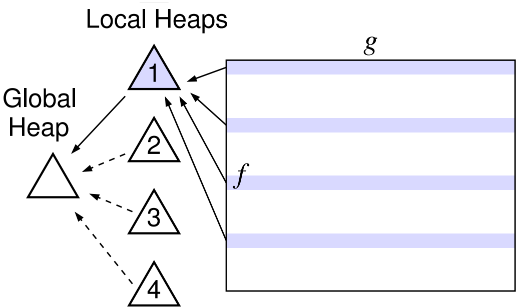
Parrallelizing Multiplication Using Heaps



Intel Core i7.

Parallel Algorithm

One heap per core, merge results in a global heap.

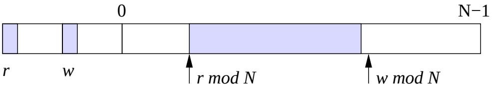


Don't waste real or cpu time:

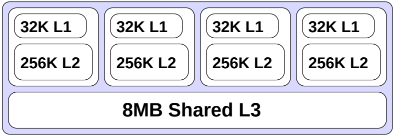
- partition terms
- transfer data
- balance load

Transferring Data

Threads write to a finite circular buffer.

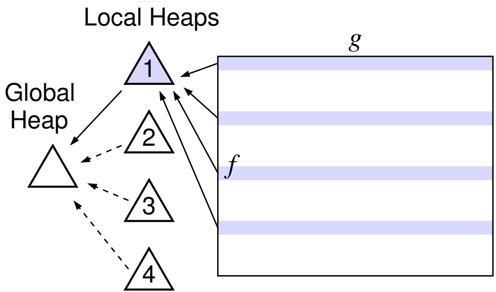



```
#define N    32768  /* size in words (256 K) */
#define CLINE  64   /* bytes per cache line  */
struct buffer {
     long r;         /* words read   */
     char pad1[CLINE - sizeof(long)];
     long w;         /* words wrote */
     char pad2[CLINE - sizeof(long)];
     long data[N];   };
```



Intel Core i7

Load Balancing



- threads try to acquire a lock for the global heap
- one thread per core avoids context switches and OS
- threads independently adjust their share of global work

buffer full	$\longrightarrow$	do more global work
buffer empty	$\longrightarrow$	do less global work

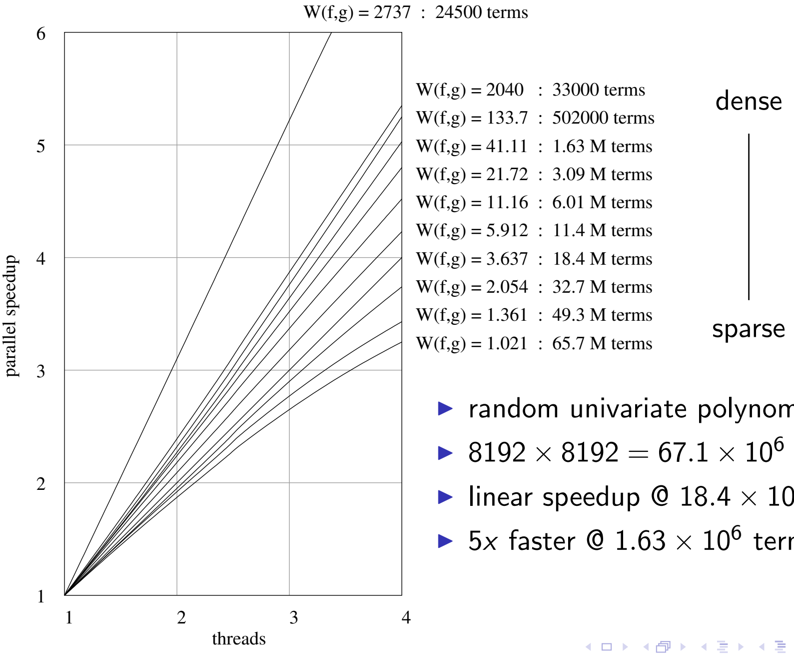
Dense Benchmark

$$f = (1 + x + y + z + t)^{30} \quad g = f + 1$$

$46376 \times 46376 = 635376$ terms $W(f, g) = 3332$

	threads	Core i7 2.66GHz		Core 2 2.4GHz	
sdmp	4	11.48 s	6.15x	14.15 s	4.25x
(packed)	3	16.63 s	4.24x	19.43 s	3.10x
	2	28.26 s	2.50x	28.29 s	2.13x
	1	70.59 s		60.25 s	
Trip 1.0 beta2 recursive dense	4	23.76 s	3.89x	26.86 s	3.94x
	3	31.05 s	2.97x	35.65 s	2.97x
	2	46.56 s	1.98x	52.98 s	1.99x
	1	92.38 s		105.78 s	
Trip 1.0 beta2 recursive sparse	4	29.36 s	3.26x	31.95 s	3.38x
	3	36.00 s	2.66x	39.96 s	2.71x
	2	50.96 s	1.88x	56.68 s	1.91x
	1	95.74 s		108.15 s	
Magma 2.15-8	1	526.12 s			
Pari/GP 2.3.3	1	642.74 s		707.61 s	
Singular 3-1-0	1	744.00 s		1048.00 s	
Maple 13	1	5849.48 s		9343.68 s	

Parallel Speedup: Core i7



- random univariate polynomials
- $8192 \times 8192 = 67.1 \times 10^6$ products
- linear speedup @ 18.4×10^6 terms
- 5x faster @ 1.63×10^6 terms