Assignment 1 Question 1

Part (a)

Calculate the product of $a \times b$ using Karatsuba's algorithm where.

```
> a := 5432;
```

$$a := 5432$$

```
> b := 3829;
```

$$b := 3829$$

So we can write $a = a_1 \cdot 10^2 + a_2 = 54 \cdot 10^2 + 32)$.
And writing $b = b_1 \cdot 10^2 + b_2 = 38 \cdot 10^2 + 29$.
Then Karatsuba's identity is that the product

$$a \cdot b = a_1 \cdot b_1 \cdot 10^4 + [(a_1 - a_2) \cdot (b_2 - b_1) + a_1 \cdot b_1 + a_2 \cdot b_2] \cdot 10^2 + a_2 \cdot b_2 .$$

We have

```
> a1 := 54;
  a2 := 32;
  b1 := 38;
  b2 := 29;
```

$$a1 := 54$$

$$a2 := 32$$

$$b1 := 38$$

$$b2 := 29$$

The three distinct products are

```
> a1b1 := expand( a1*b1 );
```

$$a1b1 := 2052$$

```
> a2b2 := expand( a2*b2 );
```

$$a2b2 := 928$$

```
> amid := expand( (a1-a2)*(b2-b1) );
```

$$amid := -198$$

From which we form the result

```
> expand( a1b1*10^4 + (amid+a1b1+a2b2)*10^2 + a2b2 );
```

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```
> a*b;
```

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Now each of these three two digit multiplications should also be done using Karatsuba's algorithm recursively. The first one $(54) \cdot (38)$ is given by

```
> 5*3*10^2+( (5-4)*(8-3) + 5*3 + 4*8 )*10 + 4*8;
```

2052

Part (b)

Letting $n = 2^k$ we have

$$T(n) = 3 \cdot T\left(\frac{n}{2}\right) + c \cdot n$$

$$3 \cdot T\left(\frac{n}{2}\right) = 3^2 \cdot T\left(\frac{n}{4}\right) + 3 \cdot c \cdot \left(\frac{n}{2}\right)$$

$$3^2 \cdot T\left(\frac{n}{4}\right) = 3^3 \cdot T\left(\frac{n}{8}\right) + 3^2 \cdot c \cdot \left(\frac{n}{4}\right)$$

...

$$3^{k-1} \cdot T\left(\frac{n}{2^{(k-1)}}\right) = 3^k \cdot T\left(\frac{n}{2^k}\right) + \frac{3^{(k-1)} \cdot c \cdot n}{2^{(k-1)}} = 3^k \cdot T(1) + \left(\frac{3}{2}\right)^{k-1} \cdot c \cdot n$$

$$3^k \cdot T(1) = 3^k \cdot d.$$

Now adding the left hand sides and right hand sides and cancelling terms we have

$$T(n) = c \cdot n + \frac{3}{2} \cdot c \cdot n + \ldots + \left(\frac{3}{2}\right)^{k-1} \cdot c \cdot n + 3^k \cdot d$$

$$= 3^k \cdot d + c \cdot n \cdot \left(1 + \frac{3}{2} + \ldots + \left(\frac{3}{2}\right)^{k-1}\right)$$

This geometric sum is given by

```
> sum( (3/2)^i, i=0..k-1 );
```

$$2\left(\frac{3}{2}\right)^k - 2$$

Noting that $2^k = n$ we get

$$T(n) = 3^k \cdot d + c \cdot n \cdot \left(\frac{2 \cdot 3^k}{2^k} - 2\right) = 3^k \cdot (d + 2 \cdot c) - 2 \cdot c \cdot n$$

Replacing $3^k = n^{\log_2 3}$ gives $T(n) = (2 \cdot c + d) \cdot n^{\log_2 3} - 2 \cdot c \cdot n$ as required.

Part (c)

Show that $\frac{T(2 \cdot n)}{T(n)} \sim 3$, that is, $\lim_{n \to \infty} \frac{T(2 \cdot n)}{T(n)} = 3$.

Replacing n with 2^k we have

```
> Tk := proc(k) (2*c+d)*3^k-2*c*2^k end;
```

$$Tk := \textbf{proc}(k)\ \ (d + 2 * c) * 3 \wedge k - 2 * c * 2 \wedge k\ \textbf{end proc}$$

```
> Tk(k);
```

$$3^k (d + 2c) - 2c\,2^k$$

Now doubling n so that $n \to 2 \cdot n$ is equivalent to $k \to k + 1$.

We want to determine $\lim_{n\to\infty}\frac{T(2\cdot n)}{T(n)} = \lim_{k\to\infty}\frac{T(k+1)}{T(k)} = 3.$

```
> T(k+1)/T(k);
```

$$\frac{(d+2\,c)\,3^{k+1}-2\,c\,2^{k+1}}{3^{k}\,(d+2\,c)-2\,c\,2^{k}}$$

Dividing the numerator and denominator by $n = 2^k$ we have this equals

```
> ( (d+2*c)*3^(k+1)/2^k-4*c ) / ( (d+2*c)*3^k/2^k - 2*c );
```

$$\frac{\frac{(d+2\,c)\,3^{k+1}}{2^{k}}-4\,c}{\frac{(d+2\,c)\,3^{k}}{2^{k}}-2\,c}$$

This equals

```
> ((d+2*c)*3*(3/2)^k - 4*c) /( (d+2*c)*(3/2)^k - 2*c );
```

$$\frac{3\,(d+2\,c)\left(\frac{3}{2}\right)^{k}-4\,c}{(d+2\,c)\left(\frac{3}{2}\right)^{k}-2\,c}$$

Now dividing numerator and denominator by (3/2)^k gives

```
> ((d+2*c)*3 - 4*c/(3/2)^k ) / ( (d+2*c) - 2*c/(3/2)^k );
```

$$\frac{3\,d+6\,c-\frac{4\,c}{\left(\frac{3}{2}\right)^{k}}}{d+2\,c-\frac{2\,c}{\left(\frac{3}{2}\right)^{k}}}$$

Now we see that as k gets large this converges to (3d+6c)/(d+2c) = 3