Assignment 2, MACM 204, Fall 2016

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Due Thursday October 6th at 2:30pm.
Late penalty: -20% for up to 24 hours late. 0 after that.

Please attempt each question in a seperate worksheet.
Print your Maple worksheets and hand in in the MACM 204 drop off box outside AQ 4115.

There are 10 questions.

The purpose of this assignment is to get you to use the the main graphics commands in Maple for 2-dimensional and 3-dimensional plots, practice your programming skills, compute and visualize partial derivatives and tangent planes.

Question 1

Given a list **L** of values and a value *x*, write a Maple procedure

```
    position := proc(x,L::list) ... end;
```

so that `position(x,L)` returns the position (the index) of the first occurence of *x* in **L**.
If *x* is not in **L**, return 0. For example for

```
> L := [1,5,4,6,2,4];
```

$$[1, 5, 4, 6, 2, 4]$$

```
> position(4,L);
```

should return 3.

Question 2

Give a list L of integers write a Maple procedure

```
     allprimes := proc(L::list(integer)) ... end;
```

that returns true if all integers in L are prime and false otherwise. For example

```
> L := [3,5,7,9];
```

$$L := [3, 5, 7, 9]$$

```
> allprimes(L);
```

should return false.

Question 3

In number theory the function $\pi(x)$ counts the number of primes $\leqslant x$.
For example $\pi(10) = 4$ since there are 4 primes 2,3,5,7 less than 10.
Write a Maple procedure `pi(x)` that computes $\pi(x)$. Calculate these values

```
> pi(10);
  seq( pi(1000*i), i=1..10 );
```

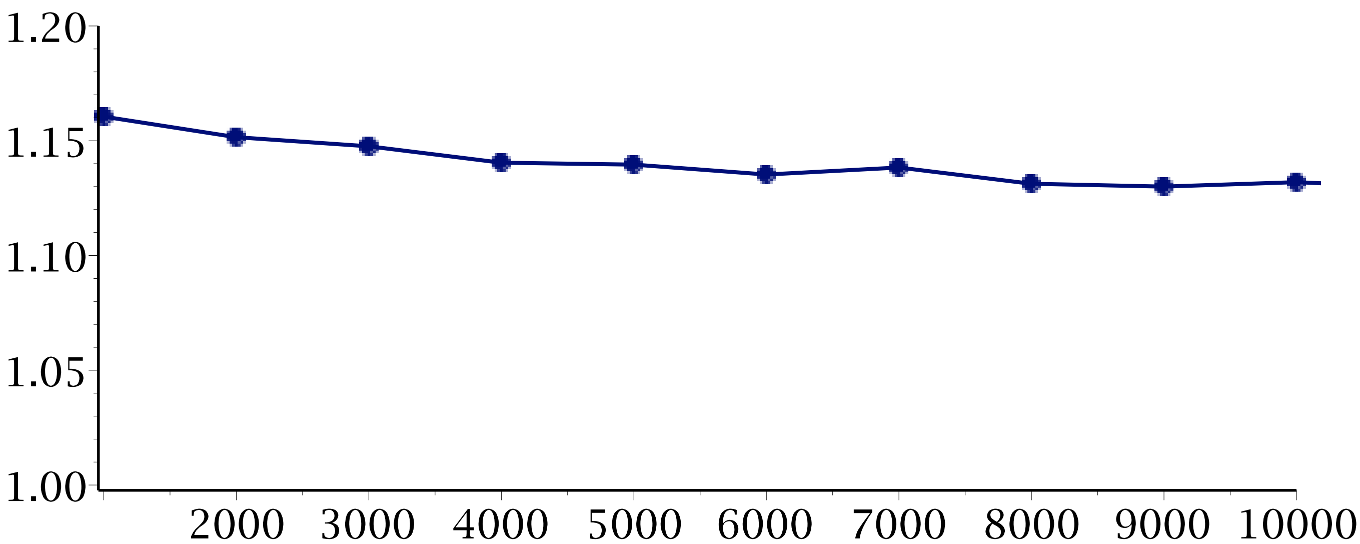
$$4$$

$$168, 303, 430, 550, 669, 783, 900, 1007, 1117, 1229$$

The prime number theorem says that $\pi(x) \backsim \frac{x}{\ln(x)}$ or equivalently $\lim_{x\to\infty} \frac{\pi(x)\cdot\ln(x)}{x} = 1$.

Calculate $\frac{\pi(x) \cdot \ln(x)}{x}$ for $x = 1000, 2000, 3000, ..., 20000$.

Using the dataplot command, graph the points $\left(x, \frac{\pi(x) \cdot \ln(x)}{x}\right)$ on a graph. You should get this plot



So the limit approaches 1 slowly. There is a graph of this on the Wiki page https://en.wikipedia.org/wiki/Prime_number_theorem

Question 4

The roots of the polynomials $x^n - 1$ for $n = 1, 2, 3, ...$ are called the roots of unity because they satisfy $x^n = 1$. They have some special properties. One of the properties is that if $n > 1$ then the roots add up to 0. For example, the roots of $x^4 - 1$ are $+1, -1, +i, -i$ which obviously add up to 0. Let's check this in Maple.

```
> L := [solve(x^4-1=0,x)];
```

$$L := [1, -1, I, -I]$$

```
> add( L[i], i=1..4 );
```

$$0$$

Notice that the imaginary number i is I in Maple. Now your first task is to check this property for $2 \leq n \leq 11$. Do this using a for loop of the form **for n from 2 to 11 do ... od** .

Your second task is to find out what happens when you multiply the roots. For $n = 4$ the product is -1 as shown below.

```
> mul( L[i], i=1..4 );
```

$$-1$$

What is the property for the product of the roots for $n \geq 1$?

Question 5

Peter Borwein has a fractal image on his home page [under Vis Number Theory if you care to look.] He created it by computing the complex roots of lots of polynomials then plotting them. The polynomials he chose were restricted to have coefficients 1 or 0 only e.g. $x^6 + x^5 + x^3 + x^2 + 1$. He simply graphed the complex roots in the complex plane and out popped a beautiful fractal image. It was totally unexpected. Do this in Maple.

To create a polynomial of degree *d* with coefficients 1 or 0 chosen at random use

```
poly := Randpoly(d,x) mod 2;
```

To compute the complex roots of a polynomial use this command

```
fsolve( poly, x, complex );
```

To plot n complex numbers $z_1, z_2, ..., z_n$ in the complex plane you can use this command

```
plots[complexplot]( [z1,z2,...,zn], style=point );
```

For example

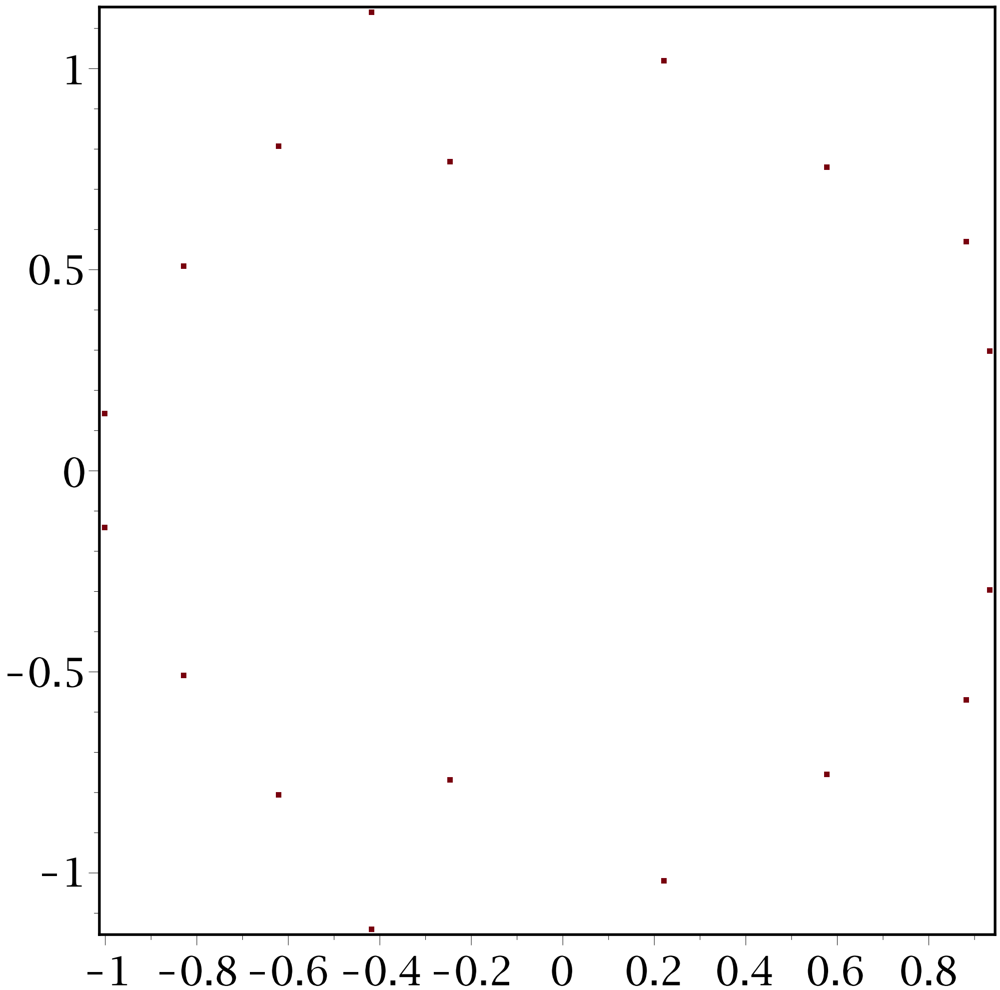
```
> poly := Randpoly(18,x) mod 2;
```

$$poly := x^{18} + x^{17} + x^{16} + x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^6 + x^5 + x^2 + x + 1$$

```
> R := [fsolve(poly=0,x,complex)];
```

$$R := [-1.0013633366604912499 - 0.14168707657366172749\,I,\\ -1.0013633366604912499 + 0.14168707657366172749\,I,\\ -0.82879637827964159036 - 0.50869099345964559008\,I,\\ -0.82879637827964159036 + 0.50869099345964559008\,I,\\ -0.62076765514032157840 - 0.80676306793526027725\,I,\\ -0.62076765514032157840 + 0.80676306793526027725\,I,\\ -0.41753859898347021978 - 1.1399829210884093913\,I,\\ -0.41753859898347021978 + 1.1399829210884093913\,I,\\ -0.24578517991207810284 - 0.76831077507102118224\,I,\\ -0.24578517991207810284 + 0.76831077507102118224\,I,\\ 0.22121118841704460225 - 1.0198891803155255995\,I, 0.22121118841704460225\\ + 1.0198891803155255995\,I, 0.57717632254843755389\\ - 0.75458224056185120728\,I, 0.57717632254843755389\\ + 0.75458224056185120728\,I, 0.88260602792291299338\\ - 0.56968122865446260039\,I, 0.88260602792291299338\\ + 0.56968122865446260039\,I, 0.93325761008760759176\\ - 0.29679528502802180945\,I, 0.93325761008760759176\\ + 0.29679528502802180945\,I]$$

```
> plots[complexplot]( R, style=point, symbol=point, axes=box );
```



Now generate at least 1000 polynomials **of degree 18** in a loop, compute their complex roots and graph them all on the same plot. You'll need to create a single list of the roots of all 1000 polynomials, so that's 18,000 complex numbers. Note, to get a good image, 10,000 polynomials is better. But you may have trouble printing the plot for 10,000 polynomials so just print it for 1000 polynomials.

Question 6

Consider the function $g(a, x) = a^2 \cdot x \cdot (1 - x) \cdot (1 - a \cdot x + a \cdot x^2)$. We want to study the solutions of $g(a, x) = x$ for $0 \leq a \leq 4$ and $0 \leq x \leq 1$.
One way is to create an implicit plot of $g(a, x) = x$. Do this using the **implicitplot** command in the **plots** package.

Another way is to graph the function $g(a, x) - x$ in 3 dimensions for $0 \leq a \leq 4$ and $0 \leq x \leq 1$ and see where the *z* co-ordinate is 0. To do this visually we can graph the 0 function. Do this using the **plot3d** command (graph $g(a, x) - x$ and 0 on the same plot) . Rotate the plot so that it matches the implicit plot.

Question 7

Consider the function $f(x, y) = 2 \cdot x^2 + 3 \cdot y^2 - x \cdot y - 4$.
We want to visualize the partial derivatives at the point $x = 1, y = 1$.
First use Maple to compute the partial derivatives $\frac{\partial}{\partial x} f(1, 1)$ and $\frac{\partial}{\partial y} f(1, 1)$. You should get 3 and 5 respectively so both slopes are positive.
Now generate a 3 dimensional plot of $f(x, y)$ (using the **plot3d** command) and the curves $f(x, 1)$ and $f(1, y)$ (using the **spacecurve** command) and display all three plots on the same graph (using the **display** command in the plots package).

Question 8

Consider the function $f(x, y) = 2 \cdot x^4 + 3 \cdot y^4 - x \cdot y - 4$.
We would like to visualize the tangent plane at $x = 1, y = 1$.

Use Maple to construct the linear Taylor polynomial *T* for $f(x, y)$ about $x = 1$, $y = 1$.
Graph $f(x, y)$ and $T(x, y)$ on the same plot using the plot3d command.

Question 9

Suppose we want to construct the Taylor polynomial $P(x)$ of degree n for $f(x)$ about the point $x = a$. We can do this using Maple's taylor command which computes a truncated **Taylor series** for $f(x)$ about $x = a$. For example, to get the Taylor polynomial of degree 3 for e^x about $x = 0$ we compute the Taylor series to order $O(x^4)$ as follows.

```
> taylor( exp(x), x=0, 4 );
```

$$1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + O(x^4)$$

So the Taylor polynomial is $1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3$

The general formula for the Taylor polynomial $P(x)$ of degree n is given by

$$P(x) = f(a) + f'(a) \cdot (x - a) + \frac{f''(a) \cdot (x - a)^2}{2} + \ldots + \frac{f^{(n)} \cdot (x - a)^n}{n!}$$

Write a Maple procedure TaylorPolynomial(f, x, a, n) that uses this general formula to compute $P(x)$. Test your Maple procedure on the following

```
> TaylorPolynomial( exp(x), x, 0, 5 );
> TaylorPolynomial( sin(x), x, 0, 8 );
> TaylorPolynomial( ln(x), x, 1, 5 );
> TaylorPolynomial( sqrt(x), x, 0, 3 );
```

The last one should produce an error. Explain why.

Question 10

Let $A = \int_a^b f(x)\, dx$. Recall that the value of A may be approximated by the **Midpoint rule** on *n* intervals of width $h = \frac{(b - a)}{n}$ using the formula

$$M_n = h \cdot \left(f\left(a + \frac{h}{2}\right) + f\left(a + \frac{3}{2} \cdot h\right) + f\left(a + \frac{5}{2} \cdot h\right) + \ldots + f\left(a + \frac{2 \cdot n - 1}{2} \cdot h\right) \right).$$

Write a Maple procedure `MidpointRule( f(x), x, a, b, n, digits )` that computes M_n and uses *d* digits of precision for arithmetic calculations, i.e., your Maple procedure should begin by initializing **`Digits := d;`** Execute the following

```
> MidpointRule( sin(x), x, 0, 1,  4, 15 );
  MidpointRule( sin(x), x, 0, 1,  8, 15 );
  MidpointRule( sin(x), x, 0, 1, 16, 15 );
```

0.460897009411942

0.459997112932708

0.459772523245456

Now compute the error in these approximations as follows.

```
> A := evalf( int(sin(x),x=0..1), 15 );
```

$$A := 0.459697694131860$$

```
> for n in [4,8,16,32,64,128] do
      e := abs( A-MidpointRule( sin(x), x, 0, 1, n, 15 ) );
      printf( "n=%3d  error=%12.10f\n", n, e );
  od:
```

Based on the data that you get, what happens to the error as we double the number of intervals n ? Obviously the error should decrease.