The Polynomial Part $\int P,\ P \in F[\theta],\ \theta = \log u$

Let $P = p_\ell \theta^\ell + \cdots + p_1\theta + p_0$ where $p_i \in F$.
Liouville's theorem says if $\int P$ is elementary then

$$(*)\quad \int P = \underset{F(\theta)}{V_0(\theta)} + \sum \underset{\bar{K}}{c_i} \log \underset{F(\theta)}{v_i(\theta)} \qquad \text{WLOG } v_i \in F[\theta]$$

Let $V_0(\theta) = \frac{a(\theta)}{b(\theta)}$ where $\gcd(a,b)=1$ and $lc_\theta\, b = 1$. in $F[\theta]$

WLOG assume $V_i(\theta)$ are monic, irreducible in $F[\theta]$
$\log(AB) = \log(A) + \log(B)$

$$(*)' \Rightarrow P = p_\ell\theta^\ell + \cdots + p_0 = \frac{a'(\theta)}{b(\theta)} - \frac{b'(\theta)a(\theta)}{b(\theta)^2} + \sum c_i \frac{V_i'(\theta)}{V_i(\theta)}$$

If $\deg_\theta b > 0$ the terms in the PFD of cannot cancel out.
$\Rightarrow \deg_\theta b = 0 \Rightarrow b \in F \Rightarrow v_0(\theta) \in F[\theta]$.

$$(*)' \quad P = p_\ell\theta^\ell + \cdots + p_0 + \underset{\text{by Th 12.2 } \hat{F}[\theta]\,!}{v_0'(\theta)} + \sum c_i \frac{V_i'(\theta)}{V_i(\theta)}$$

$\leftarrow$ deg $n_i - 1$ by Th 12.2
monic irreducible deg n_i

If $\deg_\theta V_i > 0$ then $V_i(\theta)^{-1}$ cannot cancel out $\Rightarrow v_i \in F$.

$$\Rightarrow \int P = \underset{F[\theta]}{V_0(\theta)} + \sum c_i \log \underset{F}{V_i}$$

$$\Rightarrow P = p_\ell\theta^\ell + \cdots + p_0 = \underset{\hat{F}[\theta]}{v_0'(\theta)} + \underset{\hat{F}}{\sum L'}$$

Th 12.2 $\Rightarrow \deg_\theta V_0 \le \ell + 1$ hence

$$\int P = \int p_\ell\theta^\ell + \cdots + p_0 = \underset{\hat{K}}{q_{\ell+1}}\theta^{\ell+1} + \underset{\hat{F}}{q_\ell}\theta^\ell + \cdots + \underset{\hat{F}}{q_0} + \sum \underset{\hat{K}}{c_i} \log \underset{\hat{F}}{v_i}$$