Multivariate Polynomial Rings 2.6

Let R be a ring and $a \in R[x_1, x_2, \ldots, x_n]$ a multivariate polynomial.
Let $a = \sum_{i=1}^{t} a_i x_1^{e_{1i}} \cdot x_2^{e_{2i}} \cdot \ldots \cdot x_n^{e_{ni}}$ where $a_i \in R$.

Def. The vectors $[e_{1i}, e_{2i}, \ldots, e_{ni}]$ are called <u>exponent vectors</u>.
E.g. $a = 7y^4 + 3x^2y^2 + 2xy^4 - 7xy^2 - 2x^3$ in $\mathbb{Z}[x,y]$
$[0,4] \quad [2,2] \quad [1,4] \quad [1,2] \quad [3,0]$

Def The <u>total degree</u> $\deg(a) = \max_{i=1}^{t} \sum_{j=1}^{n} e_{ji}$.
If R is an integral domain then $\deg(ab) = \deg(a) + \deg(b)$

The terms of $a \in R[x_1, x_2, \ldots, x_n]$ are usually ordered in some descending order on the exponent vectors.

<u>Pure lexicographical order.</u> If u and v are two exponent vectors then $u > v$ if $u_k > v_k$ and $u_j = v_j$ for $1 \le j < k$.
E.g. $a = -2x^3 + 3x^2y^2 + 2xy^4 - 7xy^2 + 7y^4$
$[3,0] > [2,2] > [1,4] > [1,2] > [0,4]$

<u>Graded lexicographical order</u> (Maple 17,18) Terms are ordered in decreasing total degree with ties broken using pure lex. order.
E.g. $a = 2xy^4 + 3x^2y^2 + 7y^4 + 2x^3 - 7xy^2$
deg: $5 > 4 \quad 4 > 3 \quad 3$

For <u>both orderings</u> lc, lm, lt generalize in the obvious way.
E.g. for $\mathrm{lc}(a) = 2$, $\mathrm{lm}(a) = xy^4$, $\mathrm{lt}(a) = 2xy^4$. Moreover

(i) $\mathrm{lt}(ab) = \mathrm{lt}(a)\,\mathrm{lt}(b)$
(ii) $\mathrm{lc}(ab) = \mathrm{lc}(a) \cdot \mathrm{lc}(b)$ provided R is ______________
(iii) $\mathrm{lm}(ab) = \mathrm{lm}(a)\,\mathrm{lm}(b)$