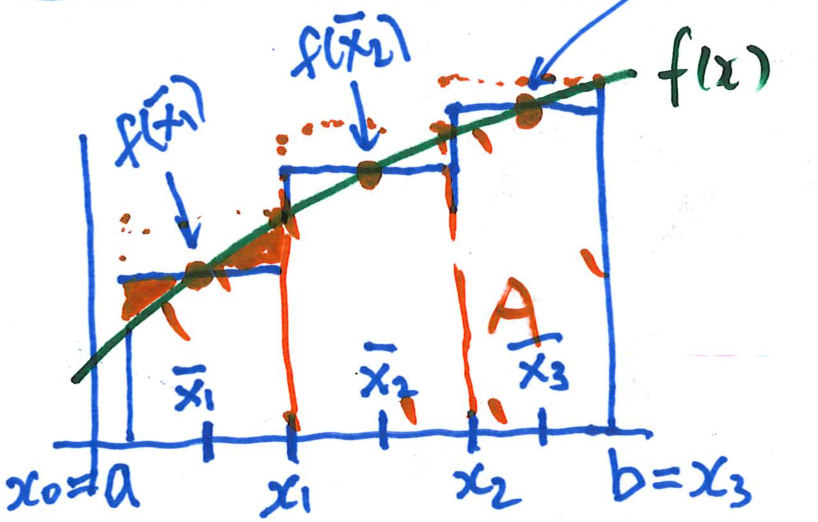
5.2 The Definite Integral

Assignment #1 due Monday 11pm.

The Midpoint rule M_n



$n = 3$ $\quad x_1 = a + \Delta x$

Divide $[a,b]$ into n subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ of equal width

$$\Delta x = \frac{b-a}{n} \quad \text{with } x_i = a + i \cdot \Delta x.$$

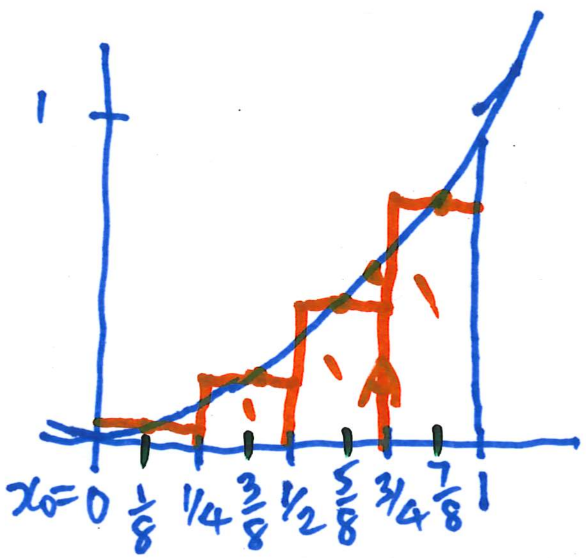
Let $\bar{x}_i$ be the midpoint of $[x_{i-1}, x_i]$.

So $\bar{x}_i = \dfrac{x_{i-1} + x_i}{2}$.

Approximate A with n rectangles

$$M_n = \Delta x f(\bar{x}_1) + \Delta x f(\bar{x}_2) + \cdots + \Delta x f(\bar{x}_n)$$
$$= \Delta x \left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n) \right]$$

Example $f(x) = x^2$, $a = 0$, $b = 1$.



$n = 4$

$\Delta x = \dfrac{1-0}{4}$

$$M_4 = \frac{1}{4}\left[f(\tfrac{1}{8}) + f(\tfrac{3}{8}) + f(\tfrac{5}{8}) + f(\tfrac{7}{8}) \right]$$
$$= 1/4 \left[\frac{1}{64} + \frac{9}{64} + \frac{25}{64} + \frac{49}{64} \right] = \frac{84}{256} = .328125$$

$L_4 = .21875$ $\qquad A = \frac{1}{3} = .333\ldots$

$R_4 = .41875$

M_n is much more accurate than L_n and R_n

$$A = \lim_{n\to\infty} M_n = \lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$$

Riemann. Pick x_i^* anywhere on $[x_{i-1}, x_i]$ i.e. $x_{i-1} \le x_i^* \le x_i$.

Approximate A with n rectangles

$$S_n = \sum_{i=1}^{n} \Delta x\, f(x_i^*) \leftarrow \text{Riemann sum.}$$

Theorem. $\lim_{n\to\infty} S_n = A$. If $x_i^* = \bar{x}_i$ then $S_n = M_n$

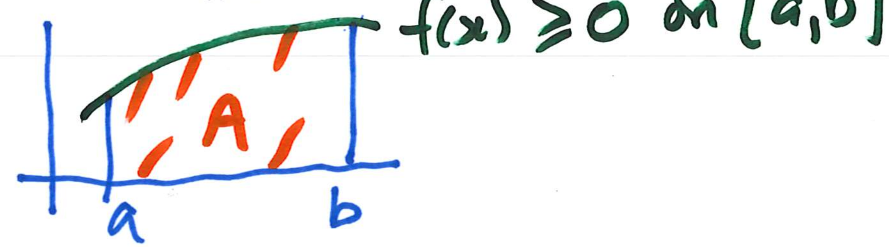
Leibniz Let $f(x)$ be continuous on $[a,b]$. Define the definite integral.

upper limit.

integrand

integral symbol $\rightarrow$

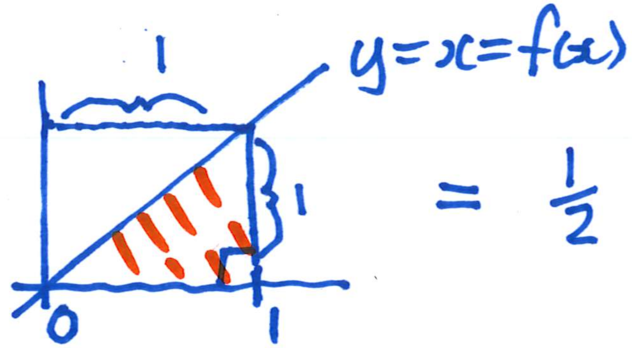
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underline{\Delta x}\, f(x_i^*) =$$



lower limit

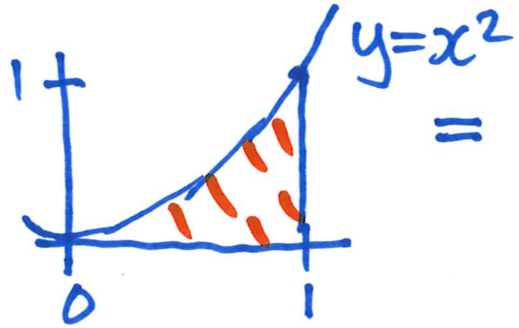
?

What is $\displaystyle\int_0^1 x\,dx =$



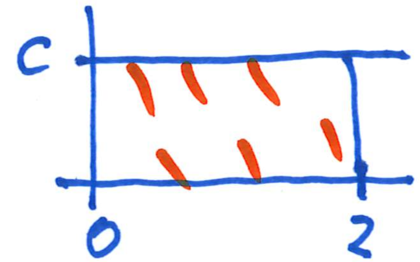
$= \frac{1}{2}$

What is $\int_0^1 x^2\,dx =$

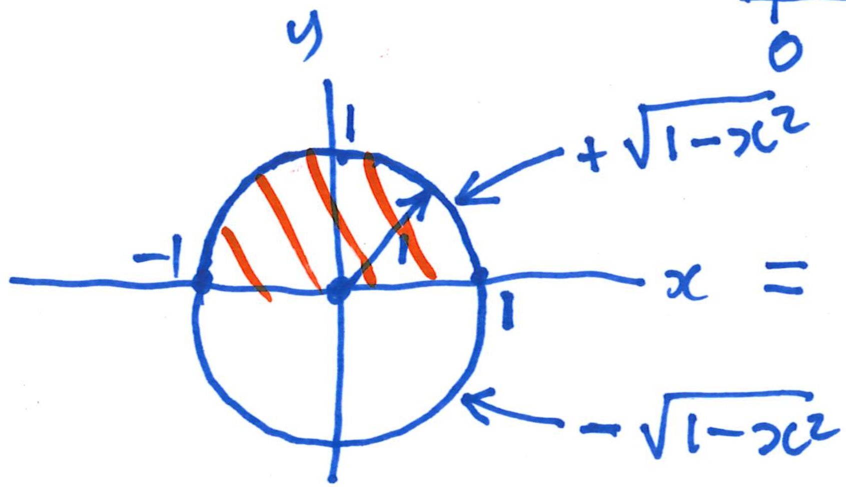


$= \frac{1}{3}$

What is $\int_0^2 c\,dx =$



$= 2\cdot c$



$= \frac{\pi}{2} = \int_{-1}^{1} \sqrt{1-x^2}\,dx$

$x^2 + y^2 = 1$

$y^2 = 1 - x^2$

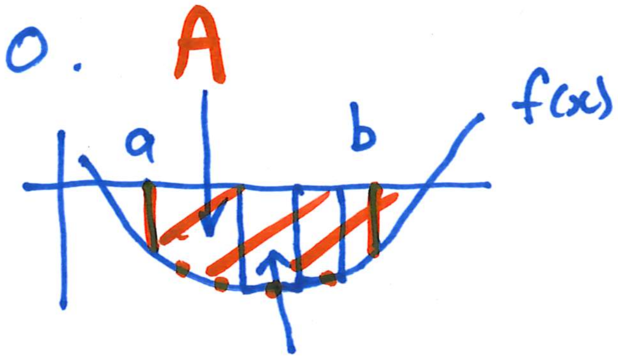
$y = \pm\sqrt{1-x^2}$

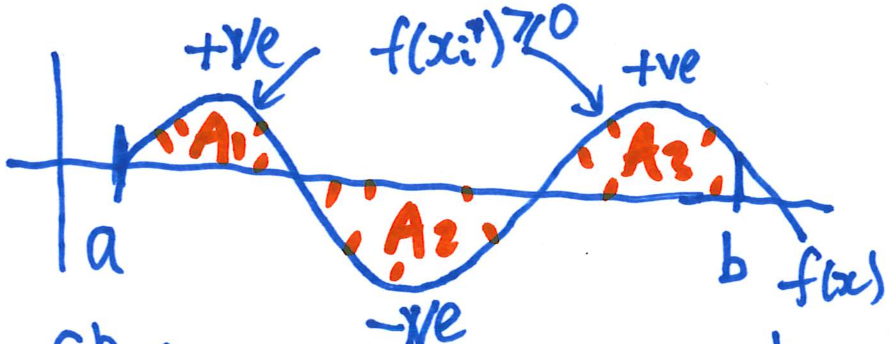
Properties of Definite Integrals $\quad b \geq a.$

① If $f(x) \geq 0$ on $[a,b]$ Then $\int_a^b f(x)\,dx \geq 0.$

② If $f(x) \leq 0$ on $[a,b]$ Then $\int_a^b f(x)\,dx \leq 0.$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i^*)}_{<0} \cdot \underbrace{\Delta x}_{>0} < 0 = -A$$



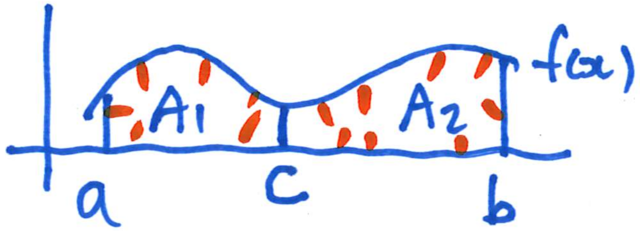


$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3.$$

③ $\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

④ $\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx.$

$$\int_a^b c\, f(x)\,dx = c \leftarrow \lim_{n\to\infty} c \leftarrow \sum_{i=1}^{n} c\,\Delta x\, f(x_i^*) = c \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\, f(x_i^*)} = c\int_a^b f(x)\,dx$$

⑤ 

$$= A_1 + A_2 = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

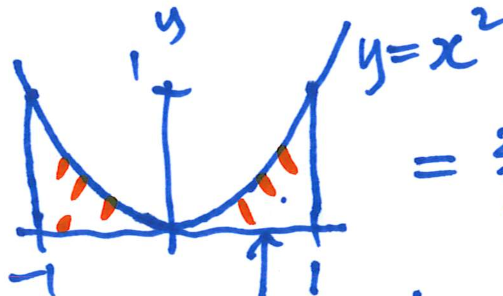
Ex 1 Calculate

$$\int_0^1 (2x^2+x)\,dx \overset{\text{P3}}{=} \int_0^1 2x^2\,dx + \int_0^1 x\,dx \overset{\text{P4}}{=} 2\underbrace{\int_0^1 x^2\,dx}_{=\frac{1}{3}} + \underbrace{\int_0^1 x\,dx}_{=\frac{1}{2}}$$

$$= \frac{2}{3} + \frac{1}{2} = \frac{4+3}{6} = \frac{7}{6}.$$

Ex 2 Calculate

$$\int_{-1}^1 x^2\,dx =$$

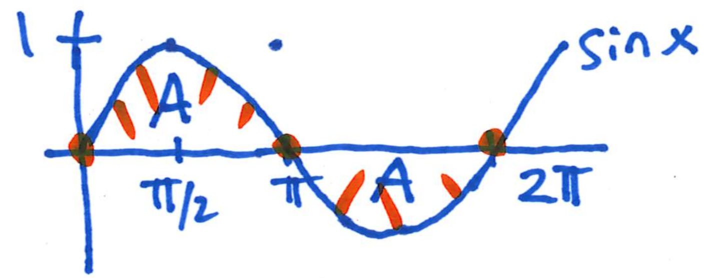


$$= \frac{2}{3}$$

$$= \int_{-1}^0 x^2\,dx + \int_0^1 x^2\,dx = \frac{1}{3} + \frac{1}{3}.$$

Ex 3 Calculate

$$\int_0^{2\pi} \sin x\,dx =$$



$$= A - A = 0$$