MATH 340 Assignment 5 Solutions, Fall 2017

Michael Monagan

Section 2.5: Complex Numbers

1. Let $i^2 = -1$, $a = (2 + 3i)$ and $b = (1 - 2i)$.
   Calculate $a + b$, $a\,b$, a^{-1}, $|a|$ and $\overline{b}$.
   Draw the points $a, b, a\,b, a^{-1}, \overline{b}$ in the complex plane.

   I get $a + b = 3 + i$, $a\,b = 8 - i$, $a^{-1} = \frac{2}{13} - \frac{3}{13}i$, $|a| = \sqrt{13}$, $\overline{b} = 1 + 2i$.

2. Let $x, y \in \mathbb{C}$. Show that $xy = yx$ and $|xy| = |x||y|$ and $\overline{xy} = \overline{x}\ \overline{y}$.

   Let $x = a+bi$ and $y = c+di$. Then $xy = (ac-bd)+(ad+bc)i = (ca-db)+(da+cb)i$ since multiplication in $\mathbb{R}$ is commutative. Also $yx = (c+di)(a+bi) = (ca-db)+(cb+da)i$.
   We have $|xy| = |(ac-bd)+(ad+bc)i| = \sqrt{(ac-bd)^2 + (ad+bc)^2} = \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}$.
   And $|x||y| = \sqrt{(a^2+b^2)}\sqrt{(c^2+d^2)} = \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}$.
   Finally $\overline{xy} = \overline{(ac-bd)+(ad+bc)i} = (ac-bd)-(ad+bc)i$. $\overline{x}\ \overline{y} = (a-bi)(c-di) = (ac-bd)-(ad+bc)i$.

3. If $f(x) = x^3 - 6x^2 + 13x - 10$ and $2 + i$ is a root of $f(x)$, find the other roots of $f(x)$ and factor $f(x)$ over $\mathbb{C}$.

   Since $f \in \mathbb{R}[x]$ and we are given $z = 2 + i$ is a root of $f(x)$ we have $\bar{z} = 2 - i$ is also a root of $f(x)$ so $g(x) = (x - (2+i))(x - (2-i)) = x^2 - 4x + 5$ divides $f(x)$. We just need to determine the third linear factor $f(x)$. Dividing $f(x)$ by $g(x)$ using long division gives the quotient $x - 2$ and remainder 0. Thus the other root is 2 and $f(x) = (x-2)(x-(2+i))(x-(2-i))$.

4. Let $\mathbb{Z}[i]$ be the subset of complex numbers $\mathbb{C}$ given by $\mathbb{Z}[i] = \{a+bi : a, b \in \mathbb{Z} \text{ and } i^2 = -1\}$. The set $\mathbb{Z}[i]$ is called the set of Gaussian integers. Show that $\mathbb{Z}[i]$ is a subring of $\mathbb{C}$. See Lemma 2.2.4 (i).

   Let $x, y \in \mathbb{Z}[i]$ with $x = a + bi$ and $y = c + di$ and $a, b, c, d \in \mathbb{Z}$. Now $x + y = (a+c) + (b+d)i$ and $a + c, b + d \in \mathbb{Z}$ thus $\mathbb{Z}[i]$ is closed under addition. Now $xy = (ac - bd) + (ad + bd)i$. Since $\mathbb{Z}$ is closed under addition and multiplication we have $ac - bd, ad + bd \in \mathbb{Z}$ thus $\mathbb{Z}[i]$ is closed under multiplication. And $-x = -a - bi$ and $-a, -b \in \mathbb{Z}$ so $\mathbb{Z}[i]$ is also closed under negation. Since $0 + 0i \in \mathbb{Z}[i]$ we have $\mathbb{Z}[i]$ is a non-empty set, thus $\mathbb{Z}[i]$ is a subring.

Section 2.7: Construction of Fields

Consider the ring $R = \mathbb{Z}_2[x]/(x^3 + x^2 + x)$.

(i) What are the congruence classes of R?

Answer: $R = \{[0], [1], [x], [x+1], [x^2], [x^2+1], [x^2+x], [x^2+x+1]\}$

(ii) Find a zero divisor in R.

Answer Since $f = x^3 + x^2 + x = x(x^2 + x + 1)$ in $\mathbb{Z}_2[x]$, $[x]$ is a zero divisor in R. Because $[x] \cdot [x^2 + x + 1] = [x\,(x^2 + x + 1)] = [x^3 + x^2 + x] = [0]$ in R.

(iii) Use the extended Euclidean algorithm to find $[x+1]^{-1}$ in R.

Answer: Let $f = x^3 + x^2 + x$ and $a = x + 1$. We must solve $sf + tg = \gcd(f, g)$ for $s, t \in \mathbb{Z}_2[x]$. I get $(x^2+1)a + 1b = 1$ thus $[x^2+1]$ is the inverse of $[x+1]$. To check this we have $[x^2+1][x+1] = [x^3 + x^2 + x + 1]$. Now dividing $x^3 + x^2 + x + 1$ by $x^3 + x^2 + x$ we get the remainder 1 so $[x^3 + x^2 + x + 1] = [1]$.

.