The parallelogram theorem from CLO section 6.4

```
> interface(imaginaryunit=_i):
  with(PolynomialIdeals):
> h1 := x2-u3;
  h2 := (x1-u1)*u3-u2*u3;
  h3 := x1*x4-x2*x3;
  h4 := u3*(u1-x3)-x4*(u1-u2);
```

$$h1 := x2 - u3$$

$$h2 := (x1 - u1)\, u3 - u2\, u3$$

$$h3 := x1\, x4 - x2\, x3$$

$$h4 := u3\, (u1 - x3) - x4\, (u1 - u2)$$

```
> g1 := x3^2+x4^2 - ((x1-x3)^2+(x2-x4)^2);
  g2 := (u1-x3)^2+x4^2 - ((x3-u2)^2+(x4-u3)^2);
```

$$g1 := x3^2 + x4^2 - (x1 - x3)^2 - (x2 - x4)^2$$

$$g2 := (u1 - x3)^2 + x4^2 - (x3 - u2)^2 - (x4 - u3)^2$$

```
> I := <h1,h2,h3,h4>;
```

$$I := \langle x2 - u3, u3\, (u1 - x3) - x4\, (u1 - u2), (x1 - u1)\, u3 - u2\, u3, x1\, x4 - x2\, x3 \rangle$$

Note, by default, all uknowns are treated as variables, so $I \subseteq \mathbb{Q}[x1, x2, x3, x4, u1, u2, u3]$.

```
> IdealInfo[Variables](I);
```

$$\{u1, u2, u3, x1, x2, x3, x4\}$$

```
> IdealInfo[Parameters](I);
```

$$\{\ \}$$

The test if g1 and g2 are in I is false

```
> IdealMembership(g1,I);
  IdealMembership(g2,I);
```

false

false

The test if they are in $\sqrt{I}$ also fails

```
> R := <h1,h2,h3,h4,1-g1*y>;
```

$$R := \langle x2 - u3, u3\, (u1 - x3) - x4\, (u1 - u2), (x1 - u1)\, u3 - u2\, u3, -\left(x3^2 + x4^2 - (x1 - x3)^2 - (x2 - x4)^2\right) y + 1, x1\, x4 - x2\, x3 \rangle$$

```
> IdealMembership(1,R);
```

false

The problem is that u1 and u3 can be 0.

```
> G := Groebner[Basis](I,tdeg(x1,x2,x3,x4,u1,u2,u3));
```

$$G := [x2 - u3, -u1\, u3 - u2\, u3 + u3\, x1, -u1\, u3 + u1\, x4 - u2\, x4 + u3\, x3, -u3\, x3 + x1\, x4, u1\, u3^2 + 2\, u2\, u3\, x4 - 2\, u3^2\, x3, -u1^2\, u3 - u1\, u2\, u3 + 2\, u1\, u3\, x3]$$

```
> factor(G);
```

$$[x2 - u3, -u3\,(-x1 + u1 + u2), -u1\,u3 + u1\,x4 - u2\,x4 + u3\,x3, -u3\,x3 + x1\,x4, u3\,(u1\,u3 + 2\,u2\,x4 - 2\,u3\,x3), -u1\,u3\,(u1 + u2 - 2\,x3)]$$

Let's specify that $u1 \neq 0$ and $u3 \neq 0$.

```
> J := <h1,h2,h3,h4,1-u1*u3*t>;
```

$$J := \langle x2 - u3, u3\,(u1 - x3) - x4\,(u1 - u2), (x1 - u1)\,u3 - u2\,u3, -t\,u1\,u3 + 1, x1\,x4 - x2\,x3\rangle$$

```
> IdealInfo[Variables](J);
```

$$\{t, u1, u2, u3, x1, x2, x3, x4\}$$

Okay, so we want to eliminate t. We compute $I \cap \mathbb{Q}[x1, x2, x3, x4, u1, u2, u3]$.

```
> J := EliminationIdeal(J,{x1,x2,x3,x4,u1,u2,u3});
```

$$J := \langle x2 - u3, 2\,x3 - x1, 2\,x4 - u3, -x1 + u1 + u2\rangle$$

```
> IsRadical(J);
```

$$true$$

```
> PrimeDecomposition(J);
```

$$\langle x2 - u3, 2\,x3 - x1, 2\,x4 - u3, -x1 + u1 + u2\rangle$$

Well, the ideal is now radical and prime (and linear) so it should be easy

```
> IdealMembership(g1,J);
  IdealMembership(g2,J);
```

$$true$$

$$true$$

If we do the test in $\mathbb{Q}(u1, u2, u3)[x1, x2, x3, x4]$ we don't need to say $u \neq 0$ and $u3 \neq 0$.

```
> K := <h1,h2,h3,h4,(variables={x1,x2,x3,x4})>;
```

$$K := \langle x2 - u3, u3\,(u1 - x3) - x4\,(u1 - u2), (x1 - u1)\,u3 - u2\,u3, x1\,x4 - x2\,x3\rangle$$

```
> IdealInfo[Variables](K);
```

$$\{x1, x2, x3, x4\}$$

```
> IdealInfo[Parameters](K);
```

$$\{u1, u2, u3\}$$

```
> IsRadical(K);
```

$$true$$

```
> IdealMembership(g1,K);
  IdealMembership(g2,K);
```

$$true$$

$$true$$

The test for $g1 \in \sqrt{K}$ is if $1 \in \sqrt{I + \langle 1 - g1\,y\rangle}$

```
> R := <h1,h2,h3,h4,1-g1*y,(variables={x1,x2,x3,x4,y})>;
```

$$R := \langle x2 - u3, u3\,(u1 - x3) - x4\,(u1 - u2), (x1 - u1)\,u3 - u2\,u3, -(x3^2 + x4^2 - (x1 - x3)^2 - (x2 - x4)^2)\,y + 1, x1\,x4 - x2\,x3\rangle$$

```
> IdealMembership(1,R);
```

$$true$$

Well, if $1 \in \sqrt{K}$ then how come $R \neq \langle 1 \rangle$. Because "forming" and ideal with < .... > does not automatically cause a Groebner basis computation. But this does

```
> Simplify(R);
```

$$\langle 1 \rangle$$