Testing ideals for primality (examples over *k* = <)

```
> F := [x^2+y+z,x+y^2+z,x+y+z^2];
```

$$F := [x^2+y+z,\ y^2+x+z,\ z^2+x+y]$$

```
> G := Groebner[Basis](F,plex(x,y,z));
```

$$G := [z^6-z^4+4z^3-2z^2+4z,\ z^4+2yz^2+z^2+2y,\ y^2-z^2-y+z,\ z^2+x+y]$$

```
> factor(G[1]);
```

$$z(z+2)(z^2+1)(z^2-2z+2)$$

Since G[1] factors over **<** this means I is not prime over **<**.

```
> F := [x^2+1,y^2+1,z^2+1];
```

$$F := [x^2+1,\ y^2+1,\ z^2+1]$$

```
> Groebner[Basis](F,plex(x,y,z));
```

$$[z^2+1,\ y^2+1,\ x^2+1]$$

Notice that F is a Groebner basis for $\langle x^2+1,\ y^2+1,\ z^2+1\rangle$ wrt. any monomial ordering by Proposition 4 of 2.9 since the leading monomials of the generators are x^2, y^2, z^2 for any monomial ordering. Let's try a linear transformation on the ideal *I* of the form

y / *y* + *a x*, *z* / *z* + *b x* + *c y*.

```
> Ft := subs(y=y+3*x,z=z+5*x,F);
```

$$Ft := [x^2+1,\ (y+3x)^2+1,\ (z+5x)^2+1]$$

```
> Gt := Groebner[Basis](Ft,plex(x,y,z));
```

$$Gt := [z^4+52z^2+576,\ -yz^3+40y^2-76yz-320,\ z^3+240x+76z]$$

```
> factor(Gt[1]);
```

$$(z^2+16)(z^2+36)$$

```
> Ft := subs(y=y+3*x,z=z+5*x-3*y,F);
```

$$Ft := [x^2+1,\ (y+3x)^2+1,\ (z+5x-3y)^2+1]$$

```
> Gt := Groebner[Basis](Ft,plex(x,y,z));
```

$$Gt := [z^8+824z^6+238864z^4+28590336z^2+1194393600,\ -505z^7-320432z^5 -60738160z^3+7057290240y-4785954048z,\ 97z^7+83384z^5+26660368z^3 +28229160960x+4752112896z]$$

```
> f := Gt[1];
```

$$f := z^8+824z^6+238864z^4+28590336z^2+1194393600$$

The Groebner basis is of the form $[f(z),\ a\cdot y+g(z),\ b\cdot x+h(z)]$ for some constants $a, b \in \mathbb{Q}$. Thus I is prime $\Leftrightarrow$ $f(z)$ is irreducible over $\mathbb{Q}$. Also I is radical $\Leftrightarrow f(z)$ is square-free (i.e., has no repeated factors).

```
> factor(f);
```

$$(z^2+144)(z^2+256)(z^2+100)(z^2+324)$$

Hence we conclude I is radical but it is not prime over **<**.