MATH 895, Assignment 2, Summer 2009

Instructor: Michael Monagan

Please hand in the assignment by 9:30am on Monday June 6th by 5pm.
Late Penalty -20% off for up to 24 hours late. Zero after that.
Please submit a printout of a Maple worksheet containing Maple code and output.

Download and read the papers "Sparse Polynomial Arithmetic" by Stephen Johnson and "Polynomial Division using Dynamic Arrays, Heaps and Packed Exponent Vectors" by Monagan and Pearce from

`http://www.cecm.sfu.ca/~mmonagan/teaching/TopicsinCA11/`

The goal of this assignment is to study how the heap data structure can be used for polynomial multiplication and division using a sparse distributed representation.

1 Multiplication.

Let $f, g \in R[x_1, x_2, ..., x_n]$ where $f = f_1 + f_2 + ... + f_{\#f}$ and $g = g_1 + g_2 + ... + g_{\#g}$. To compute $h = f \times g$ we suggested using the following divide an conquer approach, namely, if $\#f > 1$ compute

$$A = (f_1 + f_2 + ... + f_k) \times g \quad \text{and} \quad B = (f_{k+1} + ... + f_{\#f}) \times g \quad \text{where} \quad k = \lfloor \#f/2 \rfloor$$

recursively then add $A + B$ using a merge. Determine the number of comparisons this algorithm does in the worst case. Express your answer in big O notation in terms of $\#f$ and $\#g$.

2 Division.

Let $A, B \in \mathbb{Q}[x, y, z, ...]$ and let Q be the quotient of A divided B. Represent a polynomial as a Maple list of terms sorted in descending *graded lexicographical* order. Represent each term in the form $[c, e]$ where $c \in \mathbb{Q}$ is a coefficient and e, the exponent vector, is encoded as an integer. E.g. the monomial $x^i y^j z^k$ with exponent vector $[i, j, k]$ would be represented as the integer $(i + j + k)B^2 + iB + j$ where $B = 2^L$ bounds the total degree d of any monomial that appears in the division algorithm. Implement the following Maple procedures where X is a list of variables.

```
SDMP2Maple(a,X,d)
Maple2SDMP(A,X)
```

E.g. `A := SDMP2Maple(a,[x,y,z],d)` converts a Maple polynomial $a(x, y, z)$ into the SDMP data structure and `Maple2SDMP(A,[x,y,z])` converts it back. Note, to convert an integer E to base B in Maple use `convert(E,base,B);` Now implement the following algorithms.

1 Johnson's heap multiplication algorithm: $f \times g = \sum_{i=1}^{\#f} f_i \times g$,

2 Division using the repeated merging algorithm: $((f - q_1 \times g) - q_2 \times g) - q_3 \times g - ...$

3 Johnson's quotient heap division algorithm: $f - \sum_{i=1}^{\#q} q_i \times g$,

For the heap algorithms you may use Maple's `heap` package. See `?heap`.
Execute your algorithm on the following sparse problem using your distributed data structure for each algorithm.

```
> X := [u,v,w,x,y,z];
> a := randpoly(X,degree=10,terms=2500):
> b := randpoly(X,degree=5,terms=8):
> c := expand(a*b):
> nops(a), nops(b), nops(c);
                                        2479, 8, 16401

> d := degree(a)+degree(b);
> A := Maple2SDMP(a,X,d):
> B := Maple2SDMP(b,X,d); # show your data structure for this one
> C := Maple2SDMP(c,X,d):
> H := MULTIPLY(A,B): evalb(H=C);
> H := MULTIPLY(B,A): evalb(H=C);
> Q := DIVIDE(C,A); evalb(Q=B); # show output for Q
> Q := DIVIDE(C,B): evalb(Q=A);
```

Compute and print (i) N = the number of monomial comparisons each algorithm makes, (ii) M = the number of coefficient multiplications + divisions each algorithm makes and (iii) the quantity $S = N/M$ which measures the monomial comparisons relative to the coefficient arithmetic cost. In a few sentences discuss whether the results agree with the theoretical cost estimates of the algorithms.

Final question. If R is a ring and $f \in K[x_1, \ldots, x_n]$ has total degree d, find a formula in n and d for the maxium number of terms that f can have?