Upper Bounds for the Derivative of Exponential Sums

Peter Borwein and Tamás Erdélyi
Department of Mathematics
Statistics and Computing Science
Dalhousie University
Halifax, Nova Scotia
Canada B3H 3J5

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Abstract. The equality

$$\sup_{p} \frac{|p'(a)|}{\|p\|_{[a,b]}} = \frac{2n^2}{b-a}$$

is shown, where the supremum is taken for all exponential sums $p$ of the form

$$p(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, \quad a_j \in \mathbb{R},$$

with nonnegative exponents $\lambda_j$. The inequalities

$$\|p'\|_{[a+\delta,b-\delta]} \leq 4(n+2)^3\delta^{-1}\|p\|_{[a,b]}$$

and

$$\|p'\|_{[a+\delta,b-\delta]} \leq 4\sqrt{2}(n+2)^3\delta^{-3/2}\|p\|_{L_2[a,b]}$$

are also proved for all exponential sums of the above form with arbitrary real exponents. These results improve inequalities of Lorentz and Schmidt and partially answer a question of Lorentz.

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1. Introduction and Notation
Let \( \Lambda_n := \{ \lambda_1 < \lambda_2 < \cdots < \lambda_n \} \), \( \lambda_j \neq 0 \), \( j = 1, 2 \ldots, n \),

\[
E(\Lambda_n) = \{ f : f(t) = a_0 + \sum_{j=1}^{n} a_j e^{\lambda_j t}, \ a_j \in \mathbb{R} \}
\]

and

\[
E_n := \bigcup_{\Lambda_n} E(\Lambda_n) = \{ f : f(t) = a_0 + \sum_{i=1}^{n} a_j e^{\lambda_i t}, \ a_j, \lambda_j \in \mathbb{R} \}.
\]

We will use the norms

\[
\|f\|_{[a,b]} := \max_{x \in [a,b]} |f(x)|
\]

and

\[
\|f\|_{L_2[a,b]} := \left( \int_{a}^{b} |f(x)|^2 \, dx \right)^{1/2}
\]

for functions \( f \in C[a,b] \).

Schmidt [3] proved that there is a constant \( c(n) \) depending only on \( n \) so that

\[
\|p'\|_{[a+\delta,b-\delta]} \leq c(n)\delta^{-1}\|p\|_{[a,b]}
\]

for every \( p \in E_n \) and \( \delta \in (0,(b-a)/2) \). Lorentz [2] improved Schmidt’s result by showing that for every \( \alpha > \frac{1}{2} \) there is a constant \( c(\alpha) \) depending only on \( \alpha \) so that \( c(n) \) in the above inequality can be replaced by \( c(\alpha)n^{\alpha \log n} \), and he speculated that there may be an absolute constant \( c \) so that Schmidt’s inequality holds with \( c(n) = cn \). Theorem 2 of this paper shows that Schmidt’s inequality holds with \( c(n) = 4(n+2)^3 \). Our first theorem establishes the sharp inequality

\[
|p'(a)| \leq \frac{2n^2}{b-a} \|p\|_{[a,b]} 
\]

for every \( p \in E_n \) with nonnegative exponents \( \lambda_j \).

2. New Results

Theorem 1. We have

\[
\sup_p \frac{|p'(a)|}{\|p\|_{[a,b]}} = \frac{2n^2}{b-a}
\]

for every \( a < b \), where the supremum is taken for all exponential sums \( p \in E_n \) with nonnegative exponents. The equality

\[
\sup_p \frac{|p'(a)|}{\|p\|_{[a,b]}} = \frac{2n^2}{a(\log b - \log a)}
\]
also holds for every $0 < a < b$, where the supremum is taken for all Müntz polynomials of the form
\[ p(x) = a_0 + \sum_{j=1}^{n} a_j x^{\lambda_j}, \quad a_j \in \mathbb{R}, \quad \lambda_j \geq 0. \]

**Theorem 2.** The inequalities
\[ \|p'\|_{[a+\delta,b-\delta]} \leq 4(n+2)^3 \delta^{-1} \|p\|_{[a,b]} \]
and
\[ \|p'\|_{[a+\delta,b-\delta]} \leq 4\sqrt{2}(n+2)^3 \delta^{-3/2} \|p\|_{L_2[a,b]} \]
hold for every $p \in E_n$ and $\delta \in (0,(b-a)/2)$.

3. Proofs

To prove Theorem 1 we need some notation. If $\Lambda_n := \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\}$ is a set of positive real numbers then the real span of
\[ \{1, x^{\lambda_1}, x^{\lambda_2}, \cdots x^{\lambda_n}\}, \quad x \geq 0, \]
will be denoted by $M(\Lambda_n)$. It is well-known that these are Chebyshev spaces (see [1] for instance), so $M(\Lambda_n)$ possesses a unique Chebyshev “polynomial” $T_{\Lambda_n}$ on $[a,b]$, $0 < a < b$, with the properties
(i) $T_{\Lambda_n} \in M(\Lambda_n)$,
(ii) $\|T_{\Lambda_n}\|_{[a,b]} = 1$
and
(iii) there are $a = x_0 < x_1 < \cdots < x_n = b$ so that
\[ T_{\Lambda_n}(x_j) = (-1)^j, \quad j = 0, 1, \cdots, n. \]

It is routine to prove (see [1] again) that $T_{\Lambda_n}$ has exactly $n$ distinct zeros on $(a,b),
\[ \max_{0 \neq p \in M(\Lambda_n)} \frac{|p'(a)|}{\|p\|_{[a,b]}} = \frac{|T_{\Lambda_n}'(a)|}{\|T_{\Lambda_n}\|_{[a,b]}} = |T_{\Lambda_n}'(a)| \] \tag{1}
and
\[ \max_{0 \neq p \in M(\Lambda_n)} \frac{|p(0)|}{\|p\|_{[a,b]}} = \frac{|T_{\Lambda_n}(0)|}{\|T_{\Lambda_n}\|_{[a,b]}} = |T_{\Lambda_n}(0)|. \] \tag{2}

**Lemma 3.** Let
\[ \Lambda_n := \{\lambda_1 < \lambda_2 < \cdots < \lambda_n\} \quad \text{and} \quad \Gamma_n := \{\gamma_1 < \gamma_2 < \cdots < \gamma_n\} \]
be so that $0 < \lambda_j \leq \gamma_j$ for each $j = 1, 2, \cdots, n$. Then

$$| T^r_{\lambda_n}(a) | \leq | T^r_{\Lambda_n}(a) |.$$  \hspace{1cm} (3)

Proof. Without loss of generality we may assume that there is an index $m$, $1 \leq m \leq n$, so that $\lambda_m < \gamma_m$ and $\lambda_j = \gamma_j$ if $j \neq m$, since repeated applications of the result in this situation give the lemma in the general case. First we show that

$$| T_{\Gamma_n}(0) | < | T_{\Lambda_n}(0) |.$$  \hspace{1cm} (4)

Indeed, let $R_{\Gamma_n} \in M(\Gamma_n)$ interpolate $T_{\Lambda_n}$ at the zeros of $T_{\Lambda_n}$, and be normalized so that $R_{\Gamma_n}(0) = T_{\Lambda_n}(0)$. Then the Improvement Theorem of Pinkus and Smith [4, Theorem 2] yields

$$| R_{\Gamma_n}(x) | \leq | T_{\Lambda_n}(x) | \leq 1, \quad x \in [a, b].$$

Hence, using (2) with $\Lambda_n$ replaced by $\Gamma_n$, we obtain

$$| T_{\Lambda_n}(0) | = | R_{\Gamma_n}(0) | \leq | T_{\Gamma_n}(0) |,$$

which proves (4). Using the defining properties of $T_{\Lambda_n}$ and $T_{\Gamma_n}$, we deduce that $T_{\Lambda_n} - T_{\Gamma_n}$ has at least $n + 1$ zeros in $[a, b]$ (we count every zero without sign change twice). Now assume that (3) does not hold, then

$$| T^r_{\Lambda_n}(a) | > | T^r_{\Gamma_n}(a) |.$$

This, together with (4), implies that $T_{\Lambda_n} - T_{\Gamma_n}$ has at least one zero in $(0, a)$. Hence $T_{\Lambda_n} - T_{\Gamma_n}$ has at least $n + 2$ zeros in $(0, b]$. This is a contradiction, since

$$T_{\Lambda_n} - T_{\Gamma_n} \in \text{span} \{1, x^{\Lambda_1}, x^{\Lambda_2}, \cdots, x^{\Lambda_n}, x^{\gamma_m}\},$$

and every function from the above span can have only at most $n + 1$ zeros in $(0, \infty)$ (see [3]). \hfill \Box

Proof of Theorem 1. It is sufficient to prove only the second statement of the theorem, the first one can be obtained by the change of variable $x = e^t$. We obtain from (1) and Lemma 3 that

$$\frac{| p'(a) |}{\| p \|_{[a, b]}} \leq \lim_{\delta \to 0^+} \frac{| T^r_{\lambda_n, \delta}(a) |}{\| T_{\lambda_n, \delta} \|_{[a, b]}} = \lim_{\delta \to 0^+} \frac{| T^r_{\lambda_n, \delta}(a) |}{| T_{\lambda_n, \delta}(a) |}$$

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for every $p$ of the form

$$p(x) = a_0 + \sum_{j=1}^{n} a_j x^{\lambda_j}, \quad a_j \in \mathbb{R}, \quad \lambda_j > 0,$$

where

$$\Lambda_{n,\delta} := \{\delta, 2\delta, 3\delta, \ldots, n\delta\}$$

and $T_{n,\delta}$ is the Chebyshev “polynomial” of $M(\Lambda_{n,\delta})$ on $[a, b]$. From the definition and uniqueness of $T_{\Lambda_{n,\delta}}$ it follows that

$$T_{\Lambda_{n,\delta}}(x) = T_n \left( \frac{2}{b^\delta - a^\delta} x^\delta - \frac{b^\delta + a^\delta}{b^\delta - a^\delta} \right),$$

where $T_n(y) := \cos(n \arccos y)$. Therefore

$$|T_{\Lambda_{n,\delta}}'(a)| = |T_n'(-1)| \frac{2}{b^\delta - a^\delta} \delta a^{\delta-1}$$

$$= \frac{2n^2}{\delta - 1(b^\delta - 1) - \delta^{-1}(a^\delta - 1)} a^{\delta-1} \delta^{-1} a(\log b - \log a)$$

and the theorem is proved. \hfill \Box

To prove Theorem 2 we need two lemmas.

**Lemma 4.** For every set $\Lambda_n := \{\lambda_1 < \lambda_2 < \ldots \lambda_n\}$ of nonzero real numbers there is a point $y \in [-1, 1]$ depending only on $\Lambda_n$ so that

$$|p'(y)| \leq 2(n + 2)^3 \|p\|_{L_2[-1,1]}$$

for every $p \in E(\Lambda_n)$.

**Proof.** Take the orthonormal set $\{p_k\}_{k=0}^{n}$ on $[-1, 1]$ defined by

(i) $p_k \in \text{span}\{1, e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}\}, \quad k = 0, 1, \ldots, n,$

(ii) $\int_{-1}^{1} p_i p_j = \delta_{i,j}, \quad 0 \leq i \leq j \leq n.$

Writing $p \in E(\Lambda_n)$ as a linear combination of the functions $p_k$, $k = 0, 1, \ldots, n$, and using the Cauchy-Schwartz inequality and the orthonormality of $\{p_k\}_{k=0}^{n}$ on $[-1, 1]$, we obtain in a standard fashion that

$$\max_{p \in E(\Lambda_n)} \frac{|p'(t_0)|}{\|p\|_{L_2[-1,1]}^{1/2}} = \left( \sum_{k=0}^{n} p_k(t_0)^2 \right)^{1/2}, \quad t_0 \in \mathbb{R}.$$ 

Let

$$A_k := \{t \in [-1, 1] : |p_k(t)| \geq (n + 1)^{1/2}\}, \quad k = 0, 1, \ldots, n$$

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and

\[ B_k := \{ t \in [-1, 1] \setminus A_k : |p_k'(t)| \geq 2(n + 2)^{5/2} \}, \quad k = 0, 1, \ldots, n. \]

Since \( \int_{-1}^{1} p_k'^2 \, dt = 1 \), we have

\[ m(A_k) \leq (n + 1)^{-1}, \quad k = 0, 1, \ldots, n. \]

Since \( \text{span}\{1, e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_k t}\} \) is a Chebyshev system, each \( A_k := [-1, 1] \setminus A_k \) comprises of at most \( k+1 \) intervals, and each \( B_k \) comprises of at most \( 2(k+1) \) intervals. Therefore

\[ 2(n + 2)^{5/2} m(B_k) \leq \int_{B_k} |p_k'(t)| \, dt \leq 4(k + 1) \sqrt{n + 1}, \]

whence

\[ \sum_{k=0}^{n} m(B_k) \leq \frac{2\sqrt{n + 1}}{(n + 2)^{5/2}} \frac{(n + 1)(n + 2)}{2} < 1. \]

Now let

\[ A := [-1, 1] \setminus \bigcup_{k=0}^{n} (A_k \cup B_k). \]

Then

\[ m(A) \geq 2 - \sum_{k=0}^{n} m(A_k) - \sum_{k=0}^{n} m(B_k) \]

\[ > 2 - (n + 1)(n + 1)^{-1} - 1 > 0, \]

so there is a point \( y \in A \subset [-1, 1] \), where

\[ |p'(y)| \leq 2(n + 1)^{5/2}, \quad k = 0, 1, \ldots, n, \]

hence

\[ \left( \sum_{k=0}^{n} p_k'(y)^2 \right)^{1/2} \leq 2(n + 2)^3, \]

and the lemma is proved. \( \Box \)

**Lemma 5.** We have

\[ |p'(0)| \leq 2(n + 2)^3 \|p\|_{L_2[-2,2]} \leq 2(n + 2)^3 \|p\|_{[-2,2]} \]

for every \( p \in E_n \).
Proof. Let $\Lambda_n := \{\lambda_1 < \lambda_2 < \cdots, \lambda_n\}$ be a fixed set of nonzero real numbers, and let $y \in [-1, 1]$ be chosen by Lemma 4. Let $0 \neq p \in E(\Lambda_n)$. Then
\[ q(t) := p(t - y) \in E(\Lambda_n), \]
therefore, applying Lemma 4 to $q$, we obtain
\[ \frac{|p'(0)|}{\|p\|_{L_2[-2,2]}} \leq \frac{|p'(0)|}{\|p\|_{L_2[-1-y,1-y]}} = \frac{|q'(y)|}{\|q\|_{L_2[-1,1]}} \leq 2(n + 2)^3, \]
and the lemma is proved. \hfill \Box

Proof of Theorem 2. Let $t_0 \in [a + \delta, b - \delta]$. Applying Lemma 5 to $q(t) := p(\delta t / 2 + t_0)$, we get the theorem. \hfill \Box

References


