DENSE MARKOV SPACES AND
UNBOUNDED BERNSTEIN INEQUALITIES

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Abstract. An infinite Markov system \( \{ f_0, f_1, \ldots \} \) of \( C^2 \) functions on \([a, b]\) has dense span in \( C[a, b] \) if and only if there is an unbounded Bernstein inequality on every subinterval of \([a, b]\). That is if and only if, for each \([\alpha, \beta] \subseteq [a, b] \) and \( \gamma > 0 \), we can find \( g \in \text{span}\{ f_0, f_1, \ldots \} \) with \( \|g'\|_{[\alpha, \beta]} > \gamma \|g\|_{[\alpha, \beta]} \). This is proved under the assumption \( (f_1/f_0)' \) does not vanish on \((a, b)\).

Extension to higher derivatives are also considered. An interesting consequence of this is that functions in the closure of the span of a non-dense \( C^2 \) Markov system are always \( C^\infty \) on some subinterval.

The principal result of this paper will be a characterization of denseness of the span of a Markov system by whether or not it possesses an unbounded Bernstein Inequality. In order to make sense of this result we require the following definitions.

Definition 1 (Chebyshev System). Let \( f_0, \ldots, f_n \) be elements of \( C[a, b] \) the real valued continuous functions on \([a, b]\). Suppose that \( \text{span}\{ f_0, \ldots, f_n \} \) over \( \mathbb{R} \) is an \( n + 1 \) dimensional subspace of \( C[0,1] \). Then \( \{ f_0, \ldots, f_n \} \) is called a Chebyshev system of dimension \( n + 1 \) if any element of \( \text{span}\{ f_0, \ldots, f_n \} \) that has \( n + 1 \) distinct zeros in \([0,1]\) is identically zero. If \( \{ f_0, \ldots, f_n \} \) is a Chebyshev system, then \( \text{span}\{ f_0, \ldots, f_n \} \) is called a Chebyshev space.

Definition 2 (Markov System). We say that \( \{ f_0, \ldots, f_n \} \) is a Markov system on \([a, b]\) if each \( f_i \in C[a, b] \) and \( \{ f_0, \ldots, f_m \} \) is a Chebyshev system for every \( m \geq 0 \). (We allow \( n \) to tend \(+\infty\) in which case we call the system an infinite Markov system.) If \( \{ f_0, \ldots, f_n \} \) is a Markov system then \( \text{span}\{ f_0, \ldots, f_n \} \) is called a Markov space.

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Definition 3 (Unbounded Bernstein Inequality). Let $A$ be a subset of $C^1[a,b]$. We say that $A$ has an everywhere unbounded Bernstein inequality if for every $[\alpha, \beta] \subset [a,b]; \; \alpha \neq \beta$

$$\sup \left\{ \frac{\|p'||_{[\alpha, \beta]}}{\|p\|_{[a,b]}} : p \in A, p \neq 0 \right\} = \infty.$$ 

If for some $[\alpha, \beta]$ the above sup is finite the Bernstein inequality is said to be bounded in $[\alpha, \beta]$.

Note that the collection of all polynomials of the form

$$\{x^2p(x) : p \text{ is a polynomial}\}$$

has an everywhere unbounded Bernstein inequality on $[-1,1]$ despite the fact that every element has derivative vanishing at zero.

We now state the main result.

Theorem 1. Suppose $\mathcal{M} := \{f_0, f_1, f_2, \ldots \}$ is an infinite Markov system on $[a,b]$ with each $f_i \in C^2[a,b]$, and suppose that $(f_1/f_0)'$ does not vanish on $(a,b)$. Then span $\mathcal{M}$ is dense in $C[a,b]$ if and only if span $\mathcal{M}$ has an everywhere unbounded Bernstein inequality.

The additional assumption that $(f_1/f_0)'$ does not vanish on $(a,b)$ is quite weak. It holds, for example, for any ECT system. Note that $f_1/f_0$ is strictly monotone if $\mathcal{M}$ is a Markov system.

The proof requires examining the Chebyshev polynomials associated with a Chebyshev system. These we now discuss.

Suppose

$$H_n := \text{span}\{f_0, \ldots, f_n\}$$

is a Chebyshev space on $[a,b]$. We can define the Chebyshev polynomial

$$T_n(x) := T_n\{f_0, \ldots, f_n; [a,b]\}(x)$$

associated with $H_n$

by

$$T_n(x) = c \left( f_n(x) - \sum_{k=0}^{n-1} a_k f_k(x) \right)$$

where the $\{a_k\}_{k=0}^{n-1}$ are chosen to minimize

$$\left\| f_n - \sum_{k=0}^{n-1} a_k f_k \right\|_{[a,b]}$$
and where $c$ is a normalization constant chosen so that

$$\|T_n\|_{[a,b]} = 1 \quad \text{and} \quad T_n(b) > 0.$$ 

We will call $T_n$ the associated Chebyshev polynomial for $H_n$. This is a unique “generalized” polynomial in $\text{span}\{f_0, \ldots, f_n\}$ that alternates between $\pm 1$ exactly $n + 1$ times and has exactly $n$ zeros on $[a, b]$. With $f_i := x^i$, this generates the usual Chebyshev polynomials. These equioscillating polynomials encode much of the information of how the space $H_n$ behaves with respect to the supremum norm. See [2], [3], [4] and [6].

Suppose

$$\mathcal{M} = \{f_0, f_1, \ldots\}$$

is a fixed infinite Markov system on $[a, b]$. For each $n$

$$H_n := \{f_0, f_1, \ldots, f_n\}$$

is then a Chebyshev system. So there is a sequence $\{T_n\}$ of associated Chebyshev polynomials where, for each $n$, $T_n$ is associated with $H_n$. These we call the associated Chebyshev polynomials for the infinite Markov system $\mathcal{M}$.

Note that

$$\{T_0, T_1, \ldots\}$$

is a Markov system again with the same span as $\mathcal{M}$.

In [2] we showed that the span of a $C^1$ Markov system $\mathcal{M}$ is dense in $C[a, b]$ in the uniform norm (i.e. the uniform closure of span $\mathcal{M}$ on $[a, b]$ equals $C[a, b]$) if and only if the zeros of the associated Chebyshev polynomials are dense. To state this result, which we will need, we require the following notation.

Suppose $T_n$ has zeros $a \leq x_1 < x_2 < \cdots < x_n \leq b$, and let $x_0 := a$ and $x_{n+1} := b$. Then the mesh of $T_n$ is defined by

$$M_n := M_n(T_n : [a, b]) := \max_{1 \leq i \leq n+1} |x_i - x_{i-1}|.$$ 

For a sequence of Chebyshev polynomials $T_n$ from a fixed Markov system on $[a, b]$ we have

$$M_n \to 0 \quad \text{iff} \quad \lim M_n = 0$$

as follows from the interlacing of the zeros of $T_n$ and $T_{n+1}$ (see [6]).

Our main result requires the following theorem from [2].

**Theorem 2.** Suppose $\mathcal{M} := \{1, f_1, f_2, \ldots\}$ is an infinite Markov system on $[a, b]$ with each $f_i \in C^1[a, b]$. Then span $\mathcal{M}$ is dense in $C[a, b]$ in the uniform norm if and only if

$$M_n \to 0$$

(where $M_n$ is the mesh of the associated Chebyshev polynomials).

The next result we need shows that in most instances the Chebyshev polynomial is close to extremal for Bernstein-type inequalities.
Theorem 3. Let $H_n := \{1, f_1, \ldots, f_n\}$ be a Chebyshev system of $C^1$ functions on $[a, b]$. Let $T_n$ be the associated Chebyshev polynomial. Then

$$\frac{|p'_n(x_0)|}{\|p_n\|_{[a,b]}} \leq \frac{2}{1 - |T_n(x_0)|} |T'_n(x_0)|$$

for every $0 \neq p_n \in \text{span}\{1, f_1, \ldots, f_n\}$ and every $x_0 \in [a, b]$ with $|T_n(x_0)| \neq 1$.

Proof. Let $a = y_0 < y_1 < \ldots < y_n = b$ denote the extreme points of $T_n$, so

$$T_n(y_i) = (-1)^{n-i}, \quad i = 0, 1, \ldots, n.$$ 

Let $y_k \leq x_0 \leq y_{k+1}$ and $0 \neq p_n \in H_n$. If $p'_n(x_0) = 0$, then there is nothing to prove. So assume that $p'_n(x_0) \neq 0$. Then we may normalize $p_n$ so that

$$\|p_n\|_{[a,b]} = 1$$

and

$$\text{sign}(p'_n(x_0)) = \text{sign}(p(y_{k+1}) - p(y_k)).$$

Let $\delta := |T_n(x_0)|$. Let $\epsilon \in (0, 1)$ be fixed. Then there exists a constant $\eta$ with $|\eta| \leq \delta + (1 - \delta)/2$ so that

$$\eta + \frac{1 - \delta}{2}(1 - \epsilon)p_n(x_0) = T_n(x_0).$$

Now let

$$q_n(x) := \eta + \frac{1 - \delta}{2}(1 - \epsilon)p_n(x).$$

Then

$$\|q_n\|_{[a,b]} \leq 1,$$

$$q_n(x_0) = T_n(x_0)$$

and

$$\text{sign}(q'_n(x_0)) = \text{sign}(T'_n(x_0)).$$

If the desired inequality does not hold for $p_n$ then for a sufficiently small $\epsilon > 0$

$$|q'_n(x_0)| > |T'_n(x_0)|,$$

so

$$h_n(x) := q_n(x) - T_n(x)$$

will have at least 3 zeros in $(y_k, y_{k+1})$. But $h_n$ has at least one zero in each of $(x_i, x_{i+1})$. Hence $h_n \in H_n$ has at least $n + 2$ zeros in $[a, b]$, which is a contradiction.

We need the following technical result concerning Chebyshev polynomials.
Lemma 1. Suppose \( \mathcal{M} := \{1, f_1, f_2, \ldots \} \) is an infinite Markov system of \( C^2 \) functions on \([a, b]\) and \( f_1' \) does not vanish on \((a, b)\). Suppose that the associated Chebyshev polynomials \( \{T_n\} \) has a subsequence \( \{T_{n_i}\} \) with no zeros on some subinterval of \([a, b]\). Then there exists another subinterval \([c, d]\) and another infinite subsequence \( \{T_{n_i}\} \) so that for some \( \delta > 0, \gamma > 0 \)

\[
\|T_{n_i}\|_{[c, d]} < 1 - \delta
\]

and

\[
\|T'_{n_i}\|_{[c, d]} < \gamma
\]

for all \( n_i \).

Proof. For both inequalities we first choose a subinterval \([c_1, d_1]\) \( \subset \) \([a, b]\) and a subsequence \( \{n_{i,1}\} \) of \( \{n_i\} \) so that all oscillations of each \( T_{n_{i,1}} \) take place away from \([c_1, d_1]\). We now choose a subsequence \( \{n_{i,2}\} \) of \( \{n_{i,1}\} \) so that either each \( T_{n_{i,2}} \) is increasing or each \( T_{n_{i,2}} \) is decreasing on \([c_1, d_1]\). We treat the first case, the second one is analogous. Let \([c_2, d_2]\) be the middle third of \([c_1, d_1]\). If the first inequality fails to hold with \([c_2, d_2]\) and \( \{n_{i,2}\} \) then there is a subsequence \( \{n_{i,3}\} \) of \( \{n_{i,2}\} \) so that \( \|T_{n_{i,3}}\|_{[c_2, d_2]} \to 1 \) as \( n_{i,3} \to \infty \). Hence, there is a subsequence \( \{n_{i,4}\} \) of \( \{n_{i,3}\} \) so that either

\[
\max_{c_2 \leq x \leq d_2} T_{n_{i,4}}(x) \to 1 \quad \text{or} \quad \min_{c_2 \leq x \leq d_2} T_{n_{i,4}}(x) \to -1.
\]

Once again we treat the first case, the second one is analogous. Since each \( T_{n_{i,4}} \) is increasing on \([c_1, d_1]\),

\[
\lim_{n_{i,4} \to \infty} \|1 - T_{n_{i,4}}\|_{[c_1, d_1]} = 0.
\]

Now take \( g := a_0 + a_1 f_1 + a_2 f_2 \) so that \( g \) has two distinct zeros \( \alpha_1 \) and \( \alpha_2 \) in \([d_2, d_1]\), \( \|g\|_{[\alpha_1, \alpha_2]} < 1 \) and \( g \) is positive on \((\alpha_1, \alpha_2)\). Let \( \beta := \max_{\alpha_1 \leq x \leq \alpha_2} g(x) \) and \( \tilde{g} := g + 1 - \beta \). One can now deduce that \( T_{n_{i,4}} - \tilde{g} \) has at least \( n + 1 \) distinct zeros in \([a, b]\) if \( n_{i,4} \) is large enough, which is a contradiction.

For the second inequality, by [8], \( \{f_1', f_2', \ldots\} \) is a weak Markov system on \([a, b]\), and so is

\[
\left\{ \left( T_2 / T_1 \right)', \left( T_3 / T_1 \right)', \ldots \right\}
\]

on every closed subinterval of \((a, b)\). (In the definitions of weak Markov systems and weak Chebyshev systems we only count zeros where the sign changes.) The assumption that \( f_1' \) does not vanish on \((a, b)\) implies that \( T_1' \) does not vanish on \((a, b)\).

From this we deduce that each \( (T_{n_{i,2}}' / T_1')' \) has at most one sign change in \([c_2, d_2]\). Choose a subinterval \([c_3, d_3]\) \( \subset \) \([c_2, d_2]\) and a subsequence \( \{n_{i,5}\} \) of \( \{n_{i,2}\} \) so that none of \( (T_{n_{i,5}}' / T_1')' \) changes sign in \([c_3, d_3]\). Choose a subsequence \( \{n_{i,6}\} \) of \( \{n_{i,5}\} \) so that either each \( T_{n_{i,6}}' / T_1' \) is increasing or each \( T_{n_{i,6}}' / T_1' \) is decreasing on \([c_3, d_3]\). We only study the first case, the second one is similar. Let \([c_4, d_4]\) be the middle
third of \([c_3, d_3]\). If the second inequality fails to hold with \([c_4, d_4]\) and \(\{n_{i,\gamma}\}\) then there is a subsequence \(\{n_{i,\gamma}\}\) so that either

\[
\max_{c_4 \leq x \leq d_4} \frac{T''_{n_{i,\gamma}}(x)}{T'_1(x)} \to \infty
\]
or

\[
\min_{c_4 \leq x \leq d_4} \frac{T''_{n_{i,\gamma}}(x)}{T'_1(x)} \to -\infty.
\]

Again we treat only the first case, the second one is analogous. Then for every \(K > 0\) there is \(N\) so that for every \(n_{i,\gamma} \geq N\) we have

\[
T''_{n_{i,\gamma}}(x) > K, \quad x \in [d_4, d_3],
\]
hence

\[
K(d_3 - d_4) \leq \int_{d_4}^{d_3} T''_{n_{i,\gamma}}(x)dx = T''_{n_{i,\gamma}}(d_3) - T''_{n_{i,\gamma}}(d_4) \leq 2,
\]

which is a contradiction. \(\Box\)

**Lemma 2.** Suppose \(\mathcal{M} := \{f_0, f_1, \ldots\}\) is a \(C^1[a, b]\) infinite Markov system and suppose \(g \in C^1[a, b]\) and \(g\) is strictly positive on \([a, b]\). Then \(\mathcal{N} = \{gf_0, gf_1, \ldots\}\) is also a \(C^1[a, b]\) infinite Markov system. Furthermore span \(\mathcal{M}\) has a bounded Bernstein inequality on \([\alpha, \beta] \subset [a, b]\) if and only if span \(\mathcal{N}\) also has bounded Bernstein inequality on \([\alpha, \beta]\).

**Proof.** Consider differentiating \(gf\) with \(f \in \text{span } \mathcal{M}\) by the product rule. If span \(\mathcal{M}\) has a bounded Bernstein inequality on \([\alpha, \beta]\) then

\[
\|(gf)'\|_{[\alpha, \beta]} \leq \|g'f\|_{[\alpha, \beta]} + \|gf'\|_{[\alpha, \beta]} \leq c_1\|gf\|_{[\alpha, \beta]} + c_2\|gf\|_{[\alpha, \beta]}
\]

where the first constant arises since

\[
g'(x)/g(x)
\]
is uniformly bounded on \([a, b]\) and the second constant comes from the bounded Bernstein inequality for \(f\). \(\Box\)

**Proof of Theorem 1.** The only if part of this theorem is obvious. A good uniform approximation to a function with uniformly large derivative on a subinterval \([\alpha, \beta] \subset [a, b]\) must have large derivative at some points in \([\alpha, \beta]\).

In the other direction we first note that by Lemma 2 we may assume \(f_0 \equiv 1\). We use Theorem 2 and Lemma 1 in the following way. If span \(\mathcal{M}\) is not dense then there exists a subinterval \([\alpha, \beta] \subset [a, b]\) by Theorem 2, where a subsequence of the associated Chebyshev polynomials have no zeros. By Lemma 1 from this subsequence we can pick another subsequence \(T_{n_i}\) and a subinterval \([c, d] \subset [\alpha, \beta]\) with

\[
\|T_{n_i}\|_{[c, d]} < 1 - \delta
\]
and

\[
\|T''_{n_i}\|_{[c, d]} < \gamma
\]
for some positive constants \(\delta\) and \(\gamma\). The result now follows from Theorem 3. \(\Box\)
Corollary 1. Suppose $\mathcal{M} = \{f_0, f_1, \ldots \}$ is an infinite Markov system of $C^2$ functions on $[a,b]$ so that span $\mathcal{M}$ fails to be dense in $C[a,b]$ in the uniform norm. Then there exists a subinterval $[\alpha, \beta]$ of $[a,b]$ so that if $g$ is in the uniform closure of span $\mathcal{M}$ then $g$ is differentiable on $[\alpha, \beta]$.

Proof. By Theorem 1, there exists an interval $[\alpha, \beta]$ where $\|h'\|_{[\alpha, \beta]}/\|h\|_{[\alpha, \beta]}$ is uniformly bounded for every $h \in \text{span} \mathcal{M}$. Suppose $h_n \to g, h_n \in \text{span} \mathcal{M}$. Then we can choose $n_i$ so that

$$\|g - h_{n_i}\|_{[\alpha, \beta]} \leq \frac{1}{2^i}, \quad i = 0, 1, 2, \ldots$$

and hence

$$g = \sum_{i=1}^{\infty} (h_{n_i} - h_{n_{i-1}}) + h_{n_0}.$$

Since

$$\|(h_{n_i} - h_{n_{i-1}})'\|_{[\alpha, \beta]} \leq \frac{c}{2^i}$$

for some constant $c$ independent of $i$, if follows that $g$ is differentiable on $[\alpha, \beta]$.

Suppose $\mathcal{M} = \{f_0, f_1, \ldots \}$ is an extended complete Markov system of $C^\infty$ functions on $[a,b]$ (the extra requirement being that the multiplicity of the zeros matters in the definition: so if $f := \sum_{i=0}^{n} a_i f_i$ has $n+1$ zeros by counting multiplicities then $f = 0$ identically). In this case the differential operator $D$ defined by

$$D(f) := \left( \frac{f}{f_0} \right)'$$

maps $\mathcal{M}$ to $\mathcal{M}_D$ where

$$\mathcal{M}_D = \left\{ \left( \frac{f_1}{f_0} \right)', \left( \frac{f_2}{f_0} \right)', \ldots \right\}$$

and $\mathcal{M}_D$ is once again an extended complete Markov system of $C^\infty$ functions (see Nürnberger [5]). We define the differential operators $D^{(n)}(f)$ for $n$ times differentiable functions $f$ by

$$F_{i,0} := f_i, \quad F_{i,n} := \left( \frac{F_{i+1,n-1}}{F_{0,n-1}} \right)', \quad i = 0, 1, \ldots, \quad n = 1, 2, \ldots,$$

$$D^{(0)}(f) := f, \quad D^{(n)}(f) := \left( \frac{D^{(n-1)}(f)}{F_{0,n-1}} \right)', \quad n = 1, 2, \ldots.$$

Note that if span $\mathcal{M}_D$ is dense in $C[a,b]$ in the uniform norm then so is span $\mathcal{M}$. The “if” part of the next theorem can be proved from Theorem 1 by induction on $n$, while the “only if” part is obvious.
Theorem 4. Suppose $\mathcal{M} = \{f_0, f_1, \ldots\}$ is an extended complete Markov system of $C^\infty$ functions on $[a, b]$. Let $n$ be a fixed positive integer. Then span $\mathcal{M}$ is dense in $C[a, b]$ in the uniform norm if and only if

$$\sup \left\{ \frac{\|D^{(n)}(f)\|_{[\alpha, \beta]}}{\|f\|_{[a, b]}}, f \in \text{span } \mathcal{M}, f \neq 0 \right\} = \infty$$

for every $[\alpha, \beta] \subset [a, b]$, $\alpha \neq \beta$.

Corollary 2. Suppose $\mathcal{M}$ is an extended complete Markov system of $C^\infty$ functions on $[a, b]$ so that span $\mathcal{M}$ fails to be dense in $C[a, b]$ in the uniform norm. Then for each $n$ there exists an interval $[\alpha_n, \beta_n] \subset [a, b]$ of positive length where all elements of the uniform closure of span $\mathcal{M}$ are $n$ times continuously differentiable.

Proof. Use Theorem 4 as in Corollary 1. We omit the technical details. $\square$

Suppose that $\mathcal{M}$, as in Corollary 2, has the property that span $\mathcal{M}$ fails to be dense in the uniform norm on any proper subinterval of $[a, b]$, as in the case of Müntz systems

$$\mathcal{M} := \{x^{\lambda_0}, x^{\lambda_1}, \ldots\}, \quad 0 \leq \lambda_0 < \lambda_1 < \cdots, \quad \sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty, \quad 0 \leq a < b.$$

Then the uniform closure of span $\mathcal{M}$ on $[a, b]$ contains only functions that are $C^\infty$ on a dense subset of $[a, b]$. In this non-dense Müntz case the closure actually contains only analytic functions on $(a, b)$ (Achier [1], Schwartz [7]).

We record one final corollary.

Corollary 3. Suppose $\{\alpha_k\} \subset \mathbb{R} \setminus [-1, 1]$ is a sequence of distinct numbers. Then

$$\text{span} \left\{1, \frac{1}{x - \alpha_1}, \frac{1}{x - \alpha_2}, \ldots\right\}$$

is dense in $C[-1, 1]$ if and only if

$$\sum_{k=1}^{\infty} \sqrt{\alpha_k^2 - 1} = \infty.$$

Proof. The inequality

$$|p'(x)| \leq \frac{1}{\sqrt{1 - x^2}} \sum_{k=1}^{n} \frac{\sqrt{\alpha_k^2 - 1}}{|\alpha_k - x|} \|p\|_{[-1, 1]}$$

holds for any

$$p \in \text{span} \left\{1, \frac{1}{x - \alpha_1}, \ldots, \frac{1}{x - \alpha_n}\right\}.$$
See [3]. This together with Theorem 1 gives the “only if” part of the corollary.

In [3] the Chebyshev “polynomials” \( T_n \) (of the first kind) and \( U_n \) (of the second kind) for the Chebyshev space

\[
\text{span} \left\{ 1, \frac{1}{x - \alpha_1}, \ldots, \frac{1}{x - \alpha_n} \right\}
\]

are introduced. Properties of

\[
\tilde{T}_n(t) := T_n(\cos t)
\]

and

\[
\tilde{U}_n(t) := U_n(\cos t) \sin t
\]

established in [3] include

(1) \[ \|\tilde{T}_n\|_\mathbb{R} = 1 \quad \text{and} \quad \|\tilde{U}_n\|_\mathbb{R} = 1, \]

(2) \[ \tilde{T}_n(t)^2 + \tilde{U}_n(t)^2 = 1, \quad t \in \mathbb{R}, \]

(3) \[ \tilde{T}_n'(t)^2 + \tilde{U}_n'(t)^2 = \tilde{B}_n(t)^2, \quad t \in \mathbb{R}, \]

(4) \[ \tilde{T}_n(t) = -\tilde{B}_n(t)\tilde{U}_n(t), \quad t \in \mathbb{R}, \]

(5) \[ \tilde{U}_n'(t) = \tilde{B}_n(t)\tilde{T}_n(t), \quad t \in \mathbb{R} \]

where

\[
\tilde{B}_n(t) = \sum_{k=1}^{n} \frac{\sqrt{\alpha_k^2 - 1}}{|\alpha_k - \cos t|}, \quad t \in \mathbb{R}.
\]

Suppose

\[
\sum_{k=1}^{\infty} \sqrt{\alpha_k^2 - 1} = \infty.
\]

Then
(6) \[ \lim_{n \to \infty} \min_{t \in [\alpha, \beta]} \tilde{B}_n(t) = \infty, \quad 0 < \alpha < \beta < \pi. \]

Assume that there is a subinterval \([a, b]\) of \((-1, 1)\) so that

\[ \sup_{n \in \mathbb{N}} \|T'_n\|_{[a, b]} < \infty. \]

Let \(\alpha := \arccos b\) and \(\beta := \arccos a\). Then by properties (4) and (6)

\[ \lim_{n \to \infty} \|\tilde{U}_n\|_{[\alpha, \beta]} = 0 \]

hence by property (2)

\[ \lim_{n \to \infty} \|\tilde{T}_n^2 - 1\|_{[\alpha, \beta]} = 0. \]

Thus by properties (5) and (6)

\[ \lim_{n \to \infty} \min_{t \in [\alpha, \beta]} |\tilde{U}'_n(t)| = \infty \]

that is

\[ \lim_{n \to \infty} |\tilde{U}_n(\beta) - \tilde{U}_n(\alpha)| = \infty \]

which contradicts property (1). Hence

\[ \sup_{n \in \mathbb{N}} \frac{\|T'_n\|_{[\alpha, \beta]}}{\|T'_n\|_{[-1, 1]}} = \sup_{n \in \mathbb{N}} \|T'_n\|_{[\alpha, \beta]} = \infty. \]

for every subinterval \([a, b]\) of \((-1, 1)\) which together with Theorem 1 finishes the “if” part of the proof. \(\square\)

Corollary 3 is to be found in Achiesser [1, p. 255] proven by entirely different methods.

References


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