MATRIX TRANSFORMATIONS OF SERIES
OF ORTHOGONAL POLYNOMIALS

DAVID BORWEIN, PETER BORWEIN, AND AMNON JAKIMOVSKI

ABSTRACT. For a sequence of polynomials \((P_n)\) orthonormal on the interval \([-1,1]\), we consider the sequence of transforms \((g_n)\) of the series \(\sum_{k=0}^\infty a_k P_k(u)\) given by \(g_n(u) := \sum_{k=0}^\infty b_n b_k P_k(u)\). We establish necessary and sufficient conditions on the matrix \((b_{n,k})\) for the sequence \((g_n)\) to converge uniformly on compact subsets of the interior of an appropriate ellipse to a function holomorphic on that interior.

1. Introduction. Suppose throughout that \(1 < P \leq \infty\), \(1 < R < \infty\), and that all sequences and matrices are complex with indices running through \(0,1,2,\ldots\). We make the following definitions:

\(\mathbb{C}\) is the finite complex plane;

\(\gamma_R\) is the ellipse with foci \(\pm 1\) and half-axes \(a := \frac{1}{2}(R+R^{-1})\), \(b := \frac{1}{2}(R-R^{-1})\). Note that an ellipse with foci \(\pm 1\) having \(R\) as the sum of its two half-axes is necessarily \(\gamma_R\);

\(D_R^\gamma\) is the interior of the ellipse \(\gamma_R\), and \(D_\infty^\gamma := \mathbb{C}\);

\((P_n)\) is an orthonormal sequence of polynomials with respect to a fixed non-negative weight function \(w\) on the interval \([-1,1]\). That is, \(P_n\) is a polynomial of degree \(n\), and

\[\int_{-1}^1 P_n(u) P_m(u) w(u) du = \delta_{nm}.\]

We assume throughout that \(w \in L(-1,1)\) and \(w^{-\epsilon} \in L(-1,1)\) for some \(\epsilon > 0\).

The first of these integrability conditions is standard, and the second is imposed for the purposes of the present paper. The classical Jacobi polynomials, for which \(w(u) = (u - 1)^\alpha (u + 1)^\beta\) with \(\alpha, \beta > -1\), satisfy the conditions.

\(\mathcal{E}\) is the set of all sequences \(a = (a_n)\) such that \(\lim |a_n|^\frac{1}{1+n} = 0\);

*This research was supported in part by the Natural Sciences and Engineering Research Council of Canada.

1991 Mathematics Subject Classification. Primary 30C45, 47B37; Secondary 40G05.

Key words and phrases. Orthogonal polynomials, Jacobi, Chebyshev, matrix transforms, Nörlund.
$\mathcal{E}^\beta$ is the set of all sequences $\mathbf{a} \equiv (a_n)$ such that $\limsup |a_n|^{1+1} < \infty$;

$\mathcal{E}_R$ is the set of all sequences $\mathbf{a} \equiv (a_n)$ such that $\sum_{n=0}^\infty |a_n| R^n < \infty$;

$\mathcal{A}_R$ is the set of all sequences $\mathbf{a} \equiv (a_n)$ such that $\limsup |a_n|^{1/1} = \frac{1}{R}$;

The following lemma, the proof of which appears in [1], shows that $\mathcal{E}^\beta$ is the $\beta$-dual of $\mathcal{E}$.

**Lemma 1.** A sequence $\mathbf{b}$ has the property that $\sum_{n=0}^\infty b_n a_n$ is convergent for each $\mathbf{a} \in \mathcal{E}$ if and only if $\mathbf{b} \in \mathcal{E}^\beta$.

The following are the first three of an eight theorems we shall prove concerning matrix transformations of series of orthogonal polynomials. They are analogues of Theorems 1, 2 and 3 in [1] concerning matrix transformations of power series.

**Theorem 1.** A matrix $\mathbf{B} \equiv (b_{nk})$ has the property that whenever the sequence $\mathbf{a} \equiv (a_n) \in \mathcal{E}_R$ the sequence of functions $(g_n)$ given by

$$g_n(u) := \sum_{k=0}^\infty b_{nk} a_k P_k(u), \quad n = 0, 1, \ldots ,$$

converges uniformly on every compact subset of $D_P^2$, each series $\sum_{k=0}^\infty b_{nk} a_k P_k(u)$ of orthogonal polynomials being convergent on $D_P^2$, if and only if

(i) $\lim_{n \to \infty} b_{nk} =: b_k$ for $k = 0, 1, \ldots$;

(ii) $M(p) := \sup_{n \geq 0, k \geq 0} |b_{nk}| \left( \frac{p}{R} \right)^k < \infty$ whenever $1 < p < P$.

And then $\lim_{n \to \infty} g_n(u) = \sum_{k=0}^\infty b_k a_k P_k(u)$ on $D_P^2$.

**Theorem 2.** A matrix $\mathbf{B} \equiv (b_{nk})$ has the property that whenever the sequence $\mathbf{a} \equiv (a_n) \in \mathcal{A}_R$ the sequence of functions $(g_n)$ given by

$$g_n(u) := \sum_{k=0}^\infty b_{nk} a_k P_k(u), \quad n = 0, 1, \ldots ,$$
converges uniformly on every compact subset of \( D^\gamma_P \), each series \( \sum_{k=0}^{\infty} b_n a_k P_k(u) \) of orthogonal polynomials being convergent on \( D^\gamma_P \), if and only if

(i) \( \lim_{n \to \infty} b_{nk} =: b_k \) for \( k = 0, 1, \ldots \);

(ii) \( M(p) := \sup_{n \geq 0, k \geq 0} |b_{nk}| \left( \frac{p}{R} \right)^k < \infty \) whenever \( 1 < p < P \).

And then \( \lim_{n \to \infty} g_n(u) = \sum_{k=0}^{\infty} b_k a_k P_k(u) \) on \( D^\gamma_P \).

**Theorem 3.** A matrix \( B \equiv (b_{nk}) \) has the property that whenever the sequence \( a \equiv (a_n) \in E \) the sequence of functions \( (g_n) \) given by

\[
g_n(u) := \sum_{k=0}^{\infty} b_n a_k P_k(u), \quad n = 0, 1, \ldots,
\]

converges uniformly on every compact subset of \( \mathbb{C} \), each series \( \sum_{k=0}^{\infty} b_n a_k P_k(u) \) of orthogonal polynomials being convergent on \( \mathbb{C} \), if and only if

(i) \( \lim_{n \to \infty} b_{nk} =: b_k \) for \( k = 0, 1, \ldots \);

(ii) \( M := \sup_{n \geq 0, k \geq 0} |b_{nk}| \left( \frac{1}{P} \right)^k < \infty \).

And then \( \lim_{n \to \infty} g_n(u) = \sum_{k=0}^{\infty} b_k a_k P_k(u) \) on \( \mathbb{C} \).

These theorems show that if the series-to-sequence transform given by \( B \) is regular, then it is necessary in each case that \( \lim_{n \to \infty} b_{nk} = b_k = 1 \) for \( k = 0, 1, \ldots \), and this in turn implies that \( P \leq R \) in Theorems 1 and 2 (i.e., the sequence \( (g_n) \) cannot converge uniformly in the interior of any ellipse \( \gamma_P \) with \( P > R \)). Regular sequence-to-sequence transforms of power series have been considered by Peyerimhoff [8] and Luh [7] among others. One of the novel features of our approach is that we deal with series-to-sequence transforms rather than sequence-to-sequence transforms.

Let \( (B_n) \) be a sequence of non-zero complex numbers. The associated Nörlund series-to-sequence matrix \( N_B \) is the triangular matrix \( (b_{nk}) \) with

\[
b_{nk} := \begin{cases} \frac{B_{n-k}}{B_n} & \text{if } 0 \leq k \leq n, \\ 0 & \text{otherwise.} \end{cases}
\]

The following theorem is an immediate consequence of Theorem 1.
Theorem N. The Nörlund matrix $N_R$ has the property that whenever the sequence $a \equiv (a_n) \in \mathcal{E}_R$ the sequence of functions $(g_n)$ given by

$$g_n(u) := \frac{1}{B_n} \sum_{k=0}^{n} B_{n-k} a_k P_k(u), \quad n = 0, 1, \ldots,$$

converges uniformly on every compact subset of $D_R^\gamma$, if and only if

$$\lim_{n \to \infty} \frac{B_{n-1}}{B_n} = b \quad \text{with} \quad |b| = \frac{R}{P},$$

And then

$$\lim_{n \to \infty} g_n(u) = \sum_{k=0}^{\infty} b^k a_k P_k(u) \quad \text{on} \quad D_R^\gamma.$$

Note. In view of Theorem 2, Theorem N remains true if $\mathcal{E}_R$ is replaced by $A_R$.

2. Orthogonal polynomials.

In this section we set out some of the properties of orthogonal polynomials required in our proofs. Note that the function $u = \frac{1}{2}(z + z^{-1})$ maps the region $\{ z : |z| > 1 \}$ bijectively onto the region $\{ u : u \not\in [-1, 1] \}$, and that each circle $|z| = R$ is mapped onto $\gamma_R$. The inverse of this function is $z = u + \sqrt{u^2 - 1}$. Here and elsewhere in the paper the sign of the square root is chosen so that $|u + \sqrt{u^2 - 1}| > 1$ when $u \not\in [-1, 1]$. We then have, for $z = u + \sqrt{u^2 - 1}$, that $|z| = R$ when $u \in \gamma_R$, and $|z| < R$ when $u \in D_R^\gamma$. The function $u = \frac{1}{2}(z + z^{-1})$ maps both the top half and the bottom half of the unit circle $\{ z : |z| = 1 \}$ onto $[-1, 1]$.

Lemma 2. For $\epsilon > 0$ let the non-negative weight function $w \in L(-1, 1)$ associated with the orthonormal sequence of polynomials $(P_n)$ be such that $w^{-\epsilon} \in L(-1, 1)$, and let $|z| \geq 1$ and $u = \frac{1}{2}(z + z^{-1})$. Then

$$|P_n(u)| \leq K(\epsilon)(1 + n)^{2 + 2/\epsilon}|z|^n \quad \text{for} \quad n = 0, 1, \ldots,$$

where $K(\epsilon)$ is a positive number independent of $n$.

Proof. By Bernstein’s inequality (see [5, Theorem 7])

$$|P_n(u)| \leq \max_{-1 \leq t \leq 1} |P_n(t)||z|^n,$$

and by a result due to Erdélyi [2, Theorem 5]

$$\max_{-1 \leq t \leq 1} |P_n(t)| \leq K_1(\epsilon)(1 + n)^{2 + 2/\epsilon} \int_{-1}^{1} |P_n(t)| w(t) \, dt.$$

Finally, by the Cauchy-Schwarz inequality,

$$\int_{-1}^{1} |P_n(t)||w(t)| \, dt \leq \left( \int_{-1}^{1} P_n(t)^2 w(t) \, dt \right)^{1/2} \left( \int_{-1}^{1} w(t) \, dt \right)^{1/2} = \left( \int_{-1}^{1} w(t) \, dt \right)^{1/2}.$$

Combining the above inequalities we get the required result. \qed
Lemma 3. (Expansion of a holomorphic function in terms of orthogonal polynomials). Let the non-negative weight function \( w \in L(-1, 1) \) associated with the orthonormal sequence of polynomials \( (P_n) \) be such that \( w^{-\epsilon} \in L(-1, 1) \) for some \( \epsilon > 0 \). Let \( f(u) \) be holomorphic on the closed segment \([-1, 1]\), and let \( \gamma_R \) denote the largest ellipse with foci \( \pm 1 \) on the interior of which \( f(u) \) is holomorphic. The Fourier series expansion of \( f(u) \) on \( D_R^\gamma \), the interior of \( \gamma_R \), is given by

\[
f(u) = \sum_{k=0}^{\infty} a_k P_k(u),
\]

where

\[
a_k = \int_{-1}^{1} f(t)P_k(t)w(t)\,dt.
\]

The Fourier series is absolutely convergent on \( D_R^\gamma \), and is also uniformly convergent on compact subsets of \( D_R^\gamma \). It is divergent on the exterior of \( \gamma_R \). Further, the sum \( R \) of the semi-axes of the ellipse of convergence is given by

\[
\frac{1}{R} = \limsup_{k \to \infty} |a_k|^{1/k}.
\]

Proof. All but the statement about absolute convergence follows from Theorems 12.7.3 and 12.7.4 in [11], since the conditions on the weight \( w \) are more stringent than those in the said theorems. To prove the absolute convergence part, let

\[
\frac{1}{R} := \limsup_{k \to \infty} |a_k|^{1/k},
\]

and let \( u \in D_R^\gamma \). Then \( R > 1 \) and \( u = \frac{1}{2}(z + z^{-1}) \) with \( 1 \leq |z| < R \). Let \( |z| < R_0 < R \). Then \( |a_k| < R_0^{-k} \) for all sufficiently large \( k \). Hence, by Lemma 2,

\[
|a_k P_k(u)| = (|a_k||z|^k)|z^{-k} P_k(u)| \leq K(\epsilon)(1 + k)^{2+2/\epsilon} \left( \frac{|z|}{R_0} \right)^k
\]

for all sufficiently large \( k \), and therefore \( \sum_{k=0}^{\infty} |a_k P_k(u)| \) is convergent. \( \square \)

Lemma 4. (Cauchy-type inequalities for Fourier series). Let the non-negative weight function \( w \in L(-1, 1) \) associated with the orthonormal sequence of polynomials \( (P_n) \) be such that \( w^{-\epsilon} \in L(-1, 1) \) for some \( \epsilon > 0 \). Assume that the function \( f(u) \) is holomorphic on \( D_R^\gamma \) and continuous on \( \overline{D_R^\gamma} \), the closure of \( D_R^\gamma \). Let \( \sum_{k=0}^{\infty} a_k P_k(u) \) be its Fourier series. Then

\[
|a_n| \leq \frac{c(R)}{R^n} \cdot \max_{u \in \gamma_R} |f(u)| \text{ for } n = 0, 1, \ldots ,
\]
where $c(R) := \frac{2R}{R - 1} \left( \int_{-1}^{1} w(t) \, dt \right)^{\frac{1}{2}}$.

Proof. Suppose first that $n \geq 1$. By Lemma 3 we have

$$a_n = \int_{-1}^{1} f(t) P_n(t) w(t) \, dt = \int_{-1}^{1} (f(t) - q_{n-1}(t)) P_n(t) w(t) \, dt,$$

where $q_{n-1}(t)$ is any polynomial of degree $n - 1$. It follows that

$$|a_n| \leq E_{n-1}(f) \int_{-1}^{1} |P_n(t)| w(t) \, dt \leq E_{n-1}(f) \left( \int_{-1}^{1} w(t) \, dt \right)^{\frac{1}{2}},$$

where, in the notation of Lorentz [5],

$$E_{n-1}(f) := \inf_{q_{n-1}} \max_{-1 \leq t \leq 1} |f(t) - q_{n-1}(t)|.$$

Further, it is proved in [5, inequality (6), p. 78] that

$$E_{n-1}(f) \leq \frac{2R}{R - 1} \cdot \frac{1}{R^n} \cdot \max_{u \in \mathcal{R}} |f(u)|.$$

Combining the above inequalities we obtain the desired result for $n \geq 1$. Finally, the case $n = 0$ of the Cauchy-type inequality is easily seen to be true since, for $P_0 := P_0(t)$, we have

$$|P_0| \left( \int_{-1}^{1} w(t) \, dt \right)^{\frac{1}{2}} = 1. \quad \square$$

3. Proofs of Theorems 1, 2 and 3. In the proofs of Theorems 1, 2 and 3, $u$ and $z$ are related by $u = \frac{1}{2}(z + z^{-1})$, $z = u + \sqrt{u^2 - 1}$ with $|z| > 1$, the sign of the square root being chosen so that $|u + \sqrt{u^2 - 1}| > 1$.

Proof of Theorems 1 and 2. We prove these two theorems together.

Sufficiency. We assume that

$$\lim_{n \to \infty} b_{nk} =: b_k \text{ for } k = 0, 1, \ldots ;$$

$$M(p) := \sup_{n \geq 0, k \geq 0} |b_{nk}| \left( \frac{P}{R} \right)^{k} < \infty \quad \text{for} \quad 1 < p < P.$$

Let $a \in \mathcal{A}_R$, or $a \in \mathcal{E}_R$. For $1 < p < P$ choose $r$ so that $1 < r < R$ and $\frac{P}{r} < \frac{P}{R}$. Now choose $p_1$ so that $p < p_1 < P$ and $\frac{P}{r} = \frac{p_1}{R}$. Suppose $u \in D_{p_1}^r$. Then $u = \frac{1}{2}(z + z^{-1})$ with $1 \leq |z| < p$, and therefore, by Lemma 2,

$$|b_{nk} a_k P_k(u)| \leq K(\epsilon) |b_{nk}| |a_k|(1 + k)^{2+2/\epsilon} r^{k} = K(\epsilon) |b_{nk}| \left( \frac{p_1}{r} \right)^{k} |a_k|(1 + k)^{2+2/\epsilon} r^{k}$$

$$= K(\epsilon) |b_{nk}| \left( \frac{p_1}{R} \right)^{k} |a_k|(1 + k)^{2+2/\epsilon} r^{k} \leq K(\epsilon) M(p_1) |a_k|(1 + k)^{2+2/\epsilon} r^{k} < \infty.$$


Further, by (i) (of either Theorem 1 or Theorem 2),

$$\lim_{n \to \infty} b_n a_k P_k(u) = b_k a_k P_k(u).$$

Since $\sum_{k=0}^{\infty} |a_k|(1 + k)^{2+2/\epsilon} r^k < \infty$, and since $p$ can be chosen arbitrarily close to $P$ in $(1, P)$, it follows, by the Weierstrass M-test, that $g_n(u)$ exists for $n = 0, 1, \ldots$, and

$$\lim_{n \to \infty} g_n(u) = \lim_{n \to \infty} \sum_{k=0}^{\infty} b_n a_k P_k(u) = \sum_{k=0}^{\infty} b_k a_k P_k(u)$$

on $D_P^\gamma$, and that the sequence $(g_n)$ is uniformly convergent on compact subsets of $D_P^\gamma$. This completes the proof of the sufficiency of conditions (i) and (ii) both for Theorem 1 and Theorem 2.

Necessity. Let $a_k := \frac{1}{R^k(k+1)^2}$. Then $a \in A_R$ and $a \in E_R$. Under the hypotheses of either Theorem 1 or Theorem 2 the series

$$g_n(u) := \sum_{k=0}^{\infty} b_n a_k P_k(u)$$

is convergent on $D_P^\gamma$ and the sequence $(g_n)$ is uniformly convergent on compact subsets of $D_P^\gamma$. Therefore, by the Weierstrass double-series theorem, $(g_n)$ converges to a holomorphic function on $D_P^\gamma$. By Lemma 3, we get, for the above sequence $a$, that

$$b_n a_k = \int_{-1}^{1} g_n(t) P_k(t) \, dt \quad \text{for} \quad n = 0, 1, \ldots.$$ 

Since $g_n(t)$ converges uniformly on $[-1, 1]$ to $g(t)$ say, we get that

$$\lim_{n \to \infty} b_n a_k = \int_{-1}^{1} g(t) P_k(t) \, dt =: d_k.$$ 

Hence, for $k = 0, 1, \ldots$,

$$\lim_{n \to \infty} b_n = b_k,$$

where $b_k = d_k R^k(k+1)^2$. This proves the necessity of condition (i) in both Theorem 1 and Theorem 2.

Suppose now that $p$ and $\tilde{p}$ are fixed with $1 < p < \tilde{p} < P$. Since $a$ satisfies the hypotheses of both Theorem 1 and Theorem 2, the sequence $(g_n)$ is uniformly convergent on $D_{\tilde{p}}^\gamma$. Hence we have, for $u \in D_{\tilde{p}}^\gamma$ and $n = 0, 1, \ldots$, that $|g_n(u)| \leq M(\tilde{p}, a) < \infty$, $M(\tilde{p}, a)$ being independent of $n$. By Lemma 4 we get that

$$|b_n a_k R^k| \leq c(\tilde{p}) M(\tilde{p}, a)$$

for $n, k = 0, 1, \ldots$. 
Since \( a_k := \frac{1}{R^k(k+1)^2} \), it follows that
\[
|b_{nk}| \cdot \left(\frac{\bar{p}}{R}\right)^k \cdot \frac{1}{(k+1)^2} \leq c(\bar{p})M(\bar{p}, a) \text{ for } n, k = 0, 1, \ldots,
\]
and hence that
\[
\sup_{n \geq 0, k \geq 0} |b_{nk}| \cdot \left(\frac{\bar{p}}{R}\right)^k \leq c(\bar{p})M(\bar{p}, a) \sup_{k \geq 0} \left\{ \left(\frac{\bar{p}}{R}\right)^k (k+1)^2 \right\} < \infty.
\]
Therefore the condition
\[
\sup_{n \geq 0, k \geq 0} |b_{nk}| \cdot \left(\frac{\bar{p}}{R}\right)^k < \infty \quad \text{whenever } 1 < p < P,
\]
is necessary, i.e., condition (ii) is necessary in both Theorem 1 and Theorem 2. \( \square \)

**Proof of Theorem 3.**

**Sufficiency.** We assume that
\[
\left\{ \begin{array}{l}
\lim_{n \to \infty} b_{nk} =: b_k \text{ for } k = 0, 1, \ldots; \\
M := \sup_{n \geq 0, k \geq 0} |b_{nk}| \cdot \frac{1}{k+1} < \infty.
\end{array} \right.
\]
Let \( a \in \mathcal{E} \), and let \( u \in D_R^\gamma \). Then \( u = \frac{1}{2}(z + z^{-1}) \) with \( 1 \leq |z| < R < \infty \), and so, by Lemma 2,
\[
|b_{nk}a_kP_k(u)| \leq K(\epsilon)|b_{nk}|a_k|(1 + k)^{2 + 2/\epsilon} |z|^k \leq K(\epsilon)|b_{nk}|a_k|(1 + k)^{2 + 2/\epsilon} R^k \\
\leq K(\epsilon)M|a_k|(1 + k)^{2 + 2/\epsilon}(MR)^k < \infty.
\]
\( \hat{\epsilon} \)From (i) we get
\[
\lim_{n \to \infty} b_{nk}a_kP_k(u) = b_k a_k P_k(u).
\]
Since \( \sum_{k=0}^{\infty} |a_k|(1 + k)^{2 + 2/\epsilon}(MR)^k < \infty \), and since \( R \) can be arbitrarily large, it follows, by the Weierstrass M-test, that \( g_n(u) \) exists for \( n = 0, 1, \ldots \), and
\[
\lim_{n \to \infty} g_n(u) = \lim_{n \to \infty} \sum_{k=0}^{\infty} b_{nk} a_k P_k(u) = \sum_{k=0}^{\infty} b_k a_k P_k(u)
\]
on \( \mathbb{C} \), and that the sequence \( (g_n) \) is uniformly convergent on compact subsets of \( \mathbb{C} \).

**Necessity.** Let \( a_k := k^{-k} \), so that \( a \in \mathcal{E} \). Then, by hypothesis, the series
\[
g_n(u) := \sum_{k=0}^{\infty} b_{nk} a_k P_k(u)
\]
is convergent on \( \mathbb{C} \), and the sequence \((g_n)\) is uniformly convergent on compact subsets of \( \mathbb{C} \). By the Weierstrass double-series theorem, \((g_n)\) converges to an entire function on \( \mathbb{C} \). By Lemma 3 we have

\[
b_{nk}a_k = \int_{-1}^{1} g_n(t)P_k(t) \, dt \quad \text{for } n = 0, 1, \ldots.
\]

Since \( g_n(t) \) is uniformly convergent on \([-1, 1]\) to \( g(t) \) say, we get, for \( k = 0, 1, \ldots \), that

\[
\lim_{n \to \infty} b_{nk}a_k = \int_{-1}^{1} g(t)P_k(t) \, dt =: d_k,
\]

and hence that

\[
\lim_{n \to \infty} b_{nk} = b_k,
\]

where \( b_k = d_k k^k \) for \( k = 0, 1, 2, \ldots \). Thus condition (i) is necessary.

Suppose now that \( \mathbf{a} \) is an arbitrary sequence in \( \mathcal{E} \), and that \( R > 1 \). Since the sequence \((g_n)\) is uniformly convergent on \( \overline{D}_R \), we have, for \( u \in \overline{D}_R \) and \( n = 0, 1, \ldots \), that \( |g_n(u)| \leq M(R, \mathbf{a}) < \infty \). \( \square \) From Lemma 4 we get that

\[
|b_{nk}a_k| \leq c(R)M(R, \mathbf{a})R^{-k} \quad \text{for } n, k = 0, 1, \ldots. \tag{1}
\]

Hence \( \sum_{k=0}^{\infty} b_{nk}a_k \) is convergent whenever \( \mathbf{a} \in \mathcal{E} \), and we have, by Lemma 1, that

\[
M_n := \sup_{k \geq 0} |b_{nk}|^{1+\frac{1}{r}} < \infty \quad \text{for } n = 0, 1, \ldots.
\]

Assume now that

\[
\sup_{n \geq 0} \sup_{k \geq 0} |b_{nk}|^{1+\frac{1}{r}} = \sup_{n \geq 0} M_n = \infty.
\]

This implies that there exists a strictly increasing sequence of positive integers \((n_j)\) such that \( M_{n_j} \to \infty \). This in turn implies that there exists a sequence of non-negative integers \((k_j)\) such that

\[
|b_{n_j,k_j}|^{1+\frac{1}{r}} > \frac{1}{2} M_{n_j} \to \infty \quad \text{as } j \to \infty. \tag{2}
\]

We show now that the sequence \((k_j)\) is not bounded. Assume that it is bounded. Then there is a positive integer \( k^* \) such that \( 0 \leq k_j \leq k^* \). Since \( \lim_{n \to \infty} b_{nk} = b_k \) for \( k = 0, 1, \ldots, k^* \), it follows that the set of numbers \((b_{nk})_{n \geq 0, 0 \leq k \leq k^*}\) is bounded, and hence that the set of numbers \( \left( |b_{nk}|^{1+\frac{1}{r}} \right)_{n \geq 0, 0 \leq k \leq k^*} \) is bounded. But this contradicts (2). Therefore the sequence \((k_j)\) is not bounded. We can suppose (by considering a subsequence if necessary) that the sequence is strictly increasing. Choose

\[
a_k := \begin{cases} 
\left( \frac{1}{|b_{n_j,k}|} \right)^{\frac{k+1}{2}} & \text{if } k = k_j, \\
0 & \text{otherwise}.
\end{cases}
\]
We then have, by (2), that
\[
\left| a_{k_j} \right| \leq \frac{1}{\sqrt{b_{n_j,k_j}}} < \left( \frac{1}{\frac{1}{2}M_{n_j}} \right)^{\frac{1}{b_{j,j}}} \to 0 \text{ as } j \to \infty.
\]
Therefore \( a \in \mathcal{E} \), but
\[
\left| b_{n_j,k_j} \right| a_{k_j} = \sqrt{b_{n_j,k_j}} \to \infty \text{ as } j \to \infty,
\]
which contradicts (1). Thus the condition
\[
\sup_{n \geq 0, k \geq 0} \left| b_{n,k} \right|^{\frac{1}{b_{j,j}}} < \infty
\]
is necessary, i.e., condition (ii) is necessary.

\[
\square
\]

4. Additional Theorems. In this section we prove some theorems showing that the ellipse of convergence \( D^*_p \) specified in Theorem 2 cannot be enlarged when the matrix \( B \) satisfies conditions (i) and (ii) of that theorem together with certain other conditions. Analogous theorems concerning matrix transformations of power series appear in [1].

**Theorem 4.** Suppose that \( P \) and \( R \) are finite numbers greater than 1, and that \( B \equiv (b_{n,k}) \) is a triangular infinite matrix (i.e., \( b_{n,k} = 0 \) for \( k > n \)) satisfying
\[
M(p) := \sup_{n \geq 0, k \geq 0} |b_{n,k}| \left( \frac{p^k}{R} \right) < \infty \quad \text{for} \quad 1 < p < P.
\]
Then, for each \( a \in A_R \) and each \( R_1 \geq P \),
\[
\limsup_{n \to \infty} \max_{u \in \gamma_{R_1}} \left| \sum_{k=0}^n b_{n,k} a_k P_k(u) \right|^{\frac{1}{k}} \leq \frac{R_1}{P}.
\]

**Proof.** Choose \( R_1 \geq P > 1 \), and suppose \( a \in A_R \). Let \( \frac{1}{P} < \lambda < 1 \), and take \( p := \lambda P \). Then \( 1 < p < P \). Since \( \limsup |a_k|^{\frac{1}{b_{j,j}}} = \frac{1}{R} \), there is a positive constant \( c(\lambda) \) such that
\[
|a_k| \leq \frac{c(\lambda)}{(\lambda R)^k} \quad \text{for} \quad k \geq 0.
\]
By Lemma 2, for \( u \in \gamma_{R_1} \) we have \( |P_k(u)| \leq K(\epsilon)(1 + k)^{2+2/\epsilon} \frac{R_1}{\lambda^k} \) and hence
\[
\left| \sum_{k=0}^n b_{n,k} a_k P_k(u) \right| \leq K(\epsilon) \sum_{k=0}^n |b_{n,k}| \left( \frac{p^k}{R} \right) |a_k| R^k \left( \frac{R_1}{p} \right)^k (1 + k)^{2+2/\epsilon}
\]
\[
\leq K(\epsilon) M(p) c(\lambda) \sum_{k=0}^n \left( \frac{R}{\lambda R} \right)^k \left( \frac{R_1}{\lambda P} \right)^k (1 + k)^{2+2/\epsilon}
\]
\[
\leq K(\epsilon) M(p) c(\lambda)(1 + n)^{2+2/\epsilon} \sum_{k=0}^n \left( \frac{R_1}{\lambda^2 P^k} \right)^k.
\]
Since \( \frac{R_1}{\lambda^2 P} > \frac{R_1}{P} \geq 1 \), it follows that

\[
\limsup_{n \to \infty} \max_{u \in \gamma_R_1} \left| \sum_{k=0}^{n} b_{nk} a_k P_k(u) \right|^\frac{1}{n} \leq \lim_{n \to \infty} \left( \sum_{k=0}^{n} \left( \frac{R_1}{\lambda^2 P} \right)^k \right)^\frac{1}{n} = \frac{R_1}{\lambda^2 P}.
\]

Letting \( \lambda \nearrow 1 \) we get

\[
\limsup_{n \to \infty} \max_{u \in \gamma_R_1} \left| \sum_{k=0}^{n} b_{nk} a_k P_k(u) \right|^\frac{1}{n} \leq \frac{R_1}{P}.
\]

\( \square \)

**Remark.** Assume that a triangular matrix \( B \) satisfies

\[
M(p) := \sup_{n \geq 0, k \geq 0} |b_{nk}| \left( \frac{p}{R} \right)^k < \infty \text{ for } 1 < p < P.
\]

Then

\[
|b_{nn}|^{\frac{1}{n}} \frac{p}{R} \leq M(p) \frac{1}{n} \to 1 \text{ as } n \to \infty,
\]

and hence

\[
\limsup_{n \to \infty} |b_{nn}|^{\frac{1}{n}} \leq \frac{R}{P} \text{ for each } p \in (1, P).
\]

Letting \( p \nearrow P \) we get

\[
\limsup_{n \to \infty} |b_{nn}|^{\frac{1}{n}} \leq \frac{R}{P}.
\]

This suggests that it is not inappropriate to impose the condition

\[
\lim_{n \to \infty} |b_{nn}|^{\frac{1}{n}} = \frac{R}{P},
\]

as we do in the following theorem.

**Theorem 5.** Let \( B \) be a triangular matrix. Suppose that

\[
\lim_{n \to \infty} |b_{nn}|^{\frac{1}{n}} = \frac{R}{P},
\]

where \( P \) and \( R \) are finite numbers greater than 1. Then for each \( a \in A_R \) and each \( R_1 \geq P \) we have

\[
\limsup_{n \to \infty} \max_{u \in \gamma_R_1} \left| \sum_{k=0}^{n} b_{nk} a_k P_k(u) \right|^\frac{1}{n} \geq \frac{R_1}{P}.
\]

**Proof.** Assume that the conclusion of the theorem is not true. Then there is an \( a^* \in A_R \) and an \( R_1 \geq P > 1 \) such that

\[
\limsup_{n \to \infty} \max_{u \in \gamma_R_1} \left| \sum_{k=0}^{n} b_{nk} a^*_k P_k(u) \right|^\frac{1}{n} < \frac{R_1}{P}.
\]
Therefore there exists a number \( \tilde{R} \) such that \( 1 < \tilde{R} < R_1 \) and, for all \( n \) sufficiently large,

\[
\max_{u \in \gamma_{R_1}} \left| \sum_{k=0}^{n} b_{nk} a_k^* P_k(u) \right|^\frac{1}{n} \leq \frac{\tilde{R}}{P}, \text{ and hence } \max_{u \in \gamma_{R_1}} \left| \sum_{k=0}^{n} b_{nk} a_k^* P_k(u) \right| \leq \left( \frac{\tilde{R}}{P} \right)^n.
\]

Applying Lemma 4 to the function \( g_n(u) := \sum_{k=0}^{n} b_{nk} a_k^* P_k(u) \) we get in particular that, for all large \( n \),

\[
|b_{nn}| |a_n^*| R_1^n \leq c(R_1) \left( \frac{\tilde{R}}{P} \right)^n, \text{ and therefore } |b_{nn}|^{\frac{1}{n}} |a_n^*|^{\frac{1}{n}} R_1^{\frac{1}{n}} \leq c(R_1) \frac{\tilde{R}}{P}.
\]

From the last inequality we get that

\[
\frac{\tilde{R}}{P} \geq \limsup_{n \to \infty} \left( |b_{nn}|^{\frac{1}{n}} |a_n^*|^{\frac{1}{n}} R_1 \right) = R_1 \lim_{n \to \infty} |b_{nn}|^{\frac{1}{n}} \cdot \limsup_{n \to \infty} |a_n^*|^{\frac{1}{n}} = \frac{R_1}{P}.
\]

But this is a contradiction since \( 1 < \tilde{R} < R_1 \). Hence the conclusion of the theorem must hold.

The next two theorems are analogues of Theorems 6 and 7 (concerning matrix transformations of power series) in [1], which in turn generalize results about regular and non-regular Nörlund matrices due respectively to Luh [6] and K. Stadtmüller [9, Theorems 6 and 7]. The first of these new theorems, which follows immediately from Theorems 4 and 5, shows, inter alia, that the sequence \( (g_n) \) specified in Theorem 2 cannot converge uniformly in the interior of any ellipse \( \gamma_{P_1} \) with \( P_1 > P \) when \( B \) is a triangular matrix satisfying condition (ii) of Theorem 2 together with the diagonal condition of Theorem 5.

**Theorem 6.** Suppose that \( P \) and \( R \) are finite numbers greater than 1, and that \( B \) is a triangular matrix satisfying

\[
M(p) := \sup_{n \geq 0, k \geq 0} |b_{nk}| \left( \frac{P}{R} \right)^k < \infty \quad \text{for } 1 < p < P, \text{ and } \lim_{n \to \infty} |b_{nn}|^{\frac{1}{n}} = \frac{R}{P}.
\]

Then, for each \( a \in A_R \) and each \( R_1 \geq P \),

\[
\limsup_{n \to \infty} \max_{u \in \gamma_{R_1}} \left| \sum_{k=0}^{n} b_{nk} a_k P_k(u) \right|^{\frac{1}{n}} = \frac{R_1}{P}.
\]

The next theorem shows that the ellipse \( \gamma_{R_1} \) in the conclusion of Theorem 6 can be replaced by any arc of that ellipse (provided condition (i) of Theorem 2 is also satisfied when \( R_1 = P \)).
**Theorem 7.** Suppose that $P$ and $R$ are finite numbers greater than $1$, and that $B$ is a triangular matrix such that

$$
M(p) := \sup_{n \geq 0, k \geq 0} |b_{nk}| \left( \frac{p}{R} \right)^k < \infty \text{ for } 1 < p < P, \text{ and } \lim_{n \to \infty} \left| b_{nn} \right|^\frac{1}{n} = \frac{R}{P}.
$$

(i) Then, for each $a \in A_R$ and each $R_1 > P$,

$$
\limsup_{n \to \infty} \max_{u \in \Gamma} \left| \sum_{k=0}^{n} b_{nk} a_k P_k(u) \right|^\frac{1}{n} = \frac{R_1}{P},
$$

where $\Gamma$ is any closed non-trivial arc of $\gamma_{R_1}$.

(ii) If, in addition,

$$
\lim_{n \to \infty} b_{nk} =: b_k \text{ for } k = 0, 1, \ldots, \text{ where } b_k \neq 0 \text{ for } k > k^*,
$$

then, for each $a \in A_R$,

$$
\limsup_{n \to \infty} \max_{u \in \Gamma} \left| \sum_{k=0}^{n} b_{nk} a_k P_k(u) \right|^\frac{1}{n} = 1,
$$

where $\Gamma$ is any closed non-trivial arc of $\gamma_P$.

**Proof of (i).** By Theorem 6 we know that

$$
\limsup_{n \to \infty} \max_{u \in \Gamma} \left| \sum_{k=0}^{n} b_{nk} a_k P_k(u) \right|^\frac{1}{n} \leq \frac{R_1}{P}.
$$

Hence it is enough to prove that, for every $a \in A_R$,

$$
\limsup_{n \to \infty} \max_{u \in \Gamma} \left| \sum_{k=0}^{n} b_{nk} a_k P_k(u) \right|^\frac{1}{n} \geq \frac{R_1}{P}, \quad (3)
$$

which we now proceed to do. Assume that (3) is not true. Then there exists a sequence $a^* \in A_R$ and a number $\tilde{R}$ such that $P < \tilde{R} < R_1$ and

$$
\limsup_{n \to \infty} \max_{u \in \Gamma} |g_n(u, a^*)|^\frac{1}{n} \leq \frac{\tilde{R}}{P}.
$$

Hence given $\epsilon > 0$ we have, for $z := u + \sqrt{u^2 - 1}$ and all sufficiently large $n$,

$$
\max_{u \in \Gamma} \left| \frac{g_n(u, a^*)}{z^n} \right| \leq \left( \frac{\tilde{R}}{P} \cdot \frac{1}{R_1} \right)^n 2^n = \left( \frac{\tilde{R}}{R_1} \right)^n \left( \frac{2^n}{P} \right).
$$
Further, from Theorem 6 we get that, for all large \( n \),

\[
\max_{u \in \gamma} \left| g_n(u, a^*) \right| \leq \left( \frac{2^\epsilon}{P} \right)^n
\]

and

\[
\max_{u \in \gamma_{R_1}} \left| g_n(u, a^*) \right| \leq \left( \frac{2^\epsilon}{P} \right)^n.
\]

Let \( P < r < R_1 \). Since the function \( z = u + \sqrt{u^2 - 1} \) is holomorphic and different from zero on \( \mathbb{C} \setminus [-1, 1] \), we have, by Nevanlinna’s \( N \)-constants theorem (see [3, Theorem 18.3.3]), that there exist positive constants \( \theta_1, \theta_2, \theta_3 \) (depending on \( r \) but not on \( \epsilon \)) such that \( \theta_1 + \theta_2 + \theta_3 = 1 \) and

\[
\max_{u \in \gamma_r} \left| g_n(u, a^*) \right| \leq \left( \frac{R}{R_1} \right)^{\theta_1} \left( \frac{2^\epsilon}{P} \right)^{\theta_2} \left( \frac{2^\epsilon}{P} \right)^{\theta_3} = \left( \frac{R}{R_1} \right)^{\theta_1} \left( \frac{2^\epsilon}{P} \right)^n
\]

for all sufficiently large \( n \). Hence, choosing \( \epsilon > 0 \) so small that \( \left( \frac{R}{R_1} \right)^{\theta_1} 2^\epsilon < 1 \), we get

\[
\limsup_{n \to \infty} \max_{u \in \gamma_r} |g_n(u, a^*)|^{\frac{1}{n}} \leq \left( \frac{R}{R_1} \right)^{\theta_1} 2^\epsilon \frac{r}{P} < \frac{r}{P}.
\]

Since \( r > P \), the last inequality contradicts the conclusion of Theorem 5. Hence (3) must hold when \( R_1 > P \).

Proof of (ii). By Theorem 6 we know in this case that

\[
\limsup_{n \to \infty} \max_{u \in \Gamma} \left| \sum_{k=0}^{n} b_{nk} a_k P_k(u) \right|^{\frac{1}{n}} \leq 1.
\]

Hence it is enough to prove that, for every \( a \in \mathbf{A}_R \),

\[
\limsup_{n \to \infty} \max_{u \in \Gamma} \left| \sum_{k=0}^{n} b_{nk} a_k P_k(u) \right|^{\frac{1}{n}} \geq 1,
\]

(4)

Suppose (4) is not true. Then for some \( a^* \in \mathbf{A}_R \) we have

\[
\limsup_{n \to \infty} \max_{u \in \Gamma} \left| \sum_{k=0}^{n} b_{nk} a_k^* P_k(u) \right|^{\frac{1}{n}} < 1.
\]
Write
\[ g_n(u, \mathbf{a}^*) := \sum_{k=0}^{n} b_{nk} a_k^* P_k(u) . \]

It follows that there exists a positive number \( q < \frac{R_1}{P} = 1 \), such that, for all \( n \) sufficiently large,
\[ \sup_{u \in \gamma_P} |g_n(u, \mathbf{a}^*)| < q^n. \]

Given \( \alpha > 0 \) we get from Theorem 6 that, for all \( n \) sufficiently large,
\[ \max_{u \in \gamma_P} |g_n(u, \mathbf{a}^*)| \leq 2^{\alpha n}. \]

By Nevanlinna's \( N \)-constants theorem, there exists a positive number \( \theta < 1 \) (independent of \( \alpha \)) such that, for all large \( n \),
\[ \max_{-1 \leq u \leq 1} |g_n(u, \mathbf{a}^*)| \leq (q^{\theta 2(1-\theta)\alpha})^n. \]

Since we can choose \( \alpha > 0 \) so small that \( q^{\theta 2(1-\theta)\alpha} < 1 \), it follows that
\[ \max_{-1 \leq u \leq 1} |g_n(u, \mathbf{a}^*)| \to 0 \text{ as } n \to \infty. \]

By Lemma 3 we have
\[ b_{nk} a_k = \int_{-1}^{1} g_n(t, \mathbf{a}^*) P_k(t) dt \quad \text{for } n = 0, 1, \ldots. \]

Since \( g_n(t, \mathbf{a}^*) \) tends uniformly to 0 on \([-1, 1]\) as \( n \to \infty \), it follows that
\[ 0 = \lim_{n \to \infty} b_{nk} a_k^* = b_k a_k^* \quad \text{for } k = 0, 1, \ldots. \]

Since \( \mathbf{a}^* \in \mathbf{A}_R \) we have that \( a_k^* \neq 0 \) for some \( k > k^* \). Hence \( b_k = 0 \) for such a \( k \). But this contradicts the assumption that \( b_k \neq 0 \) for \( k > k^* \). Therefore (4) must hold.

\[ \square \]

5. Chebyshev Polynomials. In this section we restrict \( (P_n) \) to be the orthonormal sequence on \([0, 1]\) of Chebyshev polynomials of the first or second kind, the corresponding weight functions of which are respectively \( w(x) = \frac{1}{2} (1 - x^2)^{-\frac{1}{2}} \) and \( w(x) = \frac{1}{2} (1 - x^2)^{\frac{1}{2}} \). The special properties of these Chebyshev polynomials that makes them amenable to the proof of Theorem 8 (below) are the familiar identities
\[ 2P_n \left( \frac{1}{2}(z + z^{-1}) \right) = z^n + z^{-n} \quad (5) \]

when \( P_n \) is of the first kind, and
\[ (z - z^{-1}) P_n \left( \frac{1}{2}(z + z^{-1}) \right) = z^{n+1} - z^{-n-1} \quad (6) \]

when \( P_n \) is of the second kind.

The said theorem deals with the possibility of pointwise convergence of the sequence \( (g_n(u)) \) specified in Theorem 2 outside the convergence ellipse \( \gamma_P \). It’s analogue for power series is Theorem 8 in [1], which generalizes results due to Lejá [4] and Stadtmüller [9, Theorem 8] about regular and non-regular Nörlund matrices respectively.
Theorem 8. Suppose that $P$ and $R$ are finite numbers greater than 1, and that $B$ is a triangular matrix such that

(i) $\lim_{n \to \infty} b_{nk} =: b_k$ for $k = 0, 1, \ldots$ where $b_k \neq 0$ for $k > k^*$;

(ii) $M(p) := \sup_{n \geq 0, k \geq 0} |b_{nk}| \left( \frac{p}{R} \right)^k < \infty$ for $1 < p < P$; $\lim_{n \to \infty} |b_{nn}|^{\frac{1}{n}} = \frac{R}{P}$, and

(iii) $|b_{nk}| \leq c(\tilde{R})|b_{nn}| \left( \frac{P}{\tilde{R}} \right)^{n-k}$ for $1 < \tilde{R} < R$ and $0 \leq k \leq n$.

Suppose that $a \in A_R$ and that $\limsup_{n \to \infty} |a_n| R^n > 0$. Let

$$g_n(u) := \sum_{k=0}^{n} b_{nk} a_k P_k(u),$$

where $(P_k)$ is the orthonormal sequence on $[-1, 1]$ of Chebyshev polynomials of the first or second kind, and let $P_1 > P$. Then $\limsup_{n \to \infty} |g_n(u)|^{\frac{1}{n}} \leq 1$ for at most a finite number of points $u$ outside the ellipse $\gamma_{P_1}$ and hence, in particular, the sequence $(g_n)$ can converge at most at a finite number of points $u$ outside the ellipse $\gamma_{P_1}$.

Proof. Assume that $u$ is a point outside the ellipse $\gamma_{P_1}$ for which

$$\limsup_{n \to \infty} |g_n(u)|^{\frac{1}{n}} \leq 1. \quad (7)$$

Let $z := u + \sqrt{u^2 - 1}$, so that $|z| > P_1$; and let

$$\tilde{g}_n(z) := \sum_{k=0}^{n} b_{nk} a_k z^k.$$

Then, by (5),

$$2g_n(u) = 2 \sum_{k=0}^{n} b_{nk} a_k P_k(u) = \tilde{g}_n(z) + \tilde{g}_n(z^{-1})$$

when the Chebyshev polynomials $P_k$ are of the first kind; and, by (6),

$$(z - z^{-1}) g_n(u) = z \tilde{g}_n(z) - z^{-1} \tilde{g}_n(z^{-1})$$

when the Chebyshev polynomials $P_k$ are of the second kind.
Since $|z^{-1}| < P_1^{-1} < P$ it follows from Theorem 2 in [1] that $\tilde{g}_n(z^{-1})$ tends to a finite limit as $n \to \infty$, and therefore from (7) that, in either case,

$$\limsup_{n \to \infty} |\tilde{g}_n(z)|^{\frac{1}{n}} \leq 1.$$  (8)

Theorem 8 in [1] tells us that inequality (8) can hold for at most a finite number of points $z$ satisfying $|z| > P_1$, and thus (7) can hold for at most finitely many points $u$ outside the ellipse $\gamma_{P_1}$.  

**Remarks.** A Nörlund matrix $N_B$ for which

$$\lim_{n \to \infty} \frac{B_{n-1}}{B_n} = b \text{ with } |b| = \frac{R}{P}$$

satisfies all the conditions on the matrix in Theorem 8. In this case, however, the condition $\limsup |a_n|R^n > 0$ can be omitted since the corresponding version of the theorem for power series has recently been proved by K. Stadmüller and Gross-Erdman [10, Remark 3.7].

An open and challenging question is whether Theorem 8 holds for other orthogonal polynomials.

**References**


**Department of Mathematics, University of Western Ontario, London, Ontario, Canada N6A 5B7**

**Department of Mathematics and Statistics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6**

**School of Mathematical Sciences, Tel Aviv University, Tel Aviv, Israel**