REMEZ-TYPE INEQUALITY FOR NON-DENSE MÜNTZ SPACES WITH EXPLICIT BOUND

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ABSTRACT. Let $A := (\lambda_k)_{k=0}^\infty$ be a sequence of distinct nonnegative real numbers with $\lambda_0 := 0$ and $\sum_{k=1}^\infty 1/\lambda_k < \infty$. Let $\varrho \in (0,1)$ and $\epsilon \in (0,1-\varrho)$ be fixed. An earlier work of the authors shows that

$$C(A, \epsilon, \varrho) := \sup \left\{ \|p\|_{[0, \varrho]} : p \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}, m\{x \in [0, 1] : |p(x)| \leq 1\} \geq \epsilon \right\}$$

is finite. In this paper an explicit upper bound for $C(A, \epsilon, \varrho)$ is given. In the special case $\lambda_k := k^\alpha$, $\alpha > 1$, our bounds are essentially sharp.

1. INTRODUCTION

In this paper $A := (\lambda_k)_{k=0}^\infty$ always denotes a sequence of real numbers satisfying

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots.$$ 

In [1] a Remez-type inequality for Müntz polynomials:

$$p(x) = \sum_{k=0}^n a_k x^{\lambda_k}$$

or equivalently for Dirichlet sums:

$$P(t) = \sum_{k=0}^n a_k e^{-\lambda_k t}$$

is established. The most useful form of this inequality states that for every sequence $(\lambda_k)_{k=0}^\infty$ satisfying $\sum_{k=0}^\infty 1/\lambda_k < \infty$, there exists a constant $C(A, \epsilon)$ depending only on $A$ and $\epsilon$ (and not on $n$, $\varrho$, or $A$) so that

$$\|p\|_{[0, \varrho]} \leq C(A, \epsilon) \|p\|_A$$

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for every Müntz polynomial \( p \), as above, associated with the sequence \( \lambda_k^{\infty}_{k=0} \), and for every set \( A \subset [\varrho, 1] \) of Lebesgue measure at least \( \varepsilon > 0 \). Throughout this paper \( \| \cdot \|_A \) denotes the uniform norm on \( A \subset \mathbb{R} \).

Using this Remez-type inequality, we resolved two reasonably long standing conjectures in [1]. In this paper we give an explicit upper bound for the best possible \( C(\Lambda, \varepsilon) \) in the above Remez-type inequality for non-dense Müntz spaces. Theorem 2.3 extends an inequality of Schwartz [4] in two directions. Theorem 2.1 offers a more explicit bound for the sequences \( \lambda := (k^\alpha)_{k=0}^{\infty}, \alpha > 1 \). The sharpness of the Remez-type inequality of Theorem 2.1 is shown by Theorem 2.2.

2. Results

**Theorem 2.1.** Let \( \lambda_k := k^\alpha, \ k = 0, 1, \ldots, \alpha > 1 \). Let \( \varrho \in (0,1), \varepsilon \in (0,1-\varrho) \), and \( \varepsilon \leq 1/2 \). There exists a constant \( c_\alpha > 0 \) depending only on \( \alpha \) so that 
\[
\|p\|_{[\varrho,d]} \leq \exp \left( c_\alpha \varepsilon^{1/(1-\alpha)} \right) \|p\|_A
\]
for every \( p \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\} \) and for every set \( A \subset [\varrho, 1] \) of Lebesgue measure at least \( \varepsilon > 0 \).

The next theorem shows that the inequality of Theorem 2.1 is essentially the best possible.

**Theorem 2.2.** Let \( \lambda_k := k^\alpha, \ k = 0, 1, \ldots, \alpha > 1 \). For every \( \alpha > 1 \) and \( \varepsilon \in (0,1/2] \), there exists a constant \( c_\alpha > 0 \) depending only on \( \alpha \) and Müntz polynomials 
\[
0 \neq p = p_{\alpha,\varepsilon} \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\}
\]
depending only on \( \alpha \) and \( \varepsilon \) so that 
\[
|p(0)| \geq \exp \left( c_\alpha \varepsilon^{1/(1-\alpha)} \right) \|p\|_{[1-\varepsilon,1]}.
\]

Theorem 2.1 is a special case of the following more general, but less explicit result.

**Theorem 2.3.** Suppose \( 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \) and \( \sum_{k=0}^{\infty} 1/\lambda_k < \infty \). Let \( \varrho \in (0,1) \) and \( \varepsilon \in (0,1-\varrho) \). Let \( \delta := -\frac{1}{2} \log(1-\varepsilon) \). Let \( N \in \mathbb{N} \) be chosen so that 
\[
\sum_{k=N+1}^{\infty} \frac{1}{\lambda_k} \leq \frac{\delta}{3}.
\]
Let 
\[
\sigma_k := A\lambda_k \quad \text{with} \quad A := \frac{\delta}{3N}.
\]
Then, with \( c := \| t^{-1} \sin t \|_{L_2([\pi])} \), 
\[
\|p\|_{[\varrho,d]} \leq \frac{3c}{\delta} \prod_{k=1}^{N} \left( 2 + \frac{1}{\sigma_k} \right) \|p\|_A
\]
for every \( p \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots\} \) and for every set \( A \subset [\varrho, 1] \) of Lebesgue measure at least \( \varepsilon > 0 \).
3. Lemmas

Our first lemma shows that \( C(\Lambda, \epsilon) \) in the Remez-type inequality is related to a much simpler (Chebyshev-type) extremal problem. This is proved in both \([1]\) and \([2]\).

**Lemma 3.1.** Suppose \( 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \), \( \rho \in (0, 1) \), and \( \epsilon \in (0, 1 - \rho) \). Then
\[
\sup \left\{ \frac{|p(0)|}{\|p\|_{[0, \epsilon]}} : p \in \text{span} \{x^{\lambda_0}, x^{\lambda_1}, \ldots \} \right\} = \sup \left\{ \frac{|p(0)|}{\|p\|_{[1-\epsilon, 1]}} : p \in \text{span} \{x^{\lambda_0}, x^{\lambda_1}, \ldots \} \right\}.
\]

Our key lemma is the following.

**Lemma 3.2.** Suppose \( 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \) and \( \sum_{k=0}^{\infty} 1/\lambda_k < \infty \). Given \( \delta \in (0, 1) \), let \( N \in \mathbb{N} \) be chosen so that
\[
\sum_{k=N+1}^{\infty} \frac{1}{\lambda_k} \leq \frac{\delta}{12}.
\]
Let
\[
\sigma_k := A\lambda_k \quad \text{with} \quad A := \frac{\delta}{3N}.
\]
Then
\[
|P(\infty)| \leq \frac{3c}{\delta} \prod_{k=1}^{N} \left( 2 + \frac{1}{\sigma_k} \right) \|P\|_{[-\delta, \delta]}
\]
for every \( P \in \text{span} \{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \ldots \} \) with \( c := \|t^{-1} \sin t\|_{L_2(\mathbb{R})} \).

In the proof of Lemma 3.2 we will need the following observation.

**Lemma 3.3.** Let \( 0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \). Suppose
\( 1 \) \( F \in E^\delta \cap L_2(\mathbb{R}) \);
\( 2 \) \( F(i\lambda_k) = 0, \ k = 1, 2, \ldots \) (\( i \) is the imaginary unit);
\( 3 \) \( F(0) = 1 \).
Then
\[
|P(\infty)| \leq \|F\|_{L_2(\mathbb{R})} \|P\|_{L_2[-\delta, \delta]}
\]
for every \( P \in \text{span} \{e^{-\lambda_0 t}, e^{-\lambda_1 t}, \ldots \} \).

An entire function \( f \) is called a function of exponential type \( \delta \) if there exists a constant \( c \) depending only on \( f \) so that
\[
|f(z)| \leq c \exp(\delta|z|), \quad z \in \mathbb{C}.
\]
The collection of all such entire functions of exponential type \( \delta \) is denoted by \( E^\delta \).
The Paley-Wiener Theorem (see, for example, \([3]\)) characterizes the functions \( F \) which can be written as the Fourier transform of some function \( F \in L_2[-\delta, \delta] \). We will need it in the proof of Lemma 3.3.
Theorem (Paley-Wiener). Let $\delta \in (0, \infty)$. Then $f \in E^\delta \cap L_2(\mathbb{R})$ if and only if there exists an $f \in L_2[-\delta, \delta]$ so that

$$F(z) = \int_{-\delta}^{\delta} f(t) e^{itz} \, dt.$$ 

The following comparison theorem for Müntz polynomials is proved in [2]. We will need it in the proof of Theorem 2.3.

Lemma 3.4. Let $\Lambda := (\lambda_k)_{k=0}^\infty$ and $\Gamma := (\gamma_k)_{k=0}^\infty$ be increasing sequences of non-negative real numbers with $\lambda_0 = 0$, $\gamma_0 = 0$, and $\lambda_k \leq \gamma_k$ for each $k$. Let $0 < a < b$. Then

$$\max \left\{ \frac{|p(0)|}{||p||_{[a,b]}} : p \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}\} \right\} \geq \max \left\{ \frac{|p(0)|}{||p||_{[a,b]}} : p \in \text{span}\{x^{\gamma_0}, x^{\gamma_1}, \ldots, x^{\gamma_n}\} \right\}.$$

4. Proofs

Proof of Lemma 3.3. By the Paley-Wiener Theorem

$$F(z) = \int_{-\delta}^{\delta} f(t) e^{itz} \, dt$$

for some $f \in L_2[-\delta, \delta]$. Now if

$$P(t) = a_0 + \sum_{k=1}^{n} a_k e^{-\lambda_k t},$$

then

$$\int_{-\delta}^{\delta} f(t) P(t) \, dt = a_0 \int_{-\delta}^{\delta} f(t) \, dt + \sum_{k=1}^{n} a_k \int_{-\delta}^{\delta} f(t) e^{-\lambda_k t} \, dt$$

$$= a_0 F(0) + \sum_{k=1}^{n} a_k F(i\lambda_k) = a_0 = F(\infty).$$

Hence by the Cauchy-Schwartz Inequality and the $L_2$ inversion theorem of Fourier transforms, we obtain

$$|P(\infty)| \leq \|f\|_{L_2[-\delta, \delta]} \|P\|_{L_2[-\delta, \delta]} \leq \|F\|_{L_2[\mathbb{R}]} \|P\|_{L_2[-\delta, \delta]}$$

and the lemma is proved. □
Proof of Lemma 3.2. We define

$$F(z) := \frac{\sin(\delta z/3)}{\delta z/3} \prod_{k=1}^{N} \left( 1 - \frac{z}{i \lambda_k} \right) \frac{\sin(\sigma_k z / \lambda_k)}{\sigma_k z / \lambda_k} \prod_{k=N+1}^{\infty} \left( 1 - \left( \frac{\sin(z/\lambda_k)}{\sin i} \right)^4 \right),$$

where $i$ is the imaginary unit. It is a straightforward calculation that

$$F \in E^6, \quad F(0) = 1, \quad F(i \lambda_k) = 0, \quad k = 1, 2, \ldots$$

and

$$|F(t)| \leq \frac{\sin(\delta t/3)}{\delta t/3} \prod_{k=1}^{N} \left( 2 + \frac{1}{\sigma_k} \right), \quad t \in \mathbb{R}.$$ 

Hence Lemma 3.3 implies that

$$|P(\infty)| \leq \frac{3c}{\delta} \prod_{k=1}^{N} \left( 2 + \frac{1}{\sigma_k} \right) ||P||_{[-\delta, \delta]}$$

for every $P \in \text{span}\{e^{-\lambda_k t}, e^{-\lambda_k^* t}, \ldots\}$ with $c := ||t^{-1} \sin t||_{L_q(\mathbb{R})}$. \qed

Proof of Theorem 2.3. When $A = [1 - \epsilon, 1]$, the theorem follows from Lemma 3.2 by the substitution $x = e^{-\delta} e^{-t}$. The general case follows from Lemma 3.1. \qed

Proof of Theorem 2.1. Let

$$\delta := -\frac{1}{2} \log(1 - \epsilon).$$

Observe that $N$ in Theorem 2.1 can be chosen so that

$$N := \left\lfloor \left( \frac{\delta(a - 1)}{12} \right)^{1/(1-a)} \right\rfloor + 1.\quad (4.2)$$

Also, $\sigma_k$ in Lemma 3.2 is of the form

$$\sigma_k = \frac{\delta k^a}{3N}.$$ 

Let $M + 1$ be the smallest value of $k \in \mathbb{N}$ for which

$$\frac{1}{\sigma_k} < 1, \quad \text{that is,} \quad \frac{3N}{k^a \delta} \leq 1.$$ 

Note that

$$M := \left\lfloor \left( \frac{3N}{\delta} \right)^{1/a} \right\rfloor.$$
If $0 < M < N$, then
\[
\prod_{k=1}^{N} \left( 2 + \frac{1}{\sigma_k} \right) = \prod_{k=1}^{N} \left( 2 + \frac{3N}{\delta k^\alpha} \right)
\leq \left( \prod_{k=1}^{M} \frac{9N}{\delta k^\alpha} \right) \left( \prod_{k=M+1}^{N} 3 \right) \leq \left( \frac{9N}{e} \right)^M \left( \frac{M}{e} \right)^{-\alpha M} 3^{M-M}
\leq \left( \frac{9e^\alpha N}{\delta} \right)^M M^{-\alpha} 3^{N-M}
\leq \left( \frac{9e^\alpha N}{\delta} \right)^M \left( \frac{1}{2} \left( \frac{3N}{\delta} \right)^{1/\alpha} \right)^{-\alpha M} 3^{N-M}
\leq (3(2e)^\alpha)^M 3^{N-M} \leq (3(2e)^\alpha)^N,
\]
and the theorem follows by (4.1), (4.2), and Theorem 2.1.

If $N \leq M$, then
\[
\prod_{k=1}^{N} \left( 2 + \frac{1}{\sigma_k} \right) = \prod_{k=1}^{N} \left( 2 + \frac{3N}{\delta k^\alpha} \right)
\leq \left( \prod_{k=1}^{N} \frac{9N}{\delta k^\alpha} \right) \leq \left( \frac{9N}{\delta} \right)^N \left( \frac{N}{e} \right)^{-\alpha N}
\leq \left( \frac{9e^\alpha N^{1-\alpha}}{\delta} \right)^N \leq \left( \frac{9e^\alpha}{\delta} \right)^N \left( \frac{\delta(\alpha - 1)}{12} \right)^{1/\alpha (1-\alpha)} N^{(1-\alpha)N}
\leq \left( \frac{9e^\alpha}{\delta} \right)^N \left( \frac{\delta(\alpha - 1)}{12} \right)^N \leq \left( \frac{3e^\alpha (\alpha - 1)}{4} \right)^N,
\]
and the theorem follows by (4.1), (4.2), and Theorem 2.1.

If $M = 0$, then
\[
\prod_{k=1}^{N} \left( 2 + \frac{1}{\sigma_k} \right) \leq \prod_{k=1}^{N} 3 = 3^N,
\]
and the theorem follows by (4.1), (4.2), and Theorem 2.1. \hfill \Box

**Proof of Theorem 2.2.** Let $n \in \mathbb{N}$ be a fixed. We define $\gamma_k := kn^{\alpha-1}, \ k = 0, 1, \ldots$. Let $T_n(x) := (\frac{1}{2}(x-1))^n$ and
\[
Q_n(x) := T_n \left( \frac{2x^{n-1}}{1 - (1 - \epsilon)^{n^{\alpha-1}}} - \frac{1 + (1 - \epsilon)^{n^{\alpha-1}}}{1 - (1 - \epsilon)^{n^{\alpha-1}}} \right) \in \text{span}\{x^{\gamma_i}, x^{\gamma_1}, \ldots, x^{\gamma_n}\}.
\]
Then, by Lemma 3.4,
\[
\sup \left\{ \frac{|p(0)|}{\|p\|_{[1-\epsilon,1]}} : p \in \text{span}\{x^{\lambda_i}, x^{\lambda_1}, \ldots, \} \right\} \geq \frac{|Q_n(0)|}{\|Q_n\|_{[1-\epsilon,1]}} = |Q_n(0)|
= \left( \frac{1}{1 - (1 - \epsilon)^{n^{\alpha-1}}} \right)^n,
\]
Now let $n$ be the smallest integer satisfying $n^{n^{-1}} \geq \epsilon^{-1}$. Since $(1 - \epsilon)^{1/\epsilon}$ is bounded away from 0 on $(0, 1/2]$, the result follows. □

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**References**


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