On the Irrationality of A Certain Multivariate $q$ Series

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Abstract

We prove that for integers $q > 1, m \geq 1$ and positive rationals $r_1, r_2, \ldots, r_m \neq q^j, j = 1, 2, \ldots$, the series

$$\sum_{j=1}^{\infty} \frac{q^{-j}}{(1 - q^{-j}r_1)(1 - q^{-j}r_2) \cdots (1 - q^{-j}r_m)}$$

is irrational. Furthermore, if all the positive rationals $r_1, r_2, \ldots, r_m$ are less than $q$, then the series

$$\sum_{j_1, \ldots, j_m=0}^{\infty} \frac{r_1^{j_1} \cdots r_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1}$$

is also irrational.

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1 Introduction and Results

The main result of this paper is the following theorem:

Theorem 1.1: If $q$ is an integer greater than one, $m$ is a positive integer, $r_1, r_2, \ldots, r_m$ are any positive rationals such that $r_1, r_2, \ldots, r_m \neq q^j, j = 1, 2, \ldots$, then the series

$$\sum_{j=1}^{\infty} \frac{q^{-j}}{(1 - q^{-j}r_1)(1 - q^{-j}r_2) \cdots (1 - q^{-j}r_m)}$$

is irrational. Furthermore, if all the positive rationals $r_1, r_2, \ldots, r_m$ are less than $q$, then the series

$$\sum_{j_1, \ldots, j_m=0}^{\infty} \frac{r_1^{j_1} \cdots r_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1}$$

is also irrational.

This generalizes the irrationality results of the single variable case proved in Borwein [3], Erdős [6], and Erdős and Graham [7]. The approach is via Padé approximants. These provide, when appropriately specialized, rational approximations that are “too good” to allow for rationality. These methods are also used in Borwein and Zhou [4], Mahler [9], Chudnovsky and Chudnovsky [5], Walfiser [10], and Zhou and Lubinsky [11]. Unfortunately the methods are

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not sufficiently general to allow a unified treatment and each new class of functions requires considerable additional work.

As in [4] we use the standard $q$ analogues of factorials and binomial coefficients. The $q$--factorial is

$$[n]_q! := [n]! := \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q)}{(1 - q)^n},$$

(1.1)

where $[0]_q! := 1$. The $q$--binomial coefficient is

$$\left[ \begin{array}{c} n \\ k \end{array} \right]_q := \frac{[n]!}{[k]! \cdot [n-k]!}.$$

(1.2)

As

$$q^i - 1 = (q - 1)(q^{i-1} + q^{i-2} + \cdots + 1), \quad i \geq 1,$$

we have

$$\lim_{q \to 1} [n]_q! = n!, \quad \text{and} \quad \lim_{q \to 1} \left[ \begin{array}{c} n \\ k \end{array} \right]_q = \binom{n}{k}. $$

(1.3)

Note that (see Borwein [3])

$$[n]_{q-1}! = q^{-n(n-1)/2}[n]!, \quad \left[ \begin{array}{c} n \\ k \end{array} \right]_{q-1} = q^{-k(k-1)/2-n(n+1)/2}[n-k]![k]!(1 - q)^n,$$  

(1.4)

(1.5)

and (see Gasper and Rahman [8]) for $|t| < q^{-n}$,

$$\frac{1}{\prod_{h=0}^{n} (t - q^{-h})} = (-1)^{n+1} q^{n(n+1)/2} \sum_{l=0}^{\infty} \left[ \begin{array}{c} n+l \\ l \end{array} \right] t^l.$$  

(1.7)

We prove some properties of approximants to a related function in section 2, and use those properties to prove Theorem 1.1 in section 3.

2 Some Results On A Related Function

Let $q > 1$, $|x_1|, \cdots, |x_m| < q$, and integer $m \geq 1$, and let

$$L^*(x_1, \cdots, x_m) := \sum_{j_1, \cdots, j_m = 0}^{\infty} \frac{x_1^{j_1} \cdots x_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1}. $$

(2.1)

For $m = 1$, and $|x| < 1$,

$$\lim_{q \to 1} (q - 1)L^*(x) = \lim_{q \to 1} \sum_{j=0}^{\infty} \frac{(q - 1)x^j}{q^{j+1} - 1} $$

$$= \sum_{j=0}^{\infty} \frac{x^j}{j + 1} $$

$$= \frac{1}{x} \ln(1 - x). $$

(2)}
So we call $L^*(x_1, \ldots, x_m)$ a multivariate $q$ analogue of log. Now for $k \geq 1$ integer and $|x_1|, \ldots, |x_m| < q$, as

\[
L^*(q^{-1}x_1, \ldots, q^{-1}x_m) = \sum_{j_1, \ldots, j_m = 0}^{\infty} \frac{q^{-(j_1 + \cdots + j_m)x_1^{j_1} \cdots x_m^{j_m}}}{q^{j_1 + \cdots + j_m + 1} - 1} \\
= \sum_{j_1, \ldots, j_m = 0}^{\infty} \frac{(1 - q^{j_1 + \cdots + j_m + 1} + q^{j_1 + \cdots + j_m + 1}x_1^{j_1} \cdots x_m^{j_m})}{q^{j_1 + \cdots + j_m} (q^{j_1 + \cdots + j_m + 1} - 1)} \\
= \sum_{j_1, \ldots, j_m = 0}^{\infty} \frac{x_1^{j_1} \cdots x_m^{j_m}}{q^{j_1 + \cdots + j_m} - 1} - \sum_{j_1, \ldots, j_m = 0}^{\infty} \frac{x_1^{j_1} \cdots x_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1} \\
= qL^*(x_1, \ldots, x_m) = \frac{1}{(1 - q^{-1}x_1) \cdots (1 - q^{-1}x_m)},
\]

we have

\[
L^*(q^{-k}x_1, \ldots, q^{-k}x_m) = q^k L^*(x_1, \ldots, x_m) - \sum_{j=1}^{k} \frac{q^{-j}}{(1 - q^{-j}x_1) \cdots (1 - q^{-j}x_m)} \\
= q^k L^*(x_1, \ldots, x_m) - S_k(x_1, \ldots, x_m), \tag{2.2}
\]

where

\[
S_k(x_1, \ldots, x_m) := \sum_{j=1}^{k} \frac{q^{-j}}{(1 - q^{-j}x_1) \cdots (1 - q^{-j}x_m)}. \tag{2.3}
\]

From (2.2), we have

\[
L^*(x_1, \ldots, x_m) = q^{-k} L^*(q^{-k}x_1, \ldots, q^{-k}x_m) + \sum_{j=1}^{k} \frac{q^{-j}}{(1 - q^{-j}x_1) \cdots (1 - q^{-j}x_m)},
\]

and then

\[
L^*(x_1, \ldots, x_m) = \lim_{k \to \infty} q^{-k} L^*(q^{-k}x_1, \ldots, q^{-k}x_m) \\
+ \lim_{k \to \infty} \sum_{j=1}^{k} \frac{q^{-j}}{(1 - q^{-j}x_1) \cdots (1 - q^{-j}x_m)} \\
= \sum_{j=1}^{\infty} \frac{q^{-j}}{(1 - q^{-j}x_1) \cdots (1 - q^{-j}x_m)}. \tag{2.4}
\]

Now let $q > 1$, $x_1, \ldots, x_m \neq q^j$, $j = 1, 2, \ldots$, and integer $m \geq 1$, and let

\[
L(x_1, \ldots, x_m) := \sum_{j=1}^{\infty} \frac{q^{-j}}{(1 - q^{-j}x_1) \cdots (1 - q^{-j}x_m)}. \tag{2.5}
\]

Then $L(x_1, \ldots, x_m)$ is an extension of $L^*(x_1, \ldots, x_m)$, i.e.

\[
L(x_1, \ldots, x_m) = L^*(x_1, \ldots, x_m), \text{ for } |x_1|, \ldots, |x_m| < q. \tag{2.6}
\]

It is easy to see that we also have the following functional equation for $L(x_1, \ldots, x_m)$:

\[
L(q^{-k}x_1, \ldots, q^{-k}x_m) = q^k L(x_1, \ldots, x_m) - S_k(x_1, \ldots, x_m), \tag{2.7}
\]

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where \( k \geq 1 \) is an integer and \( S_k(x_1, \ldots, x_m) \) is defined by (2.3). Now we prove some properties of the function \( L(x_1, \ldots, x_m) \).

**Theorem 2.1:** Let \( n \geq 0 \) be an integer, \( L(x_1, \ldots, x_m), S_k(x_1, \ldots, x_m) \) be defined by (2.5) and (2.3) respectively. Let

\[
R_n(x_1, \ldots, x_m) := \prod_{j=1}^{n} \left( 1 - q^{-j}x_1 \right) \cdots \left( 1 - q^{-j}x_m \right),
\]

and

\[
I(x_1, \ldots, x_m) := \frac{R_n(x_1, \ldots, x_m)}{2\pi i} \int_{\Gamma} \frac{L(tx_1, \ldots, tx_m)dt}{\left( \prod_{k=0}^{n} (t - q^{-k}) \right)^{n+1}},
\]

where \( \Gamma \) is a circular contour containing \( 0, q^{-n}, \ldots, q^{0} \), and let

\[
Q(x_1, \ldots, x_m) := \frac{q^{n(n+1)/2}}{(1-q)n!n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} q^{nk+k(k+1)/2} R_n(x_1, \ldots, x_m),
\]

\[
P(x_1, \ldots, x_m) := \frac{q^{n(n+1)/2}}{(1-q)n!n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} q^{nk+k(k+1)/2} R_n(x_1, \ldots, x_m) S_k(x_1, \ldots, x_m)
\]

\[+ \frac{R_n(x_1, \ldots, x_m)}{n!} \frac{d^n}{dt^n} \left\{ L(tx_1, \ldots, tx_m) \right\}_{t=0}.\]

Then

(i) \( I(x_1, \ldots, x_m) = Q(x_1, \ldots, x_m) L(x_1, \ldots, x_m) + P(x_1, \ldots, x_m); \)

(ii) \( q^{(m-1)n(n+1)/2} \left( \prod_{j=1}^{n} (q^j - 1) \right) Q(x_1, \ldots, x_m) \in \mathbb{Z}[q, x_1, \ldots, x_m]; \)

(iii) \( q^{(m-1)n(n+1)/2} \left( \prod_{j=1}^{n+1} (q^j - 1) \right) P(x_1, \ldots, x_m) \in \mathbb{Z}[q, x_1, \ldots, x_m]; \)

(iv) For \( n \in \mathbb{N} \) fixed, and \( 0 < |x_1|, \ldots, |x_m| < q, \)

\[
|I(x_1, \ldots, x_m)| \leq \frac{c_q}{q^{2mn(n+1)}},
\]

where \( c_q \) is a constant depending only on \( q, m \), and \( x_1, \ldots, x_m \).

**Proof of Theorem 2.1:** **Proof of (i).** We can see that the integrand in (2.9) has simple poles at \( t = q^0, q^{-1}, \ldots, q^{-n} \), and a pole of order \( n + 1 \) at \( t = 0 \), inside the contour \( \Gamma \). By the residue theorem and the functional equation (2.7), and (1.6), we have

\[
I(x_1, \ldots, x_m) = \frac{R_n(x_1, \ldots, x_m)}{2\pi i} \int_{\Gamma} \frac{L(tx_1, \ldots, tx_m)dt}{\left( \prod_{k=0}^{n} (t - q^{-k}) \right)^{n+1}}.
\]
\[
R_n(x_1,\ldots,x_m) = \frac{R_n(x_1,\ldots,x_m) \sum_{k=0}^{n} \frac{L(q^{-k}x_1,\ldots,q^{-k}x_m)}{\prod_{k\neq k}^{n}(q^{-k} - q^{-k})} q^{-k(n+1)}}{R_n(x_1,\ldots,x_m) \frac{d^n}{dm} \left\{ \prod_{k=0}^{n} (t - q^{-k}) \right\}_{t=0} \sum_{k=0}^{n} \frac{L(tx_1,\ldots,tx_m)}{\prod_{k=0}^{n}(t - q^{-k})} q^{-k(n+1)/2} q^k L(x_1,\ldots,x_m)}
\]

\[
= \frac{q^{n(n+1)/2} R_n(x_1,\ldots,x_m) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} q^{nk+k(1+1)/2} q^k L(x_1,\ldots,x_m)}{(1-q)^n[n]!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} q^{nk+k(1+1)/2} S_k(x_1,\ldots,x_m)}{n! \frac{d^n}{dm} \left\{ \prod_{k=0}^{n} (t - q^{-k}) \right\}_{t=0} \sum_{k=0}^{n} \frac{L(tx_1,\ldots,tx_m)}{\prod_{k=0}^{n}(t - q^{-k})} q^{-k(n+1)/2} S_k(x_1,\ldots,x_m)}
\]

\[
= Q(x_1,\ldots,x_m) L(x_1,\ldots,x_m) + P(x_1,\ldots,x_m).
\]

**Proof of (ii).** As \(\binom{n}{k}\) is a polynomial in \(q\) with integer coefficients, and

\[
R_n(x_1,\ldots,x_m) = \prod_{j=1}^{n} ((1 - q^{-j}x_1) \cdots (1 - q^{-j}x_m))
\]

we have (2.13).

**Proof of (iii).** From (2.3) and (2.8), for \(1 \leq k \leq n\),

\[
R_n(x_1,\ldots,x_m) S_k(x_1,\ldots,x_m) = \sum_{h=1}^{k} q^{k-h} \prod_{j=1}^{n} ((1 - q^{-j}x_1) \cdots (1 - q^{-j}x_m)),
\]

so from (2.16),

\[
q^{n(n+1)/2} R_n(x_1,\ldots,x_m) S_k(x_1,\ldots,x_m) \in \mathbb{Z}[q_1,\ldots,q_m].
\]

Now for \(t < q^{-\ell}\), where \(\ell > 0\) is an integer such that \(|q^{-\ell}x_i| < q_i\) for all \(i = 1,\ldots,m\),

\[
L(tx_1,\ldots,tx_m) = L^\ell(tx_1,\ldots,tx_m) = \sum_{j_1,\ldots,j_m = 0}^{\infty} \frac{x_1^{j_1} \cdots x_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1},
\]

then from (1.7) and (2.18), for \(t < \min\{q^{-n},q^{-\ell}\},

\[
\frac{L(tx_1,\ldots,tx_m)}{\prod_{k=0}^{n}(t - q^{-k})} = (-1)^{n+1} \frac{n(n+1)/2}{l} \sum_{j_1,\ldots,j_m, l=0}^{\infty} \binom{n+l}{l} \frac{x_1^{j_1} \cdots x_m^{j_m}}{q^{j_1 + \cdots + j_m + l + 1} - 1}.
\]

So

\[
\frac{1}{n! \frac{d^n}{dm} \left\{ \prod_{k=0}^{n} (t - q^{-k}) \right\}_{t=0} \sum_{j_1,\ldots,j_m, l=0}^{\infty} \binom{n+l}{l} \frac{x_1^{j_1} \cdots x_m^{j_m}}{q^{j_1 + \cdots + j_m + l + 1} - 1}}
\]

\[
= (-1)^{n+1} \frac{n(n+1)/2}{l} \sum_{j_1,\ldots,j_m, l=0}^{\infty} \binom{n+l}{l} \frac{x_1^{j_1} \cdots x_m^{j_m}}{q^{j_1 + \cdots + j_m + l + 1} - 1}.
\]
and (2.14) follows from (2.11), (2.17) and (2.19).

**Proof of (iv).** For $R > 1$ and $\Gamma := \{ z : |z| = R \}$, we have from (2.9),

$$|I(x_1, \cdots, x_m)| \leq R \cdot \frac{R^n(x_1, \cdots, x_m) \max_{|t| = R} |L(tx_1, \cdots, tx_m)|}{R^{n+1} \prod_{k=0}^{n} (R - |q|^{-k})} \leq \frac{f_q \max_{|t| = R} |L(tx_1, \cdots, tx_m)|}{R^n \prod_{k=0}^{n} (R - q^{-k})}. \quad (2.20)$$

Now for $0 < |x_1|, \cdots, |x_m| < q$,

$$|R_n(x_1, \cdots, x_m)| = \prod_{j=1}^{n} (1 - q^{-j} x_1) \cdots (1 - q^{-j} x_m) \leq \prod_{j=0}^{\infty} (1 - q^{-j})^m := f_q, \quad (2.21)$$

where $f_q$ is a constant depending only on $q$ and $m$.

Let $R = q^{mn}$. As

$$\max_{|t| = R} |1 - q^{-j}tx_i| \geq \max_{|t| = R} |1 - q^{-j} |t||x_i| \geq |1 - q^{mn-j+1}|,$$

for $1 \leq i \leq m$, $j = 1, 2, \cdots$, and

$$q^j - 1 = q^j (1 - q^{-j}) \geq \frac{1}{2} q^j,$$

as $q$ is an integer greater than 1, then

$$\max_{|t| = R} |L(tx_1, \cdots, tx_m)| \leq \max_{|t| = R} \sum_{j=1}^{\infty} \left| \frac{q^{-j}}{(1 - q^{-j} x_1 t) \cdots (1 - q^{-j} x_m t)} \right| \leq \left( \sum_{j=1}^{mn} \frac{q^{j-mn-1}}{(q^j - 1)^m} \right) + \frac{q^{-mn-1}}{(1 - x_1/q) \cdots (1 - x_m/q)} + \left( \sum_{j=1}^{\infty} \frac{q^{-j-mn-1}}{(1 - q^{-j})^m} \right) \leq q^{-mn-1} \left( \sum_{j=1}^{mn-1} \frac{2m}{q^{(m-1)j}} + \frac{1}{(1 - x_1/q) \cdots (1 - x_m/q)} + L(1, \cdots, 1) \right) \leq \frac{q^{-mn-1}}{1 - x_1/q} \cdots (1 - x_m/q) + L(1, \cdots, 1) \leq C_1 q^{-mn}, \quad (2.22)$$

where $C_1 := 2m q + \frac{q}{(1 - x_1/q) \cdots (1 - x_m/q)} + qL(1, \cdots, 1)$ is a constant depending only on $q, m$, and $x_1, \cdots, x_m$. Now

$$R^n \prod_{k=0}^{n} (R - q^{-k}) = R^{n+1} \prod_{k=0}^{n} (1 - q^{-n-k}) \geq R^{n+1} \prod_{j=0}^{\infty} (1 - q^{-j}) \geq C_2 q^{mn(2n+1)}, \quad (2.23)$$
where \( C_2 := \prod_{j=0}^{\infty} (1 - q^{-j}) \) is a constant depending only on \( q \). Putting (2.22) and (2.23) into (2.20), we have
\[
|I(x_1, \cdots, x_m)| \leq c_q q^{-2mn(n+1)},
\]
where
\[
c_q := f_q C_1 / C_1.
\]
This completes the proof of Theorem 2.1. \( \square \)

3 Proof of Theorem 1.1

We first prove that for \( 0 < x_1, \cdots, x_m < q \), and \( q > 1 \),
\[
|I(x_1, \cdots, x_m)| \neq 0,
\]
where \( I(x_1, \cdots, x_m) \) is defined by (2.9). Note that if we choose the contour in (2.9) to be \( \Gamma = \{ z \in \mathbb{C} : |z| = 1 + \epsilon \} \), where \( \epsilon > 0 \) is small enough such that \( 0 < |tx_1|, \cdots, |tx_m| < q \) for \( t \in \Gamma \), then
\[
L(tx_1, \cdots, tx_m) = L^*(tx_1, \cdots, tx_m), \quad t \in \Gamma.
\]
Now
\[
R_n(x_1, \cdots, x_m) = \prod_{j=1}^{n} \left( (1 - q^{-j}x_1) \cdots (1 - q^{-j}x_m) \right) > 0
\]
for \( 0 < x_1, \cdots, x_m < q \), and
\[
I(x_1, \cdots, x_m) = \frac{R_n(x_1, \cdots, x_m)}{2\pi i} \int_{\Gamma} \frac{L(tx_1, \cdots, tx_m) dt}{t^{2n+2} \left( \prod_{k=0}^{n} (1 - 1/(q^k t)) \right)}
\]
\[
= \frac{R_n(x_1, \cdots, x_m)}{2\pi i} \int_{\Gamma} \frac{L(tx_1, \cdots, tx_m) dt}{t^{2n+2} \left( \sum_{j_0, \cdots, j_n \geq 0} \left( \prod_{k=0}^{n} \left( \frac{1}{q^k t} \right)^{j_k} \right) \right)}
\]
\[
= R_n(x_1, \cdots, x_m) \sum_{j_0, \cdots, j_n \geq 0} q^{-\sum_{k=0}^{n} kj_k} \cdot \frac{1}{2\pi i} \int_{\Gamma} \frac{dt}{t^{2n+2+(j_0 + \cdots + j_n)}}
\]
\[
= R_n(x_1, \cdots, x_m) \sum_{j_0, \cdots, j_n \geq 0} q^{-\sum_{k=0}^{n} kj_k} \sum_{i_1 + \cdots + i_m = 0} \frac{x_1^{i_1} \cdots x_m^{i_m}}{q^{i_1 + \cdots + i_m + 1} - 1}
\]
\[
> 0,
\]
as \( x_1, \cdots, x_m \geq 0, q > 1 \), and as infinitely many terms above are positive, so (3.1) holds.

Now let \( r_1, r_2, \cdots, r_m \) be any fixed positive rational numbers such that \( r_1, r_2, \cdots, r_m \neq q^j \) for all \( j = 1, 2, \cdots \). From (2.7), we can see that the irrationality of \( L(r_1, r_2, \cdots, r_m) \) is equivalent
to the irrationality of \( L(q^{-k}r_1, q^{-k}r_2, \ldots, q^{-k}r_m) \) for any integer \( k \geq 1 \), so we can assume that \( 0 < r_1, r_2, \ldots, r_m < q \), and then

\[
L(r_1, r_2, \ldots, r_m) = \sum_{j_1, \ldots, j_m = 0}^{\infty} \frac{r_1^{j_1} \cdots r_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1} > 0.
\]

Now let

\[
H_{m,n}(q) := q^{(m-1)n(n+1)/2} \left( \prod_{j=1}^{n+1} (q^j - 1) \right). \tag{3.3}
\]

Then

\[
0 < |H_{m,n}(q)| \leq q^{(mn+2)(n+1)/2}, \tag{3.4}
\]

and

\[
H_{m,n}(q) \cdot \{Q(r_1, \ldots, r_m), P(r_1, \ldots, r_m)\} \subset \mathbb{Z}[q, r_1, \ldots, r_m]. \tag{3.5}
\]

Now as

\[
\Delta_{m,n} := |H_{m,n}(q)Q(r_1, \ldots, r_m)L(r_1, \ldots, r_m) + H_{m,n}(q)P(r_1, \ldots, r_m)|
\]

\[
= |H_{m,n}(q)| |I(r_1, \ldots, r_m)|
\]

\[
> 0, \tag{3.6}
\]

and from (2.15) and (3.3), we have

\[
\Delta_{m,n} \leq \frac{c_q}{q^{2mn(n+1)}}
\]

\[
= \frac{c_q}{q^{3mn+2(n+1)/2}}
\]

\[
\leq \frac{c_q}{q^{mn^2}}. \tag{3.7}
\]

Finally, if

\[
r_1 := \frac{i_1}{l_1}, r_2 := \frac{i_2}{l_2}, \ldots, r_m := \frac{i_m}{l_m}, \tag{3.8}
\]

with \( i_1, \ldots, i_m \) and \( l_1, \ldots, l_m \) positive integers, then

\[
Q^*(r_1, \ldots, r_m) := (l_1 \cdots l_m)^{2n} H_{m,n}(q)Q(r_1, \ldots, r_m), \tag{3.9}
\]

and

\[
P^*(r_1, \ldots, r_m) := (l_1 \cdots l_m)^{2n} H_{m,n}(q)P(r_1, \ldots, r_m), \tag{3.10}
\]

are integers, and by (3.6) to (3.10),

\[
0 < |Q^*(r_1, \ldots, r_m)L(r_1, \ldots, r_m) + P^*(r_1, \ldots, r_m)|
\]

\[
= (l_1 \cdots l_m)^{2n} |H_{m,n}(q)| |Q(r_1, \ldots, r_m)L(r_1, \ldots, r_m) + P(r_1, \ldots, r_m)|
\]

\[
\leq (l_1 \cdots l_m)^{2n} \frac{c_q}{q^{mn^2}},
\]

which tends to zero as \( n \to \infty \). This shows that \( L(r_1, \ldots, r_m) \) is irrational, that is

\[
\sum_{j_1, \ldots, j_m = 0}^{\infty} \frac{r_1^{j_1} \cdots r_m^{j_m}}{q^{j_1 + \cdots + j_m + 1} - 1}
\]

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is irrational for \( q > 1 \) integer, \( r_1, r_2, \ldots, r_m \) positive rationals less than \( q \) and integer \( m \geq 1 \), and
\[
\sum_{j=1}^{\infty} \frac{q^{-j}}{(1 - q^{-j}r_1)(1 - q^{-j}r_2) \cdots (1 - q^{-j}r_m)}
\]
is irrational for \( q > 1 \) integer, \( r_1, r_2, \ldots, r_m \) positive rationals such that \( r_1, r_2, \ldots, r_m \neq q^j \) for all \( j = 1, 2, \ldots, \) and integer \( m \geq 1 \).

This completes the proof of Theorem 1.1. \( \square \)

Now by the standard methods (as in chapter 11 of Borwein and Borwein [1]), the estimates in the proof of Theorem 1.1 gives that, under the assumption of the theorem,
\[
|L(r_1, \ldots, r_m) - \frac{s}{t}| > \frac{1}{t^{\alpha}},
\]
for some constant \( \alpha \) and all integers \( s \) and \( t \), and hence
\[
\sum_{j_1, \ldots, j_m = 0}^{\infty} \frac{r_1^{j_1} \cdots r_m^{j_m}}{q^{r_1^{j_1} + \cdots + r_m^{j_m} + 1} - 1}
\]
is not a Liouville number.

References


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