Note

On Monochromatic Triangles

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Communicated by the Managing Editors
Received December 8, 1980

Let \( A \) and \( B \) be two disjoint finite sets in \( \mathbb{R}^2 \). Simple conditions that guarantee the existence of a triangle with vertices in one of the sets and with no points from the other set in its interior are given. The analogous problem for \( d \)-simplices in \( \mathbb{R}^d \) is treated. Conditions are derived that guarantee the existence of a triangle with vertices in one of the sets and with no points from either set on its boundary.

INTRODUCTION

Let \( A \) and \( B \) be two disjoint nonempty finite sets in \( \mathbb{R}^2 \). Under the assumption that \( A \cup B \) spans \( \mathbb{R}^2 \) Motzkin [2] proves the existence of a monochromatic line (a line through at least 2 points of one of the sets that misses the other set). A special case of this result is the following lemma.

**Lemma 1 (J. B. Kelly [3, p. 298]).** Let \( A \) and \( B \) be two finite sets in \( \mathbb{R}^n \). Suppose that every open segment joining two points of \( A \) contains a point of \( B \), and vice versa. Then the sets \( A \) and \( B \) lie on a line.

J. B. Kelly’s proof of Lemma 1 is a minimum-altitude proof based on L. M. Kelly’s proof of Sylvester’s theorem. We offer the following particularly simple proof of Lemma 1.

**Proof.** Let \( T_1 = \{p_1, p_2, p_3\} \) be a nondegenerate triangle of smallest area with all vertices either in \( A \) or all vertices in \( B \). We show that no such triangle exists. We assume that \( \{p_1, p_2, p_3\} \subset A \). By assumption there exists \( \{b_1, b_2, b_3\} \subset B \) so that each \( b_i \) lies on a different edge of \( T_1 \). The triangle \( T_2 = \{b_1, b_2, b_3\} \) now has smaller area that \( T_1 \) and contradicts the initial assumption. \( \square \)

This result motivated Baston and Bostock [1] to examine various generalizations. It is our intention to do the same.

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The first theorem concerns triangles with monochromatic interiors.

**Theorem 1.** Let \( A \) and \( B \) be two finite disjoint sets of points in \( \mathbb{R}^2 \). Suppose \( A \) contains five points in general position. Then there exists a triangle with vertices in one of the sets and with no point from the other set in its interior.

Theorems 1 and 3 are similar in flavour to results in [1]. Figure 1 shows that the assumption of five points in general position in Theorem 1 is necessary.

**Proof.** Suppose that \( \text{card}(A) = n \geq 5 \). Let \( \Gamma \) denote the convex hull of \( A \), let \( A' = A \cap \text{int}(\Gamma) \) and let \( B' = B \cap \text{int}(\Gamma) \). Let \( k = \text{card}(A') \). Suppose that \( A \) and \( B \) satisfy the conditions of the theorem but contradict the conclusion.

We observe that there is a triangulation of \( \Gamma \) consisting of \( n + k - 2 \) triangles having \( A \) as its vertex set. To see this, first, partition \( \Gamma \) into \( n - k - 2 \) triangles using the points of \( A \) on the boundary of \( \Gamma \) as vertices, then add the interior vertices one at a time. Each additional interior vertex increases the number of triangles by two.

Since each triangle in such a triangulation of \( \Gamma \) contains a point of \( B' \), it follows that \( \text{card} \,(B') \geq n + k - 2 \). Any line 1 has at least three noncollinear points of \( A \) on or to one side of it, hence, there is a point of \( B' \) not on 1. Thus, the points of \( B' \) are not all collinear, so the convex hull of \( B' \) has a triangulation consisting of at least \((n + k - 2) - 2\) triangles. This implies that \( \text{card}(A') \geq n + k - 4 = k = \text{card}(A') \) which is impossible.

**Theorem 2.** Suppose \( A \) and \( B \) are disjoint finite sets in \( \mathbb{R}^2 \). Suppose that \( A \) contains 5 points \( p_1, ..., p_5 \) that are the vertices of a strictly convex pentagon. Let \( \Pi \) denote the convex hull of \( \{p_1, ..., p_5\} \) and suppose that \( \Pi \cap A \) has no three points collinear. Then there exists a triangle \( T \) with

\[
\begin{array}{c}
\text{A} \\
\text{B} \\
\end{array}
\]
vertices only in $A$ or only in $B$ so that no other point from either $A$ or $B$ lies on any of the edges of $T$.

Proof. Assume we have sets $A$ and $B$ which satisfy the hypothesis but not the conclusion of the theorem. Let $\Gamma$ be the smallest (in area) strictly convex pentagon with vertices in $A$ and with the property that $\Gamma \cap A$ has no three points collinear. Let $A' = A \cap \Gamma$ and $B' = B \cap \Gamma$. Let $q_1, q_2, q_3, q_4, q_5$ be the vertices of $\Gamma$ in order. Notice that no other set of five points of $A'$ spans a convex pentagon.

Now any three points of $A'$ span a triangle which has no other points of $A'$ on its edges, so one edge must contain a point of $B'$. Suppose $C$ is a noncollinear subset of $B'$. The convex hull $A$ of $C$ has a triangulation whose vertex set is $B \cap A$. (See the proof of Theorem 1.) Any triangle of this triangulation must contain a point of $A'$, so $A' \cap A \neq \emptyset$.

The triangles $q_1q_2q_3$ and $q_4q_5q_1$, each contain a point of $B'$ and these two points are distinct. Let $1$ be the line joining them. Since $1$ does not contain more than two points of $A'$, we can find either three points of $A'$ on one side of 1 or two points of $A'$ on one side of 1 and one on 1. In either case we obtain a point of $B'$ not on 1. Thus, the convex hull of $B'$ is two dimensional and contains a point $r$ of $A'$ which is necessarily in int $\Gamma$ and hence, is distinct from $q_1, q_2, q_3, q_4, q_5$.

The convex hull of $r$ and three consecutive vertices of $\Gamma$ is a quadrilateral, since if $r$ were within the triangle spanned by $q_1, q_2, q_3$, say, then $r, q_1, q_3$, $q_4, q_5$ would span a smaller convex pentagon, which is impossible.

Consider the five radial segments $rq_i$. Suppose two of these that are not adjacent each contain a point of $B'$, say $rq_1$ and $rq_2$, both meet $B'$. Then a third point of $B'$ can be found on the triangle $q_1q_2q_3$. These three points of $B'$ cannot be collinear and hence, there is a point $s$ of $A'$ within the triangle $q_1q_2q_3$. Since $s$ is interior to the quadrilateral $rq_1rq_2q_3$ it is a new point.

If the configuration just analyzed does not exist, then there are three consecutive radial segments, say $rq_1, rq_2, rq_3$, none of which contains a point of $B'$. Then each of the segments $q_1q_2, q_2q_3, q_3q_1$ must contain a point of $B'$. These three points of $B'$ span a triangle which must contain a point $s$ of $A'$. Since $s$ is interior to the triangle $q_1q_2q_3$, it is a new point.

Thus, in either case there is a seventh point $s$ in $A'$. At least three of the $q$'s lie on one side of the line joining $r$ and $s$, and these three together with $r$ and $s$ span a convex pentagon. This, contradiction finishes the proof. \[\square\]

The example in Fig. 2 shows that we cannot weaken the assumptions of Theorem 2. Since any set of nine points in general position contain the vertices of a strictly convex pentagon [4, Prob. 31] we have

Corollary 1. Suppose that $A$ and $B$ are disjoint finite sets in $\mathbb{R}^2$ and suppose that there exists a convex set $\Gamma$ in $\mathbb{R}^2$ so that $\Gamma \cap A$ contains no
three collinear points. Then, if \( \text{card}(\Gamma \cap A) \geq 9 \) there exists a triangle \( T \) with vertices only in \( A \) or only in \( B \) so that no other point from \( A \cup B \) lies on any edge of \( T \).

We note that both Theorem 1 and Corollary 1 are valid in \( \mathbb{R}^m, m \geq 2 \). The proofs in higher dimensions follow, under careful projection, from the two dimensional cases.

The following higher dimensional analogue of Theorem 1 is valid.

**Theorem 3.** Suppose that \( A \) and \( B \) are two finite disjoint sets in \( \mathbb{R}^d \), \( d \geq 2 \). Suppose that \( A \) contains \( 2d + 1 \) points in general position. Then there exists a \( d \)-simplex with vertices in one of the sets and with no points of either set in its interior.

This follows, from Lemma 1. The arguments are analogous to those used in the proof of Theorem 1.

**Lemma 1.** Suppose \( S \) is a finite set in \( \mathbb{R}^d \) that spans \( \mathbb{R}^d \). Let \( \Gamma \) be its convex hull. Then \( \Gamma \) has a triangulation with vertices in \( S \) and with at least \( n - d + k(d - 1) \) \( d \)-simplices, where \( n = \text{card}(S) \) and \( k = \text{card}(S \cap \text{int} \Gamma) \).

There are many obvious related questions: What happens if we consider quadrilaterals (pentagons, etc.) in Theorem 1? What conditions yield a result like Theorem 2 with the conclusion that there exists a triangle with both monochromatic edges and monochromatic interior? What is a correct analogue to Theorem 3 if we consider three sets instead of just two?

**Acknowledgment**

I would like to thank Professor Gleason for suggestions and observations that have considerably improved this paper.
REFERENCES