MARKOV’S AND BERNSTEIN’S INEQUALITIES ON DISJOINT INTERVALS

PETER B. BORWEIN

1. Introduction. In 1889, A. A. Markov proved the following inequality:

INEQUALITY 1. (Markov [4]). If \( p_n \) is any algebraic polynomial of degree at most \( n \) then

\[
\| p'_n \|[_{[a,b]}] \leq \frac{2n^3}{b-a} \| p_n \|_{[a,b]}
\]

where \( \| \cdot \|_a \) denotes the supremum norm on \( A \).

In 1912, S. N. Bernstein established

INEQUALITY 2. (Bernstein [2]). If \( p_n \) is any algebraic polynomial of degree at most \( n \) then

\[
| p'_n (x) | \leq \frac{n}{(x-a)(b-x)} \left\| p_n \right\|_{[a,b]}
\]

for \( x \in (a,b) \).

In this paper we extend these inequalities to sets of the form \([a,b] \cup [c,d]\). Let \( \Pi_n \) denote the set of algebraic polynomials with real coefficients of degree at most \( n \).

THEOREM 1. Let \( a < b \leq c < d \) and let \( p_n \in \Pi_n \). Then

\[
| p'_n (x) | \leq \frac{\left( \frac{c-x}{d-x} \right)^{1/3}}{(b-x)(x-a)} \frac{n}{((b-x)(x-a))^{1/2}} \| p_n \|_{[c,d]} \cup [a,b]}
\]

for \( x \in (a,b) \).

We note that Inequality 2 is a special case \((b = c = d)\) of the above theorem.

COROLLARY 1. Let \( a < b \leq c < d \) and let \( p_n \in \Pi_n \). Then

\[
| p'_n (x) | \leq \frac{(x-b)^{1/3}}{(x-c)} \frac{n}{((x-c)(d-x))^{1/2}} \| p_n \|_{[a,b] \cup [c,d]}
\]

for \( x \in (c,d) \).

Received May 1, 1979.
COROLLARY 2. Let \( a < b \leq c < d \) and let \( p_n \in \Pi_n \). Then,

\[
\|p_n\|_{[c,d]} \leq \left( \frac{d-b}{d-a} \right)^{1/2} \frac{2n^2}{d-c} \|p_n\|_{[a,b]} \cup [c,d].
\]

Thus, we obtain sharper bounds than those we achieve by applying Inequality 1 or Inequality 2 directly to \([c,d]\).

On sets of the form \([-b, -a] \cup [a, b]\) we can derive an asymptotically "best possible" form of Markov’s inequality.

**Theorem 2.** a) If \( 0 < a < b < n \) is even and \( p_n \in \Pi_n \), then

\[
\|p_n\|_{[-b, a] \cup [b, a]} \leq \left( 1 + \frac{9}{n^2} \right) \frac{n^2 b}{b^2 - a^2} \|p_n\|_{[-b, a] \cup [b, a]}
\]

provided that \( n \) is large enough to satisfy

\[
\frac{b^2 - a^2}{3abn} + \left( \frac{b + a}{2b} \right) \left( 1 + \frac{6}{n} \right)^2 e^{6b^2 - 9a^2} 3abn \leq 1.
\]

b) For each even \( n \) there exists \( p_n \in \Pi_n \) so that

\[
\|p_n\|_{[-b, a] \cup [a, b]} = \frac{n^2 b}{b^2 - a^2} \|p_n\|_{[-b, a] \cup [b, a]}.
\]

**Corollary 3.** Suppose \( n \) is even and \( n \geq 50 \). If \( p_n \in \Pi_n \) then

\[
\|p_n\|_{[-2, -1] \cup [1, 2]} \leq \left( 1 + \frac{9}{n^2} \right) \frac{2n^2}{3} \|p_n\|_{[-2, -1] \cup [1, 2]}.
\]

2. Characterizing polynomials that maximize Markov’s or Bernstein’s inequalities. In this section we show that polynomials that maximize \( \|p_n(t)\|_{I} \), subject to \( \|p_n\|_{I} \leq 1 \) where \( I \) is compact, must be of the form

\[
ax^n + \beta x^{n-1} - g_{n-1}(x)
\]

where \( g_{n-2} \in \Pi_{n-2} \) is the best approximation to \( ax^n + \beta x^{n-1} \) on \( I \). In particular, we show, as Bernstein did for the interval \([-1, 1]\) (see [2]), that the polynomial that satisfies \( \|p_n\|_{I} = 1 \) and has maximum derivative at \( max I \) is of the form

\[
p_n(x) = ax^n - g_{n-1}(x)
\]

where \( g_{n-1} \in \Pi_{n-1} \) and \( g_{n-1} \) is the best approximation to \( ax^n \) on \( I \).

**Theorem 3.** Let \( I \) be any infinite compact set of real numbers and let \( \xi \in R \). Suppose \( p_n \in \Pi_n \) satisfies

\[
\frac{\|p_n(\xi)\|}{\|p_n\|_{I}} = \max_{q \in \Pi_n} \frac{|q(\xi)|}{\|q\|_{I}}.
\]
Then, there exist \( \alpha \) and \( \beta \) so that \( p_n(x) = ax^n + \beta x^{n-1} - s_{n-1}(x) \) where \( s_{n-1} \in \Pi_{n-1} \) is the best Chebyshev approximation to \( ax^n + \beta x^{n-1} \) on \( I \). (The best Chebyshev approximation is the one that minimizes the supremum norm.)

We need the following lemma for the proof of this theorem:

**Lemma 1.** Let \( p_n \in \Pi_n \) and let \( \xi \) be any point that is not a root of \( p_n \). Suppose that there exist at most \( k \leq n - 2 \) points \( x_1 < x_2 < \ldots < x_k \) where \( p_n \) changes sign. Then there exists \( q_n \in \Pi_n \) so that

a) \( \text{sgn } q_n(x) = \text{sgn } p_n(x) \),

b) \( \text{sgn } q_n(x + \xi) = -\text{sgn } p_n(x) \), except possibly at the roots of \( q_n \).

**Proof.** Let

\[
s(x) = - (\text{sgn } p_n(-\infty)) (-1)^k \prod_{i=1}^k (x - x_i)
\]

and consider \( q_n(x) = s(x) (x - y)^2 \). Then, if \( s(\xi) \neq 0 \),

\[
\frac{dq_n(x)}{dx} = (\xi - y)(2s(\xi) + (\xi - y)s'(\xi))
\]

which as a function of \( y \) changes sign at \( \xi \). Thus, for an appropriate \( y \) close to \( \xi \), \( q_n \) satisfies a) and b).

**Proof of Theorem 3.** Let \( p_n \) satisfy the assumptions of the theorem (that such a \( p_n \) exists is a simple consequence of \( \Pi_n \) being finite dimensional).

Suppose \( p_n \) has at most \( n - 2 \) changes of sign and suppose \( p_n(\xi) \neq 0 \). If \( q_n \) satisfies the conclusion of Lemma 1, then for sufficiently small \( \epsilon > 0 \),

\[
\|p_n + \epsilon q_n\|_I \leq \|p_n\|_I \quad \text{and} \quad \|p_n'(\xi) + \epsilon q_n'(\xi)\| > \|p_n'(\xi)\|
\]

which contradicts the assumption that \( p_n \) satisfies (1). Now suppose \( p_n(\xi) = 0 \) and \( p_n \) changes sign at \( x_1 < \ldots < x_k \). If

\[
q_n(x) = - (\text{sgn } p_n(-\infty)) (-1)^k \left( \prod_{i=1}^k (x - x_i) \right)(x - \xi)^2
\]

then, for sufficiently small \( \epsilon > 0 \),

\[
\|p_n + \epsilon q_n\|_I < \|p_n\|_I \quad \text{and} \quad \|p_n'(\xi) + \epsilon q_n'(\xi)\| > \|p_n'(\xi)\|
\]

which also contradicts the assumption that \( p_n \) satisfies (1). Thus, \( p_n \) has at least \( n - 1 \) sign changes.

We now suppose that the coefficient of \( x^n \) is non-zero for \( p_n \). It follows that \( p_n \) has \( n \) real roots \( x_1 < x_2 < \ldots < x_k \). We claim that in each interval \( (x_i, x_{i+1}) \) there exists a point \( y_i \in I \) so that

(2) \( \|p_n(y_i)\| = \|p_n\|_I \).

If (2) is false then as in the proof of the lemma, we can, for a suitably
chosen \( y \), construct
\[
g_n(x) = -(\text{sgn } p_n(-\infty))(-1)^n \left( \prod_{i=1}^{n-1} (x - x_i) \right) \left( \prod_{i=n+2}^n (x - x_i) \right) \times (x - y)^n
\]
where
a) \( \text{sgn } g_n'(\xi) = \text{sgn } p_n'(\xi) \)
and
b) \( \text{sgn } g_n(x) = -\text{sgn } p_n(x) \)
except possibly for \( x \in [x_1, \ldots, x_n, y] \cup [x_j, x_{j+1}] \). We note that since the \( y \) of Lemma 1 can be chosen from an interval, we may assume that \(|p_n(y)| \neq \|p_n\|_I\). It follows from a), b) and the assumption
\[
\|p_n\|_{[x_j, x_{j+1}]} < \|p_n\|_I
\]
that for sufficiently small \( \epsilon > 0 \),
\[
\|p_n + \epsilon g_n\|_I < \|p_n\|_I
\]
and
\[
|p_n'(\xi) + \epsilon g_n'(\xi)| \geq |p_n'(\xi)|.
\]
This contradiction establishes (2).
We may by a similar argument show that there exists \( y_n \) so that
\[
y_n \in I \cap (-\infty, x_1) \quad \text{or} \quad y_n \in I \cap (x_n, \infty)
\]
and
\[
|p_n(y_n)| = \|p_n\|_I.
\]
Thus, if \( p_n(x) = \alpha x^n + \beta x^{n-1} - s_{n-2}(x) \) where \( \alpha \neq 0 \), then \( p_n \) achieves its maximum norm, with alternate sign, at \( n \) points \( y_1 < y_2 < \ldots < y_n \) in \( I \). This suffices to establish the theorem.

If \( p_n \) is actually of degree \( n - 1 \), then \( p_n(x) = \beta x^{n-1} - q_{n-1}(x) \). A similar argument shows that \( q_{n-1}(x) \) is the best approximation to \( \beta x^{n-1} \) on \( I \).

**Theorem 4.** Let \( I \) be any infinite compact set and let \( \xi \geq \delta = \max I \). Suppose \( p_n \in \Pi_n \) satisfies
\[
|p_{n}^{\prime} (\xi)| \leq \max_{x \in \partial I} \frac{|g_{n}^{\prime} (\xi)|}{|g_{n}^{\prime} |},
\]
Then \( p_n(x) = \alpha x^n - q_{n-1}(x) \) where \( q_{n-1} \in \Pi_{n-1} \) and \( q_{n-1} \) is the best Chebyshev approximation to \( \alpha x^n \) on \( I \).
Proof. Let \( \gamma = \min I \). The preceding theorem guarantees the existence of \( n-1 \) points \( \gamma < x_1 < \ldots < x_{n-1} < \delta \) where \( p_n \) changes sign. We first show that \( p_n \) has \( n \) distinct roots in \( [\gamma, \delta] \). Suppose \( p_n \) does not change sign at any point in \( [\gamma, \delta] \) other than \( x_1, \ldots, x_{n-1} \). Consider

\[
q_n(x) = -\text{sgn} \ (p_n(\delta)) \left( \prod_{k=1}^{n-1} (x - x_k) \right) (y - x)
\]

\[
= s_n(x)(y - x)
\]

then

\[
\frac{d}{dx} q_n(x) \bigg|_x = s_n'(x)(y - x) - s_n(x).
\]

Since \( \text{sgn} \ s_n'(x) = \text{sgn} \ s_n(x) \neq 0 \) we may, for a suitable choice of \( y > \delta \), set \( t_n = q_n(x) \) where

a) \( \text{sgn} \ t_n'(x) = \text{sgn} \ p_n'(\gamma) \)

b) \( \text{sgn} \ t_n = -\text{sgn} \ p_n \) on \( I \).

Thus, for sufficiently small \( \epsilon > 0 \),

\[
\|p_n + \epsilon t_n\|_I < \|p_n\|_I \quad \text{and} \quad |p_n'(\gamma) + \epsilon t_n'(\gamma)| > |p_n'(\gamma)|
\]

which is a contradiction. Thus, \( p_n \) has \( n \) distinct roots \( \gamma \leq x_1 < x_2 < \ldots < x_n \leq \delta \). We now show that

\[
|p_n(\delta)| = |p_n(\gamma)| = \|p_n\|_I.
\]

This, coupled with (2) of the proof of Theorem 3, suffices to complete the result. We will only show that \( |p_n(\delta)| = \|p_n\|_I \) since the proof that \( |p_n(\gamma)| = \|p_n\|_I \) is similar. Suppose \( |p_n(\delta)| < \|p_n\|_I \). Let

\[
q_n(x) = -(\text{sgn} \ p_n(\infty))(-1)^{n-1} \left( \prod_{k=1}^{n-1} (x - x_k) \right) (y - x)
\]

where, as before, \( y > \delta \) is chosen so that

\[
\text{sgn} \ q_n'(x) = \text{sgn} \ p_n'(\gamma).
\]

Then, for sufficiently small \( \epsilon > 0 \), \( p_n + \epsilon q_n \) contradicts the assumption that \( p_n \) satisfies (1).

3. Bernstein's inequality on \([a, b] \cup [c, d]\).

Proof of Theorem 1. Let \( A = [a, b] \cup [c, d] \) and let \( \tau \in A \). Let \( p_n \in \Pi_n \) satisfy

\[
\frac{|p_n'(\tau)|}{\|p_n\|_A} = \max_{x \in I_n} \frac{|q_n'(x)|}{\|q_n\|_A}
\]
and
\[ \|p_\alpha\|_A = 1. \]

We may, by the proof of Theorem 3, assume that \( p_\alpha \) has all its roots in \( A \) with the possible exceptions of a root \( \lambda_1 \in (b, c) \) and a root \( \lambda_2 > d \) or \( \lambda_3 < a \). We treat the case where \( \lambda_1 \in (b, c) \) and \( \lambda_3 > d \). The other cases proceed analogously. We observe that if we increase \( c \) or \( a \) and if we decrease \( b \) or \( d \) we strengthen the inequality in the statement of the theorem. Thus, we may also assume that for \( y \in \{a, b, c, d\} \),
\[ |p_\alpha(y)| = 1 \quad \text{and} \quad |p_\alpha'(y)| \neq 0. \]

(If there is no point \( z \in (b, c) \) where \( |p_\alpha(z)| \geq 1 \) then we can deduce the result from Inequality 2.) We have guaranteed the existence of points
\[ b < \epsilon_1 < \delta_1 < \lambda_1 < \delta_2 < \epsilon_2 < c \]
and
\[ d < \epsilon_3 < \delta_3 < \lambda_2 < \delta_4 \]
so that
\[ |p_\alpha' (\epsilon_i) | = 0 \quad i = 1, 2, 3 \]
and
\[ |p_\alpha (\delta_i) | = 1 \quad i = 1, 2, 3, 4. \]

We deduce from Theorem 3 and a comparison of roots and leading terms that
\[ (p_\alpha'(x))^2 (x - a) (x - b) (x - c) (x - d) (x - \delta_1) (x - \delta_2) (x - \delta_3) \times (x - \delta_4) \]
\[ = n^2 ((p_\alpha(x))^2 - 1) (x - \epsilon_1)^2 (x - \epsilon_2)^2 (x - \epsilon_3)^2. \]

Thus, if \( \tau \in (a, b) \),
\[ (p_\alpha' (\tau))^2 \leq \frac{n^2 (\tau - \epsilon_1)^2}{|\tau - a| (\tau - b) (\tau - c) (\tau - d)|} \cdot \frac{(\tau - \epsilon_2)^2}{|\tau - \delta_1| (\tau - \delta_2)|} \cdot \frac{(\tau - \epsilon_3)^2}{|\tau - \delta_3| (\tau - \delta_4)|} \leq \frac{n^2 (\tau - c)^2}{|\tau - a| (\tau - b) (\tau - c) (\tau - d)|} \]
and the result now follows.

Corollary 1 follows immediately from Theorem 1. Corollary 2 is a consequence of Corollary 1 and the next inequality.

**Inequality 3.** (Schur [3] p. 41). If \( p_{n-1} \in \Pi_{n-1} \) and
\[ |p_{n-1}(x)| \leq \frac{L}{((x - a)(b - x))^{1/2}} \quad \text{for} \quad a < x < b, \]
then
\[ \|p_{n-1}(x)\|_{[a,b]} \leq \frac{2L_n}{b-a}. \]

4. Markov's inequality on \([-b, -a] \cup [a, b]\). We require the following results for the proof of Theorem 2.

Theorem 5. (Achieser [1], p. 287). Let \( n \) be an even integer. The polynomial \( p_n \in P_n \) with leading coefficient 1 that deviates least from zero on \([-b, -a] \cup [a, b]\) is
\[ S_n(x) = \left( \frac{b^2 - a^2}{2^{n/2}} \right) T_{n/2} \left( \frac{2x^2 - b^2 - a^2}{b^2 - a^2} \right) \]
where \( T_n \) is the \( n \)th Chebyshev polynomial \( (T_n = \cos n \cos^{-1} x) \).

Lemma 2. Let \( n \) be even and let \( S_n \) be defined as in Theorem 5. Then,
\[ \frac{\|S'(b)\|_{[-b,-a] \cup [a,b]}}{\|S\|_{[-b,-a] \cup [a,b]}} = \frac{n^2 b}{b^2 - a^2}. \]
The proof of Lemma 2 is straightforward and is omitted.

Lemma 3. Suppose \( n \) is even. Then
\[ \max_{p_n \in P_n} \frac{|p_n(b)|}{\|p_n\|_{[-b,-a] \cup [a,b]}} = \frac{n^2 b}{b^2 - a^2}. \]
Proof. This is a direct consequence of Theorem 4, Theorem 5 and Lemma 2.

Lemma 4. (Sobolev [5]). If \( p_n \in P_n \) has non-negative coefficients then, for \( x > 0 \)
\[ |p_n'(x)| \leq \frac{n}{x} |p_n(x)|. \]

Proof of Theorem 2. Suppose \( p_n \in P_n \) satisfies
\[ \frac{\|p_n\|_{[-b,-a] \cup [a,b]}}{\|p_n\|_{[-b,-a] \cup [a,b]}} = \frac{\|p_n'\|_{[-b,-a] \cup [a,b]}}{\|p_n'\|_{[-b,-a] \cup [a,b]}}. \]
Suppose \( \xi \in [a, b] \) is a point where
\[ |p'(\xi)| = \|p'\|_{[-b,-a] \cup [a,b]} \]
and
\[ |p_n'(\xi)| \geq \frac{n^2 b}{b^2 - a^2} \|p_n\|_{[-b,-a] \cup [a,b]}. \]
Then, by Inequality 2 applied to \([a, b]\)
\[
\frac{n^2b}{b^2 - a^2} \leq \frac{n}{((b - \xi - \xi - a))^{1/2}}
\]
and
\[
(b - \xi)(\xi - a) \leq \frac{(b^2 - a^2)^2}{n^2b^2}.
\]
Since either \((b - \xi) \geq \frac{1}{2}(b - a)\) or \((\xi - a) \geq \frac{1}{2}(b - a)\), either
\[
(b - \xi) \leq \frac{2(b + a)(b^2 - a^2)}{b^2n^2} \quad \text{or} \quad (\xi - a) \leq \frac{2(b + a)(b^2 - a^2)}{b^2n^2}.
\]
Suppose
\[
(2) \quad (b - \xi) \leq \frac{2(b + a)}{b} \cdot \frac{(b^2 - a^2)}{bn^2} \leq \frac{4(b^2 - a^2)}{bn^2}.
\]
Then, by Lemma 3 and (2), for \(n \geq 10\),
\[
\max_{\rho_n \in \Pi_{n-1}} \left| \frac{\rho_n'(-\xi - a)}{||\rho_n'||_{[-\xi - a] \cup [a, b]}} \right| \leq \max_{\rho_n \in \Pi_{n-1}} \left| \frac{\rho_n'(\xi)}{||\rho_n'||_{[-\xi - a] \cup [a, b]}} \right| \leq \frac{n^2b}{\xi - a} \leq \frac{4(b^2 - a^2)}{bn^2}.
\]
Suppose now that \((\xi - a) \leq 4(b^2 - a^2)/bn^2\). Write \(\rho_n(x) = \eta_n(x)\rho_n(x)\) where \(\eta_n(x)\) has all its roots in \([-b, -a]\) and \(\rho_n(x)\) has no roots in \([-b, -a]\). By Theorem 3, \(\rho_n\) oscillates between its maximum and minimum at least \(n\) times on \([-b, -a] \cup [a, b]\). Hence, \(\rho_n\) has at least \(n - 2\) distinct roots in \([-b, -a] \cup [a, b]\). By the proof of Theorem 3, between any two roots of \(\rho_n\) there is a point of \([-b, -a] \cup [a, b]\) where \(\rho_n\) attains its norm. Suppose now that \(m \geq 2 + n/2\). Then, \(\rho_n(x) - \rho_n(-x) \in \Pi_{n-1}\) has at least \(n/2\) roots in \([-b, -a]\) and at least \(n/2\) roots in \([a, b]\) and hence, \(\rho_n(x) = -\rho_n(-x)\). However, if \(\rho_n\) is even, then it follows from Theorem 3, Theorem 5 and Lemma 3 that \(\rho_n = \rho_n\) and we are done. Thus, we may assume \(m \geq n/2 + 1\). Similarly, since \(r_n(x)\) has at least \(k - 2\) roots in \([a, b]\), we may assume that \(k \leq n/2 + 3\). We may also assume that \(n \geq 10\).

\[
(3) \quad \left| \frac{\eta_n'(\xi)}{\eta_n(\xi)} \right| \leq \frac{m}{a + x} \leq \frac{m + 2}{4a} \left| \eta_n(\xi) \right|.
\]
Also, since \(\eta_n(x) = a\Pi(x + x_i)\) with \(x_i \geq a\),
\[
(4) \quad \left| \frac{\eta_n'(\xi)}{\eta_n(a)} \right| = \Pi \left( \frac{\xi + x_i}{a + x_i} \right) \leq \Pi \left( 1 + \frac{\xi - a}{a + x_i} \right) \leq \left( 1 + \frac{2(b^2 - a^2)}{abn^2} \right)^{(k+2)/2} \leq e^{(b^2-a^2)/4bn}.\]
By Inequality 1,

\[(5) \quad |\tau'(x)| \leq \frac{2b^2}{b-a} ||\tau||_{[a,b]} \leq \frac{2\left(\frac{n}{2} + 3\right)^2}{b-a} ||\tau||_{[a,b]}.
\]

Thus, by (3), (4) and (5),

\[|\phi^*(x)| \leq |\phi^*(x)| |\tau(x)| + |\tau'(x)| \frac{b}{a} |\phi(x)|
\]

\[\leq \frac{n + 2}{4a} ||\phi||_{[a,b]} + \frac{2\left(\frac{n}{2} + 3\right)^2}{b-a} ||\tau||_{[a,b]} \frac{b}{a} |\phi(x)|
\]

\[\leq \frac{n + 2}{4a} ||\phi||_{[a,b]} + \frac{2\left(\frac{n}{2} + 3\right)^2}{b-a} \frac{b}{a} ||\phi||_{[a,b]} |\phi(x)|
\]

\[= \left(\frac{b^2 - a^2}{3abn} + \frac{(b + a)}{2b}\left(1 + \frac{6}{n}\right)e^{\frac{6(b^2-a^2)}{3abn}}\right) \frac{n^2b}{b^2 - a^2} \frac{b}{a} ||\phi||_{[a,b]}.
\]

References


University of British Columbia,
Vancouver, British Columbia