ON APPROXIMATION BY
TRIGONOMETRIC LAGRANGE INTERPOLATING POLYNOMIALS II

P.B. Borwein, T.F. Xie and S.P. Zhou

We show that trigonometric Lagrange interpolating approximation with arbitrary
real distinct nodes in $L^p$ space for $1 \leq p < \infty$, as that with equally spaced nodes
in $L^p$ space for $1 < p < \infty$ in an earlier paper by T.F. Xie and S.P. Zhou, may
also be arbitrarily "bad". This paper is a continuation of this earlier work by Xie
and Zhou, but uses a different method.

Let $L^p_2\pi$, $1 \leq p \leq \infty$ be the class of real integrable functions of power $p$ and of
period $2\pi$ and let $L^\infty_2\pi = C_{2\pi}$ the class of all real continuous functions of period $2\pi$.

For $f \in L^1_2\pi$, $S_n(f, x)$ is the $n$th partial sum of the Fourier series of $f(x)$; for
$f \in L^p_2\pi$, $E_n(f)_p$ is the $n$th best approximation of $f(x)$ in $L^p$; for $f \in C_{2\pi}$, $L^\infty_n(f, x)$
is the $n$th trigonometric Lagrange interpolating polynomial of $f(x)$ with distinct nodes
$X_n = \{x_{n,j}\}_{j=0}^{2n}$ (by $a \neq b$ we mean that $a \neq b (\text{mod } 2\pi)$). In particular,

$$L_n(f, x) = \sum_{k=0}^{2n} f(x_k) l_k(x)$$

is the $n$th trigonometric Lagrange interpolating polynomial of $f(x)$ with equally spaced
nodes, where

$$l_k(x) = \frac{1}{2n+1} \frac{\sin (n+1/2)(x - x_k)}{\sin (x - x_k)/2}$$

$$x_k = \frac{2k\pi}{2n+1}, \quad k = 0, 1, \ldots, 2n.$$

The norm of $f \in L^p_2\pi$ is defined as follows.

$$\|f\|_{L^p} = \left( \int_0^{2\pi} |f(x)|^p dx \right)^{1/p}, \quad 1 \leq p < \infty,$$

$$\|f\| = \|f\|_{L^\infty} = \max_{0 \leq x \leq 2\pi} |f(x)|.$$
Although

\[ \|L_n\| = \sup \{ \|L_nf\| : \|f\| = 1 \} \sim \|S_n\| \sim \log(n+1), \]

(whereby \(A_n \sim B_n\) we indicate that there exists a positive constant \(M\) independent of \(n\) such that \(M^{-1} \leq A_n/B_n \leq M\)) the story for the behaviour of these two linear operators in \(L^p\) space is different. Throughout the paper, \(C(x)\) always indicates a positive constant depending upon \(x\) and \(C\) indicates a positive absolute constant, which may have different values at different places. For Fourier partial sums, by applying the well-known Riesz theorem (see, for example, Zygmund [4]) one has

\[ \|f - S_n(f)\|_{L^p} \leq C(p)E_n(f)_p, \quad 1 < p < \infty, \]

while for Lagrange interpolation with equally spaced nodes, the work [3] proved that there exists an infinitely differentiable function \(f \in C_{2*}\) such that

\[ \limsup_{n \to \infty} \frac{\|f - L_n(f)\|_{L^p}}{\lambda_n^{-1}E_n(f)_p} > 0, \quad 1 < p < \infty, \]

where \(\{\lambda_n\}\) is any given positive decreasing sequence with

\[ n^s\lambda_n \to 0 \]

for any \(s > 0\).

One might ask what happens in \(L^1\) space? Though in many cases \(L^1\) possesses similar properties to \(L^\infty\) by duality, it appears not to happen in this case. Furthermore, what happens for Lagrange interpolation with arbitrary real distinct nodes in \(L^p\) space for \(1 \leq p < \infty\)? Since the constructive method used in [3] is no longer valid in these cases, the present paper will use a different idea to construct the required counterexample.

**Theorem.** Let \(1 \leq p < \infty\). Suppose that \(\{X_n\}\) is a given sequence of real distinct nodes and \(\{\lambda_n\}\) is any given positive decreasing sequence. Then there exists an infinitely differentiable function \(f \in C_{2*}\) such that

\[ \limsup_{n \to \infty} \frac{\|f - L_n^X(f)\|_{L^p}}{\lambda_n^{-1}E_n(f)_p} > 0. \]

**Corollary.** Let \(1 \leq p < \infty\). Suppose that \(\{X_n\}\) is a given sequence of real distinct nodes and \(\{\lambda_n\}\) is any given positive decreasing sequence. Then there exists an infinitely differentiable function \(f \in C_{2*}\) such that

\[ \limsup_{n \to \infty} \frac{\|f - S_n(f)\|_{L^p}}{\lambda_n^{-1}E_n(f)_p} > 0. \]
Lemma 1. Let $1 \leq p < \infty$. Suppose that $X_n = \{x_{n,j}\}_{j=0}^{2n}$ is a sequence of real distinct nodes and $N_n$ is a natural number. Then there exists a function $h_n \in C_{2\pi}$ such that

$$h_n(x_{n,0}) = 0,$$

$$1 \leq h_n(x_{n,j}) \leq \|h_n\| \leq 2n, \quad j = 1, 2, \ldots, 2n,$$

and

$$\|h_n\|_{L^p} \leq C_n N_n^{-3/p}.$$

Proof: Because of the period $2\pi$, without loss of generality we can assume that

$$0 = x_{n,0} < x_{n,1} < x_{n,2} < \cdots < x_{n,2n} < 2\pi.$$

Let

$$N_{n,j}^* = \frac{2\pi - x_{n,j}}{x_{n,j}} N_n$$

for $1 \leq j \leq 2n$. Then it is clear that $z^{N_n(2\pi - z)} N_{n,j}^*$ has a maximum point $x_{n,j}$. Write

$$h_n := z_{n,j}^{N_n(2\pi - z)} N_{n,j}^*,$$

set

$$h_n(z) = \sum_{k=1}^{2n} \rho_{n,k} z^{N_n(2\pi - z)} N_{n,k}^*$$

for $z \in [0, 2\pi)$, and extend it to the whole line with period $2\pi$. Evidently, $h_n \in C_{2\pi}$ and

$$h_n(0) = h_n(2\pi) = 0.$$

We clearly have

$$h_n(x_{n,j}) \geq \rho_{n,j}^{-1} z_{n,j}^{N_n(2\pi - x_{n,j})} N_{n,j}^* = 1$$

for $1 \leq j \leq 2n$. At the same time,

$$h_n(x_{n,j}) \leq \|h_n\| \leq \sum_{k=1}^{2n} \rho_{n,k}^{-1} z_{n,j}^{N_n(2\pi - x_{n,j})} N_{n,k}^* = 2n.$$

On the other hand, a calculation yields

$$\|z^{N_n(2\pi - z)} N_{n,j}^*\|_{L^p} = (2\pi)^{N_n + N_{n,j}^* + 1/p} \left( \frac{\Gamma(N_n p + 1) \Gamma(N_{n,j}^* p + 1)}{\Gamma(N_n p + N_{n,j}^* p + 2)} \right)^{1/p} \leq C \rho_{n,j} N_n^{-p/2},$$

so

$$\|h_n\|_{L^p} \leq C_n N_n^{-p/2}.$$
The proof of Lemma 1 is thus completed.

**Lemma 2.** Let \( 1 \leq p < \infty \). Suppose that \( X_n = \{x_{n,j}\}_{j=0}^{2n} \) is a sequence of real distinct nodes and that \( \{\lambda_n\} \) is a given positive decreasing sequence. Then there exists a trigonometric polynomial \( g_n(x) \) of degree \( M_n \) such that for large enough \( n \),

\[
\|g_n\| = O(n\delta_n^{-1}),
\|g_n - S_n(g_n)\|_{L^p} = O(\lambda_n),
\]

and

\[
\|g_n - L_n^X(g_n)\|_{L^p} \geq C,
\]

where

\[
\delta_n = 2^{-2n/p} \prod_{0 \leq i \neq j \leq 2n} \left\| \sin \frac{x_{n,i} - x_{n,j}}{2} \right\|^{1/p}.
\]

**Proof:** Let \( h_n(x) \) be the function defined in Lemma 1. We first establish

(3) \[
\|h_n - S_n(h_n)\|_{L^p} = O \left( n \log (n + 1)N_n^{-2/p} \right),
\]

and

(4) \[
\|h_n - L_n^X(h_n)\|_{L^p} \geq C2^{-2n/p} \eta_n^{1/p} - CnN_n^{-2/p},
\]

where

\[
\eta_n = \prod_{0 \leq i \neq j \leq 2n} \left\| \sin \frac{x_{n,i} - x_{n,j}}{2} \right\|.
\]

Inequality (3) is straightforward: we just need to apply (2) and the estimation of the Lebesgue constant. Now write

\[
L_n^X(h_n, x) = \sum_{j=0}^{2n} h_n(x_{n,j}) I_j^X(x),
\]

where

\[
I_j^X(x) = \frac{\prod_{k \neq j} \sin \frac{x - x_{n,k}}{2}}{\prod_{k \neq j} \sin \frac{x_{n,j} - x_{n,k}}{2}}.
\]

Since

\[
\sin \frac{x - x_{n,k}}{2} = \sin \frac{x_{n,j} - x_{n,k}}{2} \cos \frac{x - x_{n,j}}{2} + \cos \frac{x_{n,j} - x_{n,k}}{2} \sin \frac{x - x_{n,j}}{2},
\]

for \( x \in [x_{n,j} - n^{-1}2^{-1} \eta_n, x_{n,j} + n^{-1}2^{-1} \eta_n] \), we have

(5) \[
I_j^X(x) = 1 + O(n^{-1}).
\]
Meanwhile, for \( x \in [x_{n,j} - n^{-1}2^{-2n}\eta_n, x_{n,j} + n^{-1}2^{-2n}\eta_n] \) and \( i \neq j \),

\[
|l_i^X(x)| = \left| \prod_{k \neq i} \sin \frac{x - x_{n,k}}{2} \right| \left| x - x_{n,j} \right| \left| \frac{d}{dz} \left( \prod_{k \neq i} \sin \frac{x - x_{n,k}}{2} \right) \right| \leq \frac{1}{\eta_n} \leq 2^{-2n}.
\]

Combining (5), (6) and (1), for sufficiently large \( n \) we get

\[
\|h_n - L_n^X(h_n)\|_{L^p} \geq \left( \sum_{j=1}^{2^n} \int_{s_{n,j} - n^{-1}2^{-2n}\eta_n}^{s_{n,j} + n^{-1}2^{-2n}\eta_n} \left\| \sum_{k=0}^{2^n} h_n(z_{n,j}) l_k^X(z) \right\|^p \right)^{1/p} - \|h_n\|_{L^p} - CnN_n^{-1/(2p)}
\]

\[
\geq C2^{-2n/p}\eta_n^{1/p} - CnN_n^{-1/(2p)},
\]

that is, (4).

Without loss of generality suppose that \( \lambda_n \leq 1 \). Now choose

\[ N_n = \left[ n^{2p}\log^{2p}(n + 1)2^{4n}\eta_n^{-2}\lambda_n^{2p} + 1 \right]; \]

then (3), (4) become

\[ \|h_n - S_n(h_n)\|_{L^p} = O(\delta_n \lambda_n), \]

and

\[ \|h_n - L_n^X(h_n)\|_{L^p} \geq C\delta_n, \]

where

\[ \delta_n = 2^{-2n/p}\eta_n^{1/p}. \]

Because \( h_n \in C_{2n} \), we may select a trigonometric polynomial \( g_n^* \) with sufficiently large degree \( M_n \geq n \) such that

\[ \|h_n - g_n^*\| \leq \delta_n^2\lambda_n \min\left( \log^{-1}(n + 1), \left( \|L_n^X\| + 1 \right)^{-1} \right). \]

Hence by (7) and (9),

\[
\|g_n^* - S_n(g_n^*)\|_{L^p} \leq \|g_n^* - h_n\| + \|S_n(h_n) - S_n(g_n^*)\| + \|h_n - S_n(h_n)\|_{L^p}
\]

\[
\leq \delta_n^2\lambda_n \log^{-1}(n + 1)(1 + \|S_n\|) + C\delta_n \lambda_n
\]

\[ \leq C\delta_n \lambda_n. \]
Similarly, from (8) and (9),
\[ \|g_n^* - L_n^X(g_n^*)\|_{L^p} \geq \|h_n - L_n^X(h_n)\|_{L^p} - \|g_n^* - h_n\| - \|L_n^X(h_n) - L_n^X(g_n^*)\| \]
\[ \geq C\delta_n - \delta_n^2 \lambda_n (\|L_n^X\| + 1)^{-1} (1 + \|L_n^X\|) \]
\[ \geq C\delta_n \]
for large enough \( n \). Set
\[ g_n(x) = \delta_n^{-1} g_n^*(x); \]
then from the above discussion we get the required inequality.

PROOF OF THE THEOREM: Select a sequence \( \{n_j\} \) inductively: Let \( n_1 = 1 \). After \( n_j \), choose
\[ n_{j+1} = \left[ m_n^2 \lambda_{n_j}^{1/n_j} \left( \|L_{n_j}^X\| + \log n_j \right) + 1 \right] \]
where
\[ m_n = M_n \left( n^2 \delta_n^{-2/n} + 1 \right). \]
Define
\[ f(x) = \sum_{j=1}^{\infty} m_{n_j}^{-n_j} g_{n_j}(x). \]
Clearly \( f(x) \in C_2 \) is infinitely differentiable since \( g_{n_j}(x) \) is a trigonometric polynomial of degree \( m_{n_j} \) and \( \|g_n\| = O(n\delta_n^{-1}) \). Together with (10), Lemma 2 implies that
\[ \|f - L_{n_j}^X(f)\|_{L^p} \geq m_{n_j}^{-n_j} \|g_{n_j} - L_{n_j}^X(g_{n_j})\|_{L^p} - C \left( \|L_{n_j}^X\| + 1 \right) \sum_{k=j+1}^{\infty} m_{n_k}^{-n_k} \|g_{n_k}\| \]
\[ \geq Cm_{n_j}^{-n_j} - Cm_{n_{j+1}}^{-n_{j+1}/2} \lambda_{n_j} \geq Cm_{n_j}^{-n_j}. \]
At same time, by (10) and Lemma 2 again,
\[ \|f - S_{n_j}(f)\|_{L^p} = O \left( m_{n_j}^{-n_j} \|g_{n_j} - S_{n_j}(g_{n_j})\|_{L^p} + \|S_{n_j}\| + 1 \right) \sum_{k=j+1}^{\infty} m_{n_k}^{-n_k} \|g_{n_k}\| \]
\[ = O \left( m_{n_j}^{-n_j} \lambda_{n_j} + m_{n_{j+1}}^{-n_{j+1}/2} \right) = O \left( m_{n_j}^{-n_j} \lambda_{n_j} \right). \]
Altogether,
\[ \frac{\|f - L_{n_j}^X(f)\|_{L^p}}{\lambda_{n_j}^2 \|f - S_{n_j}(f)\|_{L^p}} \geq C > 0, \]
which is the required result.

REMARK. In spite of the counterexample in the present paper, there are several positive results in this direction. For example, [1, 2] discuss the rate of convergence of \( L_n(f, x) \) to \( f(x) \) in \( L^p \), in terms of the sequence of best approximation of the function in \( L^p \).
REFERENCES

[1] V. P. Motornyi, 'Approximation of periodic functions by interpolation polynomials in $L_1$', 

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Department of Mathematics, Statistics and Computing Science
Dalhousie University
Halifax NS
Canada B3H 3J5

Department of Mathematics
Hangzhou University
Hangzhou Zhejiang
China 310028

Department of Mathematics, Statistics and Computing Science
Dalhousie University
Halifax NS
Canada B3H 3J5