EXACT INEQUALITIES FOR THE NORMS OF FACTORS OF POLYNOMIALS

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ABSTRACT. This paper addresses a number of questions concerning the size of factors of polynomials. Let \( p \) be a monic algebraic polynomial of degree \( n \) and suppose \( q_1 q_2 \cdots q_k \) is a monic factor of \( p \) of degree \( m \). Then we can, in many cases, exactly determine

\[
\max \left\{ \frac{\|q_1\| \|q_2\| \cdots \|q_k\|}{\|p\|} \right\}.
\]

Here \( \| \cdot \| \) is the supremum norm either on \([-1,1]\) or on \( \{|z| \leq 1\} \). We do this by showing that, in the interval case, for each \( m \) and \( n \), the \( n \)-th Chebyshev polynomial is extremal. This extends work of Gel'fond, Mahler, Granville, Boyd and others. A number of variants of this problem are also considered.

1. Introduction. How large can the norms of factors of a polynomial be? Variations of this problem have attracted considerable attention over the years (see [2]–[14]). We exactly solve this problem in the following forms:

Suppose \( p = q \cdot r \) where \( p, q \) and \( r \) are polynomials of degree \( n, m \) and \( n - m \) respectively. Then for all \( m \) and \( n \)

\[
\|q\|_{[-1,1]} \|r\|_{[-1,1]} \leq K_{m,n} \|p\|_{[-1,1]}
\]

where

\[
K_{m,n} := 2^{n-1} \prod_{k=1}^{m} \left( 1 + \cos \frac{2k - 1}{2n} \pi \right) \prod_{k=1}^{n-m} \left( 1 + \cos \frac{2k - 1}{2n} \pi \right)
\]

and this bound is exactly attained by the Chebyshev polynomial of degree \( n \). (We denote by \( \| \cdot \|_{[-1,1]} \) the sup norm on the interval \([-1,1]\). This is the content of Theorem 1. (Theorem 1 is originally due to Kneser [10]; see also Aumann [2]. We offer a new proof of this result that easily modifies to the other cases we wish to consider.)

Suppose now that \( q \) is a factor of \( p \) of degree \( m \) and suppose that \( p \) and \( q \) are both monic. Then for each \( n \) and \( m \),

\[
|q(-2)| \leq 2^{n-1} \prod_{k=1}^{m} \left( 1 + \cos \frac{2k - 1}{2n} \pi \right) \|p\|_{[-2,2]}.
\]

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This bound is attained by the Chebyshev polynomial of degree \( n \) on \([-2, 2]\) with \( q \) being the factor composed of the \( m \) roots closest to 2. This is the content of Theorem 2.

We then generalize these results to the many factors case. We prove in, Theorem 4, that

\[
\|q_1\|_{[-1,1]} \cdot \|q_2\|_{[-1,1]} \cdots \cdot \|q_j\|_{[-1,1]} \leq 2^{m-n}(3, 20991 \cdots )^m \|p\|_{[-1,1]}
\]

where \( q_1 \cdots q_j \) is a monic factor of degree \( m \) of a monic polynomial \( p \) of degree \( n \). This result is sharp for \( j \geq 2 \). A version of this on the disc is given in Theorem 8.

If

\[
x = \frac{1}{2}(z + \frac{1}{z})
\]

then

\[
p(x) = \ell(z)r(\frac{1}{z})
\]

is a mapping between polynomials of degree \( n \) on the interval \([-1, 1]\) and self reciprocal polynomials, of the above form, of degree \( 2n \) on the boundary of the unit disc. So there is some equivalence between factoring self reciprocal polynomials on the disc and factoring on the interval.

From this and Theorem 1 we deduce that if \( p = q \cdot r \) are real polynomials of degree \( n \), \( m \) and \( n - m \) respectively then

\[
\|q\|_D \cdot \|r\|_D \leq (K_{m,n})^{1/2} \|p\|_D
\]

and that this bound is exactly obtained, for even \( n \) and \( m \), by \( z^n + 1 \). (Here \( \| \cdot \|_D \) is the sup norm on the complex unit disc \( D \).) This is done in Theorem 6.

Boyd [5], using a Mahler measure argument, gives a very pretty proof that

\[
\|q\|_D \cdot \|r\|_D \leq c^n \|p\|_D
\]

with \( c = 1.7916 \cdots \) (Here \( \| \cdot \|_D \) is the sup norm on the complex unit disc). This is an asymptotically best possible result, and improves on earlier bounds of \( c = 4 \) due to Gel'fond [7] and \( c = 2 \) due to Mahler [11]. However, there is no dependence on the degrees of the factors.

This result is extended to the many factor case in Boyd [6], a result we reproduce by different methods as Theorem 7.

Related problems concerning the size of single factors (with some normalization conditions) are considered in [3], [5] and [14] and various other problems on norms of factors and products are treated in [3], [4], [12] and [13].

In particular in Boyd [5] it is shown that if \( p = q \cdot r \) with \( p, q \) and \( r \) all monic then

\[
\|q\|_D \leq d^n \|p\|_D
\]

where \( d = 1.3813 \cdots := M(1 + x + y) \) where \( M \) is the Mahler measure of a polynomial. This again is asymptotically best possible and improves on earlier results of Granville [9] and Glesses [8] who derived (5) with \( d := \frac{1 + \sqrt{5}}{2} \) and \( d := \frac{3}{2} \) respectively.
Mignotte [14] derives the inequality

\[ \|q\|_0 \leq 2^n \|p\|_0 \]

(recall that \( m \) is the degree of \( q \)) and shows that this is asymptotically sharp for some \( n \) roughly of size \( m^2 \log m \).

A variant of these last results, which is sharper for certain polynomials, such as those with integer coefficients, is also given in Section 5. This is Theorem 5.

2. The two factor case. Throughout this note we will use the notation \( \| \| := \|\|_{[-1,1]} \) to denote the supremum norm on \([-1,1]\) and more generally \( \| \|_Y \) to denote the supremum norm on \( Y \). In this section and the next section the polynomials in question are algebraic polynomials with possibly complex coefficients. We denote by \( \pi_n \) the set of such polynomials of degree at most \( n \). We begin by offering a new proof of an old result.

**Theorem 1.** Let \( p \) be any polynomial of degree \( n \) and suppose \( p = q \cdot r \) where \( q \) is a factor of degree \( m \) and \( r \) is a factor of degree \( n - m \). Then

\[ \|q\|_{[-1,1]} \|r\|_{[-1,1]} \leq \frac{1}{2} C_{n,m} C_{n,n-m} \|p\|_{[-1,1]} \]

where

\[ C_{n,m} := 2^m \prod_{k=1}^{m} \left( 1 + \cos \frac{2k-1}{2n} \pi \right). \]

Furthermore, for any \( n \) and \( m \leq n \) the inequality is sharp in the case that \( p \) is the Chebyshev polynomial of degree \( n \). In this case the factor \( q \) is chosen to be the factor with \( m \) roots closest to \(-1\).

On applying the midpoint rule to \( \log(2 + 2 \cos x) \), we see that

\[ \frac{1}{n} \int_0^{m/n} \log(2 + 2 \cos \pi x) = \sum_{k=1}^{m} \log \left( 2 + 2 \cos \frac{2k-1}{2n} \pi \right) + O \left( \frac{1}{(n-m)^2} \right). \]

Thus

\[ \frac{1}{n} \int_0^{m/n} \log(2 + 2 \cos x) dx = \frac{\pi^2}{12} \cdot \frac{1}{(n-m)^2}. \]

Recall that the error in the midpoint rule is \( \mathcal{O}(\pi^2/n^2) \) and in this case \( \mathcal{O}(\pi^2/n^2) \). Thus

\[ C_{n,m} = e^{\log \prod_{k=1}^{m} \left( 1 + \cos \frac{2k-1}{2n} \pi \right)} = \left( e^{\frac{1}{n} \sum_{k=1}^{m} \log(2 + 2 \cos \frac{2k-1}{2n} \pi)} \right)^n \]

\[ = e^{0(n/(n-m)^2)} = (\mathcal{I}(n,m))^n \]

where

\[ \mathcal{I}(n,m) := \int_0^{m/n} \log(2 + 2 \cos x) dx. \]

Note that

\[ C_{n,n/2}^{1/n} \approx (e^{\int_0^{m/n} \log(2 + 2 \cos x) dx}) = (1.7916 \cdots) \]
and that

$$C_{n,\lfloor 2n/3 \rfloor}^{1/n} \sim \left( e^{\pi^2/12 \log(2 + 2\cos \pi x)} dx \right) = (1.9081 \cdots).$$

(The first constant appears in Boyd [5] as does the square root of the second constant. We would expect this from the transformation (4). Similar asymptotics are in [10].) From (7) and Theorem 1 we have

**COROLLARY 1.** Let \( p \) be any polynomial of degree \( n \) and suppose \( p = q \cdot r \) with \( q \) and \( r \) polynomials then

$$\|q\|_{[-1,1]}\|r\|_{[-1,1]} \leq 2^{n-1} \prod_{k=1}^{[n/2]} \left( 1 + \cos \frac{2k - 1}{2n} \pi \right)^2 \|p\|_{[-1,1]} \leq (3.20991 \cdots) \|p\|_{[-1,1]}$$

(with equality for the Chebyshev polynomial of degree \( n \).)

Here and throughout we use \( a_n \leq b_n \) for positive sequences to mean \( \limsup \frac{a_n}{b_n} \leq 1 \). The proof of the theorem proceeds through a number of lemmas. For the remainder of the proof we assume \( n \) and \( m < n \) fixed. Let \( p \) be of degree \( n \) and suppose \( p = q \cdot r \) where \( q \) has degree \( m \) and \( r \) has degree \( n - m \). It is an easy consequence of the finite dimensionality of the set of polynomials of degree \( n \) that

$$(*) \quad \sup \left\{ \frac{\|q\|\|r\|}{\|p\|} : p = qr, \ p \in \pi_n, \ q \in \pi_m, \ r \in \pi_{n-m} \right\}$$

is attained for some \( p, q, r \). We proceed to show that such an extremal \( p^* \) is in fact the Chebyshev polynomial of degree of \( n \) (and that the factors \( q \) and \( r \) are as advertised).

That is

$$p^*(x) = T_n(x) = \cos n \cos^{-1}(x) = \frac{1}{2^n} \prod_{k=1}^{n} 2 \left( x - \cos \frac{2k - 1}{2n} \pi \right)$$

and the extremal factors, \( q^* \) and \( r^* \), are

$$\frac{1}{\sqrt{2}} \prod_{k=1}^{m} 2 \left( x - \cos \frac{2k - 1}{2n} \pi \right) \quad \text{and} \quad \frac{1}{\sqrt{2}} \prod_{k=m+1}^{n} 2 \left( x - \cos \frac{2k - 1}{2n} \pi \right)$$

respectively. Note that

$$\|q^*\| = |q(-1)| = \frac{1}{\sqrt{2}} C_{n,m}$$

and

$$\|r^*\| = |r(1)| = \frac{1}{\sqrt{2}} C_{n,n-m}.$$ 

(These elementary properties of \( T_n \) are available in [1] or [15].)

Our first observation is that we may assume that \( |q| \) has a maximum at \(-1\) and \( |r| \) has a maximum at \( 1 \). This follows since the supremum in (*) is invariant under a change of
variables $x \mapsto ax + b$ and a corresponding change on the underlying interval. (That is, the sup in (*) is interval independent.) So if $||q|| = |q(\alpha)|$, $\alpha \in (-1, 1)$ and $||r|| = |r(\beta)|$, $\beta \in (-1, 1)$. Then

$$\sup \left\{ \frac{||q|| \cdot ||r||}{||p||} \right\} \geq \sup \left\{ \frac{||q|| \cdot ||r||}{||p||} \right\}$$

since $||p||_{[\alpha, \beta]} \leq ||p||$. But we can scale this back to the interval $[-1, 1]$ and get a new extremal of the desired type. (Note that $\alpha = \beta$ is a trivial case.)

So we now assume we have an extremal example where $q$ and $r$ attain their maxima at $-1$ and $+1$ respectively. (From this it follows that the $p$ in the sup in (*) has exact degree $n$.)

**Lemma 1.** Under the above assumption on $q$, $q$ and $r$ we may assume that $p$ has all real roots.

**Proof.** This argument (and most of the following arguments) is a perturbation argument. We show that a purely complex root may be perturbed a bit to construct an example that contradicts extremality.

So suppose $\rho$, with $\text{im}(\rho) > 0$, is a complex root of $q$ (as in the diagram)

If we rotate $\rho$ to $\rho'$ along a circle of radius $|\rho + 1|$ in a clockwise fashion. Then $|q(-1)|$ and $|r(1)|$ are unchanged and these are still maxima. However, for any $x \in (-1, 1]$ the perturbed $p(x)$ has smaller absolute value than the original $p(x)$. (This is equivalent to observing that if $\rho + 1 = re^{\theta}$ then $|\rho - x|^2 = r^2 - 2r(x + 1) \cos \theta + (x_1)^2$ is an increasing function of $\theta$ for $0 \leq \theta \leq \pi$.) This finishes the proof.

So we may add to our assumption on the extremal $p$ that $p$ has all real roots.

**Lemma 2.** Under the above assumptions on $q$, $q$ and $r$ we may assume that $p$ has all its roots in $[-1, 1]$.

**Proof.** We consider the case where $q$ has a root $\rho > 1$ and observe that a perturbation of $\rho$ to $\rho'$ where $1 \leq \rho' < \rho$ has the effect of leaving $r(1)$ unchanged while
\[ |q(-1)| / \|p\| \text{ does not increase. (And } |q(-1)| \text{ is still a maximum of } q \text{ on } [-1, 1]). \] This follows because, for fixed \( x_0 \in [-1, 1] \),
\[ \frac{\rho(-1)}{\rho - x_0} \]
is a decreasing function of \( \rho \) on \([1, \infty)\). (A similar argument holds for \( \rho \in [-\infty, 1) \) only now we perturb \( \rho \) to \( \rho' < \rho \leq -1 \).

It is now clear that the extremal \( q \) and \( r \) are chosen by splitting the roots of \( p \) so that the \( m \) roots closest to 1 are the roots of \( q \) and the remaining \( n - m \) roots are the roots of \( r \). (Otherwise one would interchange two roots, and increase both \( |q(-1)| \) and \( |r(1)| \) without altering \( \|p\| \)).

We have now shown that we may assume our extremal example is of the form
\[ p(x) = a \prod_{k=1}^{m-n} (x - \alpha_k) \prod_{k=1}^{m} (x - \beta_k) \]
\[ = ar(x)q(x) \]
where
\[-1 \leq \alpha_1 \leq \cdots \leq \alpha_{n-m} \leq \beta_1 \leq \cdots \leq \beta_m \leq 1. \]

Furthermore \( |r(x)| \) has a maximum at +1 and \( |q(x)| \) has a maximum at -1. We now also, for convenience, assume \( \|p\| = 1 \). We are ready to prove Theorem 1.

**Proof of Theorem 1.** We show three things

1) \( |p(-1)| = 1 \) and \( |p(1)| = 1 \),

2) \( |p(x)| = 1 \) for some \( x \in (\alpha_i, \alpha_{i+1}) i = 1, \ldots, n - m - 1 \)

3) \( |p(x)| = 1 \) for some \( x \in (\beta_i, \beta_{i+1}) i = 1, \ldots, m - 1 \).

These three facts show that \( p \) is indeed the Chebyshev polynomial \( \pm T_n \) (since \( T_n \) is the unique maximally oscillatory polynomial on \([-1,1]\)).

To prove 1) we observe that if \( |p(-1)| \neq 1 \) then we would increase the interval on which we consider the problem to \([-1+\epsilon, 1]\) without increasing the norm of \( p \). However since \( r \) is monotone on \([-\infty, \alpha_1]\) we have (strictly) increased the norm of \( r \) and have violated extremality. (A similar argument applies to \( p(1) \).)

To prove 2) we suppose \( \alpha_i \) and \( \alpha_{i+1} \) are adjacent roots and let
\[ s(x) = (x - \alpha_i)(x - \alpha_{i+1}). \]
Let \( \epsilon > 0 \), we can find \( 0 < \delta_1, \delta_2 < \epsilon \) so that
\[ t(x) = (x - (\alpha_i - \delta_1))(x - (\alpha_{i+1} + \delta_2)) \]
satisfies
\[ \|t(x) - s(x)\| < \epsilon \]
\[ |t(-1)| = |s(-1)| \]
and

\[ |r(x)| < |s(x)| \quad x \in (-1, \alpha_i - \delta_i) \cup [\alpha_i + \delta_i, 1). \]

Now if \( |p|_{[\alpha_n, \omega_n]} \neq 1 \) we have, for sufficiently small \( \varepsilon \) that

\[ p^\ast(x) := p(x) \cdot r(x)/s(x) \]

is a new extremal and is extremal on an interval \([-1, 1 + \delta]\) (since \( |p^\ast(1)| < 1 \)). Now if \( p^\ast \) is scaled back to the interval \([-1, 1]\) we violate the extremality of \( p \) as in the first part of this proof. (The argument for \((\beta, \beta_n)\) is identical.)

To prove 3) we consider the effect of moving \( \beta \) slightly to the right and \( \omega_n \) an equal amount to the left. If \( |p|_{[\alpha_n, \omega_n]} \neq 1 \) this increases both \( |r(1)| \) and \( |q(1)| \) but does not increase \( |p|_{[-1,1]} \). This finishes the proof.

3. The single factor case.

**Theorem 2.** Suppose that \( p \) is a monic polynomial of degree \( n \) and that \( q \) is a monic factor of \( p \) of degree \( m \). Then

\[ |q(\beta)| \leq |p|_{[-\beta, \beta]} \left( 2^{m-n} \frac{2^{n-1}}{2n-\pi} \right)^m \prod_{k=1}^{m} \left( 1 + \cos \frac{2k-1}{2n} \pi \right). \]

This bound is attained for each \( n \) and \( m \leq n \) by the Chebyshev polynomial of degree \( n \) on \([-\beta, \beta]\) (normalized to be monic). In this case \( q \) is chosen to be the factor with \( m \) roots closest to \( \beta \).

**Corollary 2.** Suppose \( p \) is a monic polynomial of degree \( n \) and \( q \) is a monic factor of \( p \) of degree \( m \). Then

\[ |q(-2)| \leq |p|_{[-2,2]} \left( 2^{m-1} \prod_{k=1}^{m} \left( 1 + \cos \frac{2k-1}{2n} \pi \right) \right) = \frac{1}{2} C_{n,m} |p|_{[-2,2]} \]

and the inequality is sharp for all \( m \) and \( n \). Furthermore for all \( m \leq n \),

\[ C_{n,m}^{1/n} \lesssim C_{n,\lfloor 2n/3 \rfloor}^{1/n} \sim (1.9081 \ldots). \]

**Proof of Corollary 2.** Take \( \beta = 2 \) in Theorem 2. Note that

\[ 2^m \prod_{k=1}^{m} \left( 1 + \cos \frac{2k-1}{2n} \pi \right) \leq 2^{m+\gamma(n)} \prod_{k=1}^{\gamma(n)} \left( 1 + \cos \frac{2k-1}{2n} \pi \right) \]

where \( \gamma(n) + 1 (\sim \frac{2n}{3}) \) is the smallest positive integer \( k \) with \( \cos \frac{2k-1}{2n} \pi \leq -1/2 \).

So

\[ C_{n,m}^{1/n} \lesssim C_{n,\lfloor 2n/3 \rfloor}^{1/n} \]

and by (8)

\[ C_{n,m}^{1/n} \lesssim (1.9081 \ldots). \]
This is the real analogue of the complex results cited in the introduction. Since if we wish to preserve monicity in the mapping (4) we must use the transformation $x = (z + \frac{1}{2})$ which maps the boundary of the disc onto $[-2, 2]$ (not $[-1, 1]$).

**Sketch of Proof of Theorem 2.** The proof of Theorem 2 is very similar to that of Theorem 1 and we only outline it.

Fix $m$ and $n$. We consider the maximization problem

$$
\max \{ |q(-\beta)| / \|p\|_{[-\beta, \beta]} : q \text{ divides } p, q \in \pi_m, p \in \pi_n, q \text{ and } p \text{ both monic} \}.
$$

We can argue now, exactly as in Lemma 1, that $p$ has all real roots and that these all lie in $[-\beta, \infty)$ since any rotation of roots as in Lemma 1 at least maintains extremality. Arguing as in Lemma 2 gives that $p$ has all roots in $[-1, 1]$. Thus $q$ must be composed of the $m$ roots of $p$ closest to 1.

The argument of the proof of Theorem 1 now applies essentially verbatim (note that all the perturbations preserve monotonicity) and proves that an extremal $p$ can be chosen to be the Chebyshev polynomial on $[-\beta, \beta]$ normalized to be monic. Thus on $[-1, 1]$,

$$p(x) = \prod_{k=1}^{n} \left(x - \cos \frac{2k - 1}{2n} \pi \right)$$

and

$$q(x) = \prod_{k=1}^{m} \left(x - \cos \frac{2k - 1}{2n} \pi \right)$$

from which the result follows (on considering $\beta^n p(x/\beta)$ and $\beta^m q(x/\beta)$ on $[-\beta, \beta]$). \[\blacklozenge\]

4. The many factor case.

**Theorem 3.** Suppose that $p$ is a monic polynomial of degree $n$ and that $q$ and $r$ are monic factors of $p$ of degree $m_1$ and $m_2$ respectively. (Suppose further that $qr$ is also a factor of $p$.) Then

$$|q(-\beta)| / \|r(\beta)||p(-\beta, \beta)| \leq \|p\|_{[-\beta, \beta]} \prod_{k=1}^{m_1} \left(1 + \cos \frac{2k - 1}{2n} \pi \right) \prod_{k=1}^{m_2} \left(1 + \cos \frac{2k - 1}{2n} \pi \right).$$

This bound is attained for the Chebyshev polynomial of degree $n$ on $[-\beta, \beta]$ normalized to be monic.

The proof of Theorem 3 is analogous to the proofs of Theorem 1 or 2 on considering the maximization problem

$$\max \{ |q(-\beta)| / \|p\|_{[-\beta, \beta]} : p, q, r \text{ as above} \}.
$$

**Theorem 4.** Suppose that $p$ is monic polynomial of degree $n$ and suppose that $p$ has a monic factor of degree $m$ of the form $q_1q_2 \cdots q_j$ where $q_j$ is of degree $d_j$ (so $m = d_1 + \cdots + d_j$). Then

$$\|q_1\|_{[-\beta, \beta]} \|q_2\|_{[-\beta, \beta]} \cdots \|q_j\|_{[-\beta, \beta]}$$

$$\leq \|p\|_{[-\beta, \beta]} \beta^{m-n} \prod_{k=1}^{m} \left(1 + \cos \frac{2k - 1}{2n} \pi \right) \prod_{k=1}^{n} \left(1 + \cos \frac{2k - 1}{2n} \pi \right)$$

$$\leq \|p\| (2\beta)^{m-n} (3.20991 \cdots)^n \quad (m, n \to \infty)$$
Norms of Factors of Polynomials

Since if we take $x = (z + \frac{1}{z})$

similar to that for monic.

that these all are extremality. (note that $\beta$ and $\beta$ can be chosen on $[-1, 1]$)

Thus

We now apply Theorem 3 to the two factors $r_1 \cdots r_j$ and $s_1 \cdots s_j$ to finish the proof.

Note that the $m = n$ case reproduces Theorem 2 of [6] in the real case.

5. Inequalities on the disc. We now derive the inequalities on the disc from those on the interval. Suppose $t \in \pi_n$ is monic and suppose $s \in \pi_m$ and $v \in \pi_{n-m}$ are monic factors of $t$. (So $t = s \cdot v$.) Of course all of $t$, $s$ and $v$ achieve their maxima on $D$ somewhere on the boundary. Now consider

\[ p(x) := t(z)\bar{t}(1/z) \]

and

\[ q(x) := s(z)s(1/z) \quad r(x) := v(z)v(1/z) \]

with

\[ x := (z + 1/z). \]

The effect of this transformation on linear factors is as follows

\[ (z - \alpha)(\frac{1}{z} - \alpha) = -\alpha(z - \frac{1 + \alpha^2}{\alpha}) \]

so $\alpha$ is of degree $n$, $\alpha$ is of degree $m$ and $\alpha$ is of degree $n - m$. Also

\[ ||p||_{[-2,2]} \leq ||\alpha||_{\overline{D}}^2 \]

and for $\alpha$ on the boundary of the unit circle

\[ |q(\alpha + \bar{\alpha}^{-1})| = |s(\alpha)s(\alpha^{-1})|. \]

Furthermore the leading coefficient of $p$ is essentially just the product of the roots of $t$ which is of size $|t(0)|$ while the leading coefficient of $q$ is of size $|s(0)|$ and the leading coefficient of $r$ is of size $|r(0)|$.

From these transformations and the interval inequalities we can prove the following theorems.
THEOREM 5. Suppose \( t = s \cdot v \) is a monic polynomial of degree \( n \) and \( s \) is a monic factor \( t \) of degree \( m \). Then
\[
|v(0)|^{1/2} \|s\|_D \leq \left( \frac{1}{2} C_{n,m} \right)^{1/2} \|t\|_D.
\]
This is sharp, for even \( n \) and \( m \), for \( t(z) = z^n + 1 \).

PROOF. We can assume, by performing an initial rotation if necessary, that
\[
\|s\|_D = |s(-1)|.
\]
So from Corollary 2 we deduce that
\[
\|s\|_D^2 = |s(-1)|^2 = |q(-2)| \leq |s(0)/t(0)| 2^{m-1} \prod_{k=1}^{m} \left( 1 + \cos \frac{2k - 1}{2n} \pi \right) \|p\|_{-2,2} \\\quad (9)
\]
\[
\leq \|s(0)/t(0)| 2^{m-1} \prod_{k=1}^{m} \left( 1 + \cos \frac{2k - 1}{2n} \pi \right) \|t\|_D^2.
\]
Now \( s(0)/t(0) = 1/\nu(0) \).

THEOREM 6. Suppose \( t = s \cdot v \) where \( s \in \pi_m \) and \( v \in \pi_{n-m} \) and where \( t, s \) and \( v \) are real polynomials. Then
\[
\|s\|_D \|v\|_D \leq \left( \frac{1}{2} C_{n,m} \cdot C_{n,n-m} \right)^{1/2} \|t\|_D.
\]
As before,
\[
C_{n,m} := 2^m \prod_{k=1}^{m} \left( 1 + \cos \frac{2k - 1}{2n} \pi \right)
\]
and
\[
(C_{n,m} \cdot C_{n,n-m})^{1/2} \lesssim (1.79162 \ldots).
\]
This is sharp, for even \( n \) and \( m \), for \( t(z) = z^n + 1 \).

PROOF. From Theorem 1 we deduce, for \( a \) and \( b \) on the boundary of the unit disc and for arbitrary \( s \) and \( v \), that
\[
|s(a) \cdot s(a^{-1})| |v(b) \cdot v(b^{-1})| = |q(a + a^{-1})| |r(b + b^{-1})|
\]
\[
\leq \frac{1}{2} C_{n,m} \cdot C_{n,n-m} \|p\|_{|a=1/a,b=1/b|} \|t\|_D \|s\|_D \|v\|_D.
\]
\[
\leq \frac{1}{2} C_{n,m} \cdot C_{n,n-m} \|p\|_{-2,2} \|t\|_D \|s\|_D \|v\|_D.
\]
The result now follows on choosing \( a \) and \( b \) to be points on \( \{ |z| = 1 \} \) where \( s \) and \( v \) achieve their norms.

The next result reproduces Theorem 2 of Boyd [6].
THEOREM 7. Suppose \( t = s_1 s_2 \cdots s_j \) are real polynomials, with \( t \) of degree \( n \), then

\[
\|s_1\|_D \|s_2\|_D \cdots \|s_j\|_D \leq \frac{1}{\sqrt{2}} \left( \prod_{k=1}^{j} 2 \left( 1 + \cos \frac{2k - 1}{2n} \pi \right) \right) \|t\|_D \\
\lesssim (1.7916 \cdots)^j \|t\|_D
\]

This is sharp for even \( n \) for \( \ell(z) = z^n + 1 \).

PROOF. Use Theorem 4 and proceed as in the previous theorem.

THEOREM 8. Suppose \( t = v s_1 s_2 \cdots s_j \) with all polynomials real. Suppose \( t \) is monic of degree \( n \) and \( s_1 s_2 \cdots s_j \) is monic of degree \( m \). Then

\[
|v(0)|^j \|s_1\|_D \|s_2\|_D \cdots \|s_j\|_D \leq \frac{1}{\sqrt{2}} \left( \prod_{k=1}^{j} 2 \left( 1 + \cos \frac{2k - 1}{2n} \pi \right) \right) \|t\|_D.
\]

This is sharp for even \( n \) and \( m \) for \( \ell(z) = z^n + 1 \) for \( j \geq 2 \).

PROOF. This follows from Theorem 4, as in Theorem 3, with the following additional detail (as in Theorem 4). We first factor each \( s_j \) as

\[
s_j = u_j v_j
\]

where \( u_j \) has all roots in \( \{ \text{Re}(z) \leq 0 \} \) and \( v_j \) has all roots in \( \{ \text{Re}(z) > 0 \} \). It now follows that

\[
\|s_1\|_D \cdots \|s_j\|_D \leq |u_1(1) \cdots u_j(1)| \ |v_1(-1) \cdots v_j(-1)|
\]

and we proceed as before.

If we don't assume that the polynomials in question are real we can get the following.

COROLLARY 3. Suppose \( t = s \cdot v \) where \( s \in \pi_m \) and \( v \in \pi_{n-m} \). Then

\[
|s(-1)| \ |v(1)| \leq \left( \frac{1}{2} C_{n,m} \cdot C_{n,n-m} \right)^{1/2} \|t\|_D
\]

and if \( t \) is monic

\[
|t(0)|^{1/2} \|s\|_D \|v\|_D \leq \frac{1}{2} (C_{n,m} \cdot C_{n,n-m})^{1/2} (\|t\|_D)^2.
\]

These are sharp, for even \( n \) and \( m \), for \( \ell(z) = z^{n+1} \).

PROOF. The first inequality follows as in the proof of Theorem 6 with \( a := -1 \) and \( b := 1 \). The second part is immediate from (9) of the proof of Theorem 5.
REFERENCES


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