Rational Interpolation to $e^x$

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1. Introduction

We derive estimates for the error in interpolating $e^x$ by rational functions of degree $n$ on intervals of length less than two. Let $\pi_n$ denote the class of all polynomials of degree at most $n$ with real coefficients. Our main result is the following:

**Theorem 1.** Let $\gamma_1, \gamma_2, \ldots, \gamma_{2n+1}$ be points (not necessarily distinct) in $[0, a]$, where $a < 2$. Choose $P_n, Q_n \in \pi_n$ so that

$$P_n(\gamma_i) - Q_n(\gamma_i) e^{-\gamma_i} = 0 \quad \text{for} \quad i = 1, 2, \ldots, 2n+1.$$

Then, for $x \in [0, a]$,

$$|P_n(x)/Q_n(x) - e^{-x}| \leq \left(\frac{2e \sqrt{n} e^2 \sqrt{\alpha_n}}{2 - \alpha}\right) \frac{n!(n+1)!}{(2n)!(2n+1)!} \left|\prod_{i=1}^{2n+1} (x - \gamma_i)\right|.$$

Furthermore, $Q_n$ has positive coefficients.

Let

$$\lambda_{n}[a, b] = \min_{\rho \in \pi_n} \|e^x - \rho_n(x)/Q_n(x)\|_{[a, b]}.$$

where $\| \cdot \|$ denotes the supremum norm on $[a, b]$.

The following conjecture was made by G. Meinardus in 1964.

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CONJECTURE [3, p. 168].

\[ \lambda_{m,n}[-1,1] = \frac{m! \; n!}{\alpha^{m+n}(m+n)!(m+n+1)!} (1 + o(1)). \]

D. J. Newman, through some clever manipulation of the Padé approximant, has recently proved

**Theorem A** [5, p. 24].

\[ \lambda_{m,n}[-1,1] \leq \frac{8m! \; n!}{\alpha^{m+n}(m+n)!(m+n+1)!}. \]

G. Németh ([4], see also Braess [1]) has shown

**Theorem B.**

\[ \lambda_{n,n}[-1,1] = \frac{n! \; n!}{4^n(2n)! (2n+1)!} (1 + O(1)). \]

If we choose the \( \gamma_i \) in Theorem 1 to be the zeros of the \((2n+1)\)st Chebyshev polynomial (shifted to \([0, \alpha]\)) then we see that, up to the "slowly growing" \( e^{\sqrt{\alpha n}} \) term, we get essentially the right order of approximation. In light of Theorems A and B it seems plausible that the initial bracketed term of the error estimate is superfluous.

### 2. Preliminaries

Suppose that \( P_n, Q_n \in \pi_n \) and suppose that \( P_n(x) - Q_n(x) e^{-x} \) has \( 2n + 1 \) zeros on the interval \([0, \alpha]\). If \( Q_n(x) = q_0 + q_1 x + \cdots + q_n x^n \) then

\[
(P_n(x) - Q_n(x) e^{-x})^{(n+1)}
= (Q_n(x) e^{-x})^{(n+1)}
= \sum_{k=0}^{n} \binom{n+1}{k} Q_n^{(k)} e^{-x} (-1)^{(n+1-k)}
= \sum_{k=0}^{n} \frac{x^k}{k!} \sum_{j=0}^{n-k} \binom{n+1}{j} (-1)^{j+1}! q_{k+j}.
\]

(1)

Since \( (Q_n(x) e^{-x})^{(n+1)} \) has \( n \) zeros on \([0, \alpha]\), we deduce that there exist \( \beta_1, \ldots, \beta_n \in [0, \alpha] \) so that

\[
\sum_{k=0}^{n} \frac{x^k}{k!} \sum_{j=0}^{n-k} \binom{n+1}{j} (-1)^{j+1}! q_{k+j} = q_n \prod_{i=1}^{n} (x - \beta_i).
\]
Thus, if \( q_n \prod_{i=1}^{n} (x - \beta_i) = b_0 + b_1 x + \cdots + b_n x^n \), we have

\[
\begin{bmatrix}
{n+1 \choose 0} - {n+1 \choose 1} + {n+1 \choose 2} - \cdots - (-1)^n {n+1 \choose n} \\
0, - {n+1 \choose 1} + {n+1 \choose 2} - \cdots - (-1)^{n-1} {n+1 \choose n-1} \\
0, 0, - {n+1 \choose 2} - \cdots - (-1)^{n-2} {n+1 \choose n-2} \\
\vdots & \vdots & \ddots & \vdots \\
0, 0, 0, - \cdots - {n+1 \choose 0} \\
\end{bmatrix}
\begin{bmatrix}
q_0 \cdot 0! \\
q_1 \cdot 1! \\
q_2 \cdot 2! \\
\vdots \\
q_n \cdot n! \\
\end{bmatrix}
= \begin{bmatrix}
b_0 \cdot 0! \\
b_1 \cdot 1! \\
b_2 \cdot 2! \\
\vdots \\
b_n \cdot n! \\
\end{bmatrix}
\]

(2)

We can invert (2) to obtain

\[
\begin{bmatrix}
{n \choose n}, {n+1 \choose n}, {n+2 \choose n}, \ldots, {2n \choose n} \\
0, {n \choose n}, {n+1 \choose n}, \ldots, {2n-1 \choose n} \\
0, 0, {n \choose n}, \ldots, {2n-2 \choose n} \\
\vdots & \vdots & \ddots & \vdots \\
0, 0, 0, \ldots, {n \choose n} \\
\end{bmatrix}
\begin{bmatrix}
b_0 \cdot 0! \\
b_1 \cdot 1! \\
b_2 \cdot 2! \\
\vdots \\
b_n \cdot n! \\
\end{bmatrix}
= \begin{bmatrix}
q_0 \cdot 0! \\
q_1 \cdot 1! \\
q_2 \cdot 2! \\
\vdots \\
q_n \cdot n! \\
\end{bmatrix}
\]

(3)

We observe that (3) can be easily derived from (2) combined with the facts that the \((m, n)\) Padé approximant to \(e^{-x}\) is given by

\[
\sum_{v=0}^{m} \frac{m}{v} \frac{(x)^v}{v!} \sum_{v=0}^{n} \frac{n}{v} \frac{x^v}{v!}.
\]

and that for the Padé approximant \(b_0 = b_1 = \cdots = b_{n-1} = 0\).

We are now in a position to prove the following:

**Lemma 1.** Suppose that \( \pi_n(x) \in \pi_n \) and suppose that \( Q_n = q_0 + \)
q_1x + \cdots + q_nx^n, \text{ where } q_0 > 0. \text{ Suppose also that } P_n(x) - Q_n(x) e^{-x} \text{ has 2n + 1 zeros at } \gamma_1, \ldots, \gamma_{2n+1} \in [0, a]. \text{ Then, if } a < 2, \text{ } Q_n \text{ has positive coefficients and}

\[ q_n \leq \left( \frac{2}{2 - \alpha} \right)^{\frac{n!}{(2n)!}} q_0. \]

Proof. The first part follows from an examination of (3) using the facts that for } i \leq n,

\[(i - 1)! |b_{i-1}| \leq (i!) |b_i| \quad \text{and} \quad \binom{n + i - 1}{n} \leq \frac{1}{2} \binom{n + i}{n}. \]

The second part is proved by noting that

\[ q_0 \geq n! |b_n| \left( \frac{2n}{n} \right) - (n - 1)! |b_{n-1}| \left( \frac{2n - 1}{n} \right) \]

\[ \geq \left( 1 - \frac{\alpha}{2} \right) \frac{(2n)!}{n!} q_n. \]

The next lemma is a slight adaptation of a result of S. N. Bernstein [2, p. 38].

**Lemma 2.** Suppose that f and g are m + 1 times continuously differentiable on [a, b] and suppose that \( f(x) = g(x) = 0 \) has m + 1 solutions on [a, b]. If

\[ |f^{(m+1)}(x)| \leq g^{(m+1)}(x) \quad \text{for } x \in [a, b] \]

then

\[ |f(x)| \leq |g(x)| \quad \text{for } x \in [a, b]. \]

**Lemma 3** [3, pp. 16 and 165]. (a) If \( \gamma_1, \ldots, \gamma_{m+n+1} \in [a, b] \) then there exist \( P_m \in \pi_m, Q_n \in \pi_n, \) so that

\[ P_m(\gamma_i) - Q_n(\gamma_i) e^{-\gamma_i} = 0 \quad \text{for } i = 1, 2, \ldots, n + m + 1. \]

(b) If \( \pi_m \neq \pi_m, Q_n \neq Q_n \) and

\[ \|e^{-x} - P_m/Q_n\|_{[a, b]} = \min_{P_m \in \pi_m, Q_n \in \pi_n} \|e^{-x} - P_m/Q_n\|_{[a, b]} \]

then \( P_m/Q_n \) interpolates \( e^{-x} \) at exactly \( n + m + 1 \) points in [a, b].
3. Proof of Theorem 1

Lemma 3 guarantees the existence of \( P_n \) and \( Q_n \) with the desired interpolation property. We may assume that

\[
Q_n(x) = q_0 + \cdots + q_{n-1}x^{n-1} + x^n.
\]

Then, as in (1), there exist \( \beta_1, \ldots, \beta_n \in [0, \alpha] \) so that

\[
(Q_n(x) e^{-x})^{(n+1)} = (-1)^{n+1} e^{-x} \prod_{i=1}^{n} (x - \beta_i)
\]

\[
= (-1)^{n+1} e^{-x} R_n(x).
\]

Hence,

\[
(Q_n(x) e^{-x})^{(2n+1)} = (-1)^{n+1} \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} e^{-x} R_n^{(n-k)}(x).
\]

Since \( R_n^{(n-k)}(x) = n!/k! \prod_{i=1}^{k} (x - \rho_{i,k}) \), where \( \rho_{i,k} \in [0, \alpha] \), we have

\[
|(Q_n(x) e^{-x})^{(2n+1)}| \leq n! \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} \alpha^k
\]

\[
\leq n! \sum_{k=0}^{n} \frac{n! \alpha^k}{k! \prod_{k=0}^{n} \frac{n! \alpha^k}{k! (n-k)!}}.
\]

By Stirling's formula, \( n^ne^{-n} < n! < e\sqrt{n} n^ne^{-n} \),

\[
\frac{n! \alpha^k}{k! \prod_{k=0}^{n} \frac{n! \alpha^k}{k! (n-k)!}} \leq \frac{e\sqrt{n} \alpha^k e^{n} n^k}{k^k k^k (n-k)^{n-k}}
\]

\[
= e\sqrt{n} \alpha^k e^{k} \frac{n^k}{k!} \left(1 + \frac{k}{n-k}\right)^{n-k}
\]

\[
\leq e\sqrt{n} \frac{(\alpha \sqrt{n})^k}{k!}.
\]

A little elementary calculus reveals that \((ae^3/n/k^2)^k\) has a maximum at \( k = \sqrt{an} \) and hence,

\[
|(Q_n(x) e^{-x})^{(2n+1)}| \leq (n+1)! \ e\sqrt{n} e^\sqrt{\alpha n}.
\]

We apply Lemma 2 using \( m = 2n + 1 \),

\[
f(x) = P_n(x) - Q_n(x) e^{-x},
\]
and

\[ g(x) = e^{\sqrt{n} e^{2^{n+1}} (x + \gamma_i) (x - \gamma_i)} \]

and deduce that for \( x \in [0, \alpha] \),

\[ |P_n(x) - Q_n(x) e^{-x}| \leq e^{\sqrt{n} e^{2^{n+1}} (n + 1)! \prod_{i=1}^{2n+1} (x - \gamma_i)} \]

We complete the result by appealing to Lemma 1 to show that for \( x \geq 0 \),

\[ Q_n(x) \geq q_o \geq \frac{(2 - \alpha) (2n)!}{2^n}. \]

The \((1, 1)\) Padé approximant to \( e^{-x} \) has denominator \( Q(x) = 1 + \frac{1}{2}x \). It follows that the \((1, 1)\) rational function that interpolates \( e^{-x} \) with multiplicity three at any point \( \beta \) will have denominator \( Q_3(x) = 1 + \frac{1}{2}(x - \beta) \). In particular if \( \beta \geq 2 \) then \( Q_o \) does not have positive coefficients. This shows that \( \alpha < 2 \) is essential, at least for the \( n = 1 \) case, in Theorem 1.

REFERENCES


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