MARKOV AND BERNSTEIN TYPE INEQUALITIES IN $L_p$ FOR CLASSES OF POLYNOMIALS WITH CONSTRAINTS

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ABSTRACT

The Markov-type inequality

$$\int_{-1}^{1} |f'(x)|^p \, dx \leq c(p) (n(k+1))^p \int_{-1}^{1} |f(x)|^p \, dx$$

is proved for all real algebraic polynomials $f$ of degree at most $n$ having at most $k$, with $0 \leq k \leq n$, zeros (counting multiplicities) in the open unit disk of the complex plane, and for all $p > 0$, where $c(p) = c(p^2 + 1 + p^2)$ with some absolute constant $c > 0$. This inequality has been conjectured since 1983 when the $L_\infty$ case of the above result was proved. It improves and generalizes many earlier results. Up to the multiplicative constant $c(p) > 0$ the above inequality is sharp. A sharp Bernstein-type analogue for real trigonometric polynomials is also established, which is interesting on its own, and plays a key role in the proof of the Markov-type inequality.

1. Introduction, notation

Bernstein's inequality [16, pp. 39–41] asserts that

$$\max_{-\pi \leq t \leq \pi} |f'(t)| \leq n \max_{-\pi \leq t \leq \pi} |f(t)| \tag{0.1}$$

for every $f \in \mathcal{T}_n$, where $\mathcal{T}_n$ denotes the set of all trigonometric polynomials of degree at most $n$ with real coefficients. The corresponding algebraic result [16, pp. 39–41], known as Markov's inequality, states that

$$\max_{-1 \leq x \leq 1} |f'(x)| \leq n^2 \max_{-1 \leq x \leq 1} |f(x)| \tag{0.2}$$

for all $f \in \mathcal{P}_n$, where $\mathcal{P}_n$ denotes the set of all algebraic polynomials of degree at most $n$ with real coefficients. The Chebyshev polynomials $Q_n \in \mathcal{T}_n$ and $T_n \in \mathcal{P}_n$ defined by

$$Q_n(t) := \cos(n t + \alpha) \quad \text{for } \alpha \in \mathbb{R}, \tag{0.3}$$

$$T_n(x) := \cos(n \arccos x) \quad \text{for } -1 \leq x \leq 1 \tag{0.4}$$
show that (0.1) and (0.2) are sharp. The substitution $x = \cos t$ in (0.1), together with (0.2), yields

$$|f'(y)| \leq \min \left\{ n^2, \frac{\sin t}{\sqrt{1-y^2}} \right\} \max_{-1 \leq x \leq 1} |f(x)| \quad \text{with} \quad -1 \leq y \leq 1 \quad (0.5)$$

for every $f \in \mathcal{P}$. The sharp $L_p$ version of Bernstein's inequality was first established by A. Zygmund [25, II (3.17) p. 11] for $p \geq 1$. It states that

$$\int_{-\pi}^{\pi} |f'(t)|^p dt \leq n^p \int_{-\pi}^{\pi} |f(t)|^p dt \quad (0.6)$$

for every $f \in \mathcal{F}_n$ and $1 \leq p < \infty$. For $0 < p < 1$, first G. Klein [14] and later P. Osval'd [21] proved (0.6) with a multiplicative constant $c(p)$. In [20] Nevai proved that $c(p) = \frac{8}{p}$ is a possible choice. Subsequently, Máté and Nevai [19] showed the validity of (0.6) with a multiplicative absolute constant, and then V. V. Arestov [1] proved (0.6) (with the best constant 1) for every $0 < p < 1$. Recently M. von Golitschek and G. G. Lorentz [13] found a very elegant proof of Arestov's Theorem.

Markov's inequality in $L_p$ gives

$$\int_{-1}^{1} |f'(x)|^p dx \leq c^{p+1} n^{2p} \int_{-1}^{1} |f(x)|^p dx \quad (0.7)$$

for every $f \in \mathcal{P}_n$, where $c > 0$ is an absolute constant. This can be proved from the above $L_p$ Bernstein-type inequalities by the substitution $x = \cos t$ and by using Nikolskii-type inequalities (cf. [19, 17]). Finding the best constant in (0.7) is still an open problem. Markov and Bernstein type inequalities in weighted spaces and in $L_p$ norms play a key role in proving inverse theorems of approximation and of course have their own intrinsic interest.

Denote by $\mathcal{P}(n, k)$ the set of all $p \in \mathcal{P}_n$ having at most $k$ zeros (counting multiplicities) in the open unit disk $\{ z \in \mathbb{C} : |z| < 1 \}$. Markov and Bernstein type inequalities for constrained polynomials have been studied in many research papers where the classes $\mathcal{P}(n, k)$ for $0 \leq k \leq n$ are of special interest. One might correctly suspect that the restrictions on the zeros of a polynomial imply an improvement in inequalities (0.5), (0.6) and (0.7). In 1940 Erdős [12] proved that there is an absolute constant $c > 0$ such that

$$|f'(y)| \leq \min \left\{ \frac{en}{2}, \frac{\sqrt{n}}{(1-y)^{\frac{3}{2}}} \right\} \max_{-1 \leq x \leq 1} |f(x)| \quad \text{with} \quad -1 \leq y \leq 1 \quad (0.8)$$

for every $f \in \mathcal{P}(n, 0)$ having only real zeros. By taking the polynomials $p_n \in \mathcal{P}(n, 0)$ defined by $p_n(x) = (1+x)^{n-1}(1-x)$, it is easy to see that the constant $\frac{\sqrt{n}}{2x}$ in (0.8) is asymptotically sharp. In 1963 G. G. Lorentz [15] showed that there is an absolute constant $c > 0$ such that

$$|f'(y)| \leq c \min \left\{ n, \frac{\sqrt{n}}{(1-y)^{\frac{3}{2}}} \right\} \max_{-1 \leq x \leq 1} |f(x)| \quad \text{with} \quad -1 \leq y \leq 1 \quad (0.9)$$
for every \( f \in \mathcal{P}_n \) of the form
\[
f(x) = \sum_{j=0}^{n} a_j (1 + x)^{j} (1 - x)^{n-j} \quad \text{with all } a_j \geq 0 \text{ or all } a_j \leq 0. \tag{0.10}
\]

By an observation of G. G. Lorentz [22], every \( f \in \mathcal{P}(n, 0) \) is of the form (0.10), and therefore (0.9) holds for every \( f \in \mathcal{P}(n, 0) \). Inequality (0.9) is sharp up to the multiplicative absolute constant \( c > 0 \); namely it is shown in [8] that there is an absolute constant \( c > 0 \) such that
\[
\sup_{f \in \mathcal{P}(n, 0)} \frac{|f''(y)|}{\max_{-1 \leq x \leq 1} |f'(x)|} \geq c \min \left\{ n, \frac{\sqrt{n}}{\sqrt{(1 - y^n)}} \right\} \tag{0.11}
\]

for every \( n \in \mathbb{N} \) and \( y \in [-1, 1] \). In 1972 Scheick [22] found the best possible constant in Lorentz's Markov-type inequality. Extending Erdős's Markov-type inequality, he proved that
\[
\max_{-1 \leq x \leq 1} |f'(x)| \leq \frac{\min \{ n \}}{\max_{-1 \leq x \leq 1} |f(x)|} \tag{0.12}
\]

for every \( f \in \mathcal{P}_n \) of the form (0.10), and hence for every \( f \in \mathcal{P}(n, 0) \). In 1980 Szabados and Varma [24] showed that there is a constant \( c(k) > 0 \) depending only on \( k \) so that
\[
\max_{-1 \leq x \leq 1} |f'(x)| \leq c(k) n \max_{-1 \leq x \leq 1} |f(x)| \tag{0.13}
\]

for every \( f \in \mathcal{P}(n, k) \) with \( 0 \leq k \leq n \), having only real zeros. Subsequently Máte [18] proved that
\[
\max_{-1 \leq x \leq 1} |f''(x)| \leq 6n \exp(\pi \sqrt{k}) \max_{-1 \leq x \leq 1} |f(x)| \tag{0.14}
\]

for every \( f \in \mathcal{P}(n, k) \) with \( 1 \leq k \leq n \), having \( n-k \) zeros in \( \mathbb{R} \setminus (-1, 1) \). Szabados' conjecture, proved by P. Borwein [2] in 1985, establishes the Markov-type inequality
\[
\max_{-1 \leq x \leq 1} |f'(x)| \leq 9n(k+1) \max_{-1 \leq x \leq 1} |f(x)| \tag{0.15}
\]

for every \( f \in \mathcal{P}(n, k) \) with \( 0 \leq k \leq n \), having \( n-k \) zeros in \( \mathbb{R} \setminus (-1, 1) \). Inequality (0.15) was extended in [6] to all \( f \in \mathcal{P}(n, k) \) with \( 0 \leq k \leq n \). Another proof of (0.15) for all \( f \in \mathcal{P}(n, k) \) with \( 0 \leq k \leq n \), is obtained in [9] with the constant 11 instead of 9. The fact that (0.15) is sharp up to the multiplicative absolute constant was shown by Szabados [23, Example 1]. While (0.15) is essentially sharp, it is a good estimate only for \( |f(y)| \) with \( |y| \) close to 1.

It was proved in [11] that there is an absolute constant \( c > 0 \) such that
\[
|f'(y)| \leq \frac{c \sqrt{(n)(k+1)^2}}{\sqrt{(1-y^n)}} \max_{-1 \leq x \leq 1} |f(x)| \quad \text{with } -1 < y < 1 \tag{0.16}
\]

for every \( f \in \mathcal{P}(n, k) \) with \( 0 \leq k \leq n \). Subsequently it was shown in [9] that there is an absolute constant \( c > 0 \) so that
\[
|f'(y)| \leq \frac{c \sqrt{(n(k+1))}}{1-y^n} \max_{-1 \leq x \leq 1} |f(x)| \quad \text{with } -1 < y < 1 \tag{0.17}
\]
for every \( f \in \mathcal{P}(n, k) \) with \( 0 \leq k \leq n \). When \( y = 0 \), inequality (0.17) is sharp up to the multiplicative constant \( c > 0 \); so it is verified in [6] that there is an absolute constant \( c > 0 \) such that
\[
\sup_{f \in \mathcal{P}(n, k)} \frac{|f'(0)|}{\max_{-1 \leq x \leq 1} |f(x)|} \geq c \sqrt{(n(k + 1))}
\] (0.18)
for every \( 0 \leq k \leq n \).

The unpleasant thing about the Bernstein-type inequalities (0.16) and (0.17) is the fact that none of them matches the inequality (0.5) in the unrestricted case \( k = n \) (note that \( ^{(n,n)} = 1 \)). In [4] the authors established the ‘right’ Markov–Bernstein type inequality in \( L_\infty \) for \( \mathcal{P}(n, k) \) with \( 0 \leq k \leq n \), which contains all of the earlier \( L_\infty \) results as special cases up to a multiplicative constant \( c > 0 \). Namely, there is an absolute constant \( c > 0 \) such that
\[
|f'(y)| \leq c \min \left\{ n(k + 1), \left( \frac{n(k + 1)}{1 - y^2} \right)^{1/2} \right\} \max_{-1 \leq x \leq 1} |f(x)| \quad \text{with} \quad -1 \leq y \leq 1 \quad (0.19)
\]
for every \( f \in \mathcal{P}(n, k) \) with \( 0 \leq k \leq n \).

The purpose of this paper is to establish the ‘right’ Markov and Bernstein type inequalities for \( \mathcal{P}(n, k) \) with \( 0 \leq k \leq n \) in \( L_p \) for every \( 0 < p \leq \infty \). Up to a multiplicative constant \( c(p) \) depending only on \( p \), our results are sharp and contain all the earlier results as special cases. Our proofs are based on a nice combination of results and methods worked out in [9, 4, 5, 19, 10]; however we have to cope with a lot of technical details.

2. Results and Proofs

Throughout this paper \( \mathbb{N} \) will denote the set of nonnegative integers.

**Theorem 1.** Let \( \chi : [0, \infty) \to \mathbb{R} \) be a convex and nondecreasing function and let \( 0 < p \leq 1 \). There is an absolute constant \( c_1 > 0 \) such that
\[
\int_{-\pi}^{\pi} \chi \left( c_1 p^2 \left| \frac{f'(t)}{\sqrt{(n(k + 1))}} \right|^p \right) dt \leq \int_{-\pi}^{\pi} \chi(|f(t)|^p) dt
\]
for every \( f \in \mathcal{T}_n \) of the form
\[
f(t) := h(\cos t) q(t), \quad (1)
\]
where \( h \in \mathcal{P}_{n-k} \) has no zeros in the open unit disk, and \( q \in \mathcal{T}_k, n, k \in \mathbb{N} \) for \( 0 \leq k \leq n \).

Using \( \chi(x) := x \) if \( 0 < p \leq 1 \), and \( \chi(x) := x^p \) if \( p > 1 \), immediately gives the following corollary.

**Corollary 2.** There is a constant \( c_2(p) > 0 \) such that
\[
\int_{-\pi}^{\pi} |f'(t)|^p dt \leq c_2(p) (n(k + 1))^{p/2} \int_{-\pi}^{\pi} |f(t)|^p dt
\]
for every \( f \in \mathcal{T}_n \) of the form (1) and for every \( p > 0 \), where \( c_2(p) := c_3^{p+1}(1 + p^{-2}) \) with some absolute constant \( c_3 > 0 \).
Theorem 3. There is a constant $c_4(p) > 0$ such that
\[
\int_{-1}^{1} |f'(x)|^p \, dx \leq c_4(p) (n(k+1))^p \int_{-1}^{1} |f(x)|^p \, dx
\]
for every $f \in \mathcal{P}_n$ of the form
\[
f(x) = h(x) q(x),
\]
where $h \in \mathcal{P}_{n-k}$ has no zeros in the open unit disk, $q \in \mathcal{P}_{k}, n, k \in \mathbb{N}$ for $0 \leq k \leq n$, and for every $p > 0$, where $c_4(p) := c_6^{p+1}(1 + p^{-2})$ with some absolute constant $c_6 > 0$.

The following examples show the sharpness of Corollary 2 and Theorem 3 up to a multiplicative positive constant depending only on $p$.

Example 4. There are $h_{n,k,p} \in \mathcal{P}_n$ of the form
\[
h_{n,k,p}(t) = (1 + \cos t)^q q_{n,k,p}(t) \text{ for } q_{n,k,p} \in \mathcal{Q}_k
\]
such that
\[
\int_{-\pi}^{\pi} |h_{n,k,p}(t)|^p \, dt \leq c_6^{p+1} p(p+1)^{-1}(n(k+1))^{p/2} \int_{-\pi}^{\pi} |h_{n,k,p}(t)|^p \, dt
\]
for all integers $0 \leq k \leq n$ and for all $p > 0$, where $c_6 > 0$ is an absolute constant.

Example 5. There are $H_{n,k,p} \in \mathcal{P}_n$ of the form
\[
H_{n,k,p}(x) = (1 + x)^Q Q_{n,k,p}(x) \text{ for } Q_{n,k,p} \in \mathcal{Q}_k
\]
such that
\[
\int_{-1}^{1} |H_{n,k,p}(x)|^p \, dx \geq c_7^{p+1} p(p+1)^{-1}(n(k+1))^p \int_{-1}^{1} |H_{n,k,p}(x)|^p \, dx
\]
for all integers $0 \leq k \leq n$ and for all $p > 0$, where $c_7 > 0$ is an absolute constant.

To prove Theorem 1 we need a series of lemmas.

Lemma 1.1. There is an absolute constant $c_8 > 0$ such that
\[
|g(z)| \leq c_8 \max_{t \in \mathbb{R}} |g(t)|
\]
for every $g \in \mathcal{P}_{(l+1)n}$ of the form
\[
g(t) = (1 - \cos (t - \beta))(1 + \cos (t - \alpha))^{m(1 - \cos (t - \alpha))}q(t),
\]
where $\alpha, \beta \in \mathbb{R}$, $q \in \mathcal{P}_{(l+1)n}, n, k, m, l \in \mathbb{N}$ with $0 \leq k \leq n$, $0 \leq m \leq n - k$, and for every $z \in \mathbb{C}$ such that
\[
|\text{Im } z| \leq 32^{-1}(l+1)^{-2}(n(k+1))^{-1/2}.
\]

The proof of the above lemma rests on the following result.
LEMMA 1.2. There is an absolute constant $c_9 > 0$ such that
\[ |P(\alpha)| \leq c_9 \max_{-1 \leq x \leq 1} |P(x)| \]
for every $P \in \mathcal{P}_{2(n+1)k}$ of the form
\[ P(x) := (x - a_1)^{2(n-k)}(x - a_2)^{2m}(x - a_3)^{2(n-k-m)}Q(x), \]  
(6)
where $a_1, a_2, a_3 \in [-1, 1]$, $Q \in \mathcal{P}_{2(n+1)k}$, $n, k, m, l \in \mathbb{N}$ with $0 \leq k \leq n$, $0 \leq m \leq n - k$, and for every
\[ \alpha \in [1, 1 + (1024(l+1)^4n(k+1))^{-1}]. \]

We reduce the proof of Lemma 1.2 to the following two results proved in [4, Lemma 13; 7, Lemma 1, Corollary 1], respectively.

LEMMA 1.3. There is an absolute constant $c_{10} > 0$ such that
\[ |P(\alpha)| \leq c_{10} \max_{-1 \leq x \leq 1} |P(x)| \]
for every $P \in \mathcal{P}_{2(n+1)k}$ of the form
\[ P(x) := (x + 1)^{2(n-k)+2m}(x - a_4)^{2(n-k-m)}Q(x), \]  
(7)
where $a_4 \in [-1, 1]$, $Q \in \mathcal{P}_{2(n+1)k}$, $n, k, m, l \in \mathbb{N}$ with $0 \leq m \leq n - k$, and for every
\[ \alpha \in [1, 1 + (4(l+1)^2n(k+1))^{-1}]. \]

LEMMA 1.4. Assume that $P \in \mathcal{P}_{2(n+1)k}$ and that
\[ |P(1)| = \max_{-1 \leq x \leq 1} |P(x)|. \]  
(8)
Then $P$ has at most $c_{11}(l+1)n^{1/2}$ zeros (counting multiplicities) in $[1 - r, 1]$ for every $r > 0$, where $c_{11} = 2 \sqrt{2}$ is a suitable choice.

Proof of Lemma 1.2. If $\frac{1}{n} \leq l \leq n$, then the conclusion of the lemma is a well-known consequence of the Chebyshev inequality [16, p. 43] for all $P \in \mathcal{P}_{2(n+1)k}$. Therefore, in what follows, let $0 \leq k < \frac{1}{n}$. Without loss of generality we may assume that $\frac{1}{2}(n-k) \leq m \leq n-k$, otherwise $\frac{1}{2}(n-k) < n-k-m \leq n-k$. Note that $0 \leq k < \frac{1}{n}$ and $\frac{1}{2}(n-k) \leq m$ imply that $\frac{1}{n} < m$. A simple variational method yields that it is sufficient to prove the lemma under the assumption that $Q \in \mathcal{P}_{(l+1)k}$ has all its zeros in $[-1, 1]$. We may also assume that
\[ \max_{-1 \leq x \leq 1} |P(x)| = 1; \]  
(9)
the general case can easily be reduced to this by a linear transformation (note that $|P|$ is increasing on $[1, \infty)$, since all the zeros of $Q$ are in $[-1, 1]$). Therefore, by Lemma 1.4, we can deduce that
\[ \max \{a_1, a_2\} \leq 1 - c_{12} \]  
(10)
with $c_{12} = (4(l+1)c_{11})^{-2} = (8.2^{1/2}(l+1))^{-2}$. For the sake of brevity let
\[ \tilde{a}_1 := \min \{a_1, a_2\} \quad \text{and} \quad \tilde{a}_2 := \max \{a_1, a_2\}. \]  
(11)
Let $P$ be of the form (6) and
\[ \tilde{P}(x) := (x - \tilde{a}_2)^{2(n-k)+2m}(x - a_3)^{2(n-k-m)}Q(x). \]  
(12)
Since \((x - \bar{a}_2)/(x - \bar{a}_1)\) is increasing on \([\bar{a}_2, \infty)\), we obtain
\[
\frac{|P(\alpha)|}{\max_{-1 \leq x \leq 1} |P(x)|} \leq \frac{|P(\alpha)|}{\max_{\bar{a}_2 \leq x \leq 1} |P(x)|} \leq \frac{|\tilde{P}(\alpha)|}{\max_{\bar{a}_2 \leq x \leq 1} |\tilde{P}(x)|}
\]
(13)
for every \(\alpha \in [1, \infty)\). From Lemma 1.3, by a linear transformation, we can easily deduce that
\[
\frac{|\tilde{P}(\alpha)|}{\max_{\bar{a}_2 \leq x \leq 1} |\tilde{P}(x)|} \leq c_{10},
\]
(14)
where
\[
1 \leq \alpha \leq 1 + (1 - \bar{a}_2)(8(l + 1)^3n(k + 1))^{-1} \leq 1 + (1024(l + 1)^4n(k + 1))^{-1}.
\]
Combining (13) and (14), we get the lemma.

Now we obtain Lemma 1.1 from Lemma 1.2 as follows.

Proof of Lemma 1.1. If \(f \in \mathcal{I}_{l+1}^n\) is of the form (5), then there is a \(Q \in \mathcal{I}_{l+1}^k\) such that
\[
g(i) g(-i) = (\cos t - \cos \beta)^{2(n-k)} (\cos t + \cos \alpha)^{2m} (\cos t - \cos \alpha)^{2(n-k-m)} q(t) g(-t)
\]
\[
= (\cos t - \cos \beta)^{2(n-k)} (\cos t + \cos \alpha)^{2m} (\cos t - \cos \alpha)^{2(n-k-m)} Q(cos t).
\]
(15)
For the sake of brevity let
\[
P(x) := (x - \cos \beta)^{2(n-k)} (x + \cos \alpha)^{2m} (x - \cos \alpha)^{2(n-k-m)} Q(x).
\]
(16)
Then Lemma 1.2 yields
\[
|\delta|^2 = |g(i\delta)|^2 = |g(i\delta) g(-i\delta)| = |P(\cos i\delta)|
\]
\[
\leq \max_{1 \leq \alpha \leq 1 + |\delta|^2} |P(\alpha)| \leq \max_{1 \leq \alpha \leq 1 + (1024(l + 1)^4n(k + 1))^{-1}} |P(\alpha)|
\]
\[
= c_{10} \max_{-1 \leq x \leq 1} |P(x)| = c_{10} \max_{t \in \mathbb{R}} |g(t) g(-t)| \leq c_{10} \max_{t \in \mathbb{R}} |g(t)|^2
\]
whenever
\[
|\delta|^2 \leq 32^{-1}(l + 1)^{-2}(n(k + 1))^{-1/2},
\]
(17)
and the lemma follows for all \(z \in \mathbb{C}\) with \(\Re z = 0\) and
\[
|\Im z| \leq 32^{-1}(l + 1)^{-2}(n(k + 1))^{-1/2}.
\]
If \(\Re z \neq 0\), then we study \(\tilde{g}(t) := g(t - \Re z)\); this is of the form (5) as well with \(\tilde{\alpha} := \alpha + \Re z, \tilde{\beta} := \beta + \Re z,\) and \(\tilde{q}(t) = q(t - \Re z) \in \mathcal{I}_{l+1}^k,\) and the case already proved gives the lemma.

From Lemma 1.1, by Cauchy’s Integral Formula, we immediately obtain the following.

Corollary 1.5. There is an absolute constant \(c_{14} > 0\) such that
\[
\max_{t \in \mathbb{R}} |g'(t)| \leq c_{14}(l + 1)^2(n(k + 1))^{1/2} \max_{t \in \mathbb{R}} |g(t)|
\]
for every \(g \in \mathcal{I}_{l+1}^n\) of the form (5).
From Corollary 1.5, by a variational method, we easily obtain the next corollary.

**Corollary 1.6.** Let \( c_{13} \) be as in Corollary 1.5. Then

\[
|g'(t_0)| \leq c_{13}(l + 1)^2(n(k + 3))^{1/2} \max_{\tau \in \mathbb{R}} |g(\tau)|
\]

for every \( g \in \mathcal{T}_{(l+1)n} \) of the form

\[
g(t) = (1 - \cos(t - \beta))^{(n-k)} h(\cos t) q(t),
\]

where \( \beta \in \mathbb{R}, q \in \mathcal{T}_{(l+1)k}, n, k, l \in \mathbb{N} \) with \( 0 \leq k \leq n \), and where \( h \in \mathcal{P}_{n-k} \) has no zeros in the open unit disk.

**Proof.** Let \( t_0 \in \mathbb{R}, n, k, l \in \mathbb{N} \) with \( 0 \leq k \leq n \), \( \beta \in \mathbb{R} \), and \( q \in \mathcal{T}_{(l+1)k} \) be fixed. By a simple compactness argument there is a function

\[
g^*(t) := (1 - \cos(t - \beta))^{(n-k)} h^*(\cos t) q(t)
\]

with \( h^* \in \mathcal{P}_{n-k} \) having no zeros in the open unit disk such that

\[
\sup_{\tau \in \mathbb{R}} \frac{|g'(t_0)|}{\max_{\tau \in \mathbb{R}} |g(\tau)|} = \frac{|g^*(t_0)|}{\max_{\tau \in \mathbb{R}} |g^*(\tau)|},
\]

where the sup in (20) is taken for all \( g \) of the form (18). We show that \( h^* \in \mathcal{P}_{n-k-m} \mathcal{P}_{n-k-2} \) has all but one of its zeros (counting multiplicities) at either \(-1\) or \(+1\). If \( f^*(x) = 0 \) and \( x \in \mathbb{C} \setminus \mathbb{R} \), then for a sufficiently small \( \varepsilon > 0 \),

\[
h^*(x) := h^*(x) \left( 1 - \frac{\varepsilon(x - \cos t_0)^2}{(x - \alpha)(x - \beta)} \right) \in \mathcal{P}_{n-k}
\]

has no zeros in the open unit disk, and

\[
g^*(t) := (1 - \cos(t - \beta))^{(n-k)} h^*(\cos t) q(t)
\]

contradicts the maximality of \( g^* \).

Now assume that there are \( \alpha, \beta \in \mathbb{R} \setminus [-1, 1] \) such that \((x - \alpha)(x - \beta)\) is a factor of \( f(x) \). Then for a sufficiently small \( \varepsilon > 0 \) it follows that

\[
h^*(x) := h^*(x) \left( 1 - \frac{\varepsilon \text{sign}(\alpha \beta)(x - \cos t_0)^2}{(x - \alpha)(x - \beta)} \right) \in \mathcal{P}_{n-k}
\]

has no zeros in the open unit disk, and (22) would contradict the maximality of \( g^* \). Also, \( \deg h^* \geq n - k - 1 \), otherwise for sufficiently small \( \varepsilon > 0 \) it would follow that

\[
h^*(x) := h^*(x) (1 - \varepsilon(x - \cos t_0)^2) \in \mathcal{P}_{n-k}
\]

has no zeros in the open unit disk, and \( g^* \) defined by (22) would contradict the maximality of \( g^* \). So now we get the desired conclusion by Corollary 1.5.

**Lemma 1.7.** There is an absolute constant \( c_{14} > 0 \) such that

\[
\max_{-\pi \leq \tau \leq \pi} |g(\tau)|^p \leq c_{14}(l + 1)^2(n(k + 3))^{1/2} \int_{-\pi}^{\pi} |g(\tau)|^p \, d\tau
\]

for every \( g \in \mathcal{T}_{(l+1)n} \) of the form (18) and for every \( 0 < p \leq 2 \).
**Proof.** Let $t_0 \in [-\pi, \pi]$ be such that

$$|g(t_0)| = \max_{-\pi \leq t \leq \pi} |g(t)|. \tag{25}$$

By Corollary 1.6 and the Mean Value Theorem we obtain that there is a $\xi$ between $t_0$ and $t$ such that

$$|g(t)| \geq |g(t_0)| - |g(t) - g(t_0)| = |g(t_0)| - |t - t_0| |g'(\xi)| = \max_{-\pi \leq t \leq \pi} |g(t)| - |t - t_0| c_{13}(l+1)^2(n(k+3))^{1/2} \max_{-\pi \leq t \leq \pi} |g(t)|$$

$$\geq 2^{-1} \max_{-\pi \leq t \leq \pi} |g(t)|$$

whenever

$$|t - t_0| \leq (2c_{13})^{-1}(l+1)^{-2}(n(k+3))^{-1/2}. \tag{26}$$

For the sake of brevity let

$$I := [t_0 - (2c_{13})^{-1}(l+1)^{-2}(n(k+2))^{-1/2}, t_0 + (2c_{13})^{-1}(l+1)^{-2}(n(k+2))^{-1/2}].$$

Then (26) and $0 < p \leq 2$ imply that

$$\int_{-\pi}^{\pi} |g(t)|^p dt \geq \int_I |g(t)|^p dt \geq c_{13}^{-1}(l+1)^{-2}(n(k+3))^{-1/2} 2^{-2} \max_{-\pi \leq t \leq \pi} |g(t)|^p,$$

and the lemma is proved.

**Lemma 1.8.** There are $f_{n,k} \in \mathcal{S}_n$ of the form

(i) $f_{n,k}(t) := (1 + \cos t)^{-k} q_{n,k}(t)$ with $q_{n,k} \in \mathcal{S}_k$

such that

(ii) $\int_{-\pi}^{\pi} (f_{n,k}(t))^2 dt = 1,$

(iii) $(f_{n,k}(0))^2 \geq c_{15}(n(k+3))^{1/2},$

(iv) $f_{n,k}(0) = 0$

hold for all integers $0 \leq k \leq n$, where $c_{15} > 0$ is an absolute constant.

**Proof.** It follows from [5, Corollary 3.3] that there are $F_{n,k} \in \mathcal{P}_n$ of the form

$$F_{n,k}(x) := (1 + x)^{-k} Q_{n,k}(x) \quad \text{with} \quad Q_{n,k} \in \mathcal{P}_k \tag{27}$$

so that

$$\int_{-1}^{1} (F_{n,k}(x))^2 dx = 1, \tag{28}$$

$$|F_{n,k}(1)|^2 \geq c_{16} n(k+3) \tag{29}$$

hold for all integers $0 \leq k \leq n$, where $c_{16} > 0$ is an absolute constant. Now let

$$f_{n,k}(t) := \left(\int_{-1}^{1} (F_{n,k}(\cos t))^2 dt\right)^{-1/2} F_{n,k}(\cos t). \tag{30}$$
Obviously $f_{n,k} \in \mathcal{F}_n$ is of the form (i), and
\[
\int_{-\pi}^{\pi} (f_{n,k}(t))^2 \, dt = 1, \tag{31}
\]
\[
f_{n,k}'(0) = 0 \tag{32}
\]
for all integers $0 \leq k \leq n$. To prove (iii), first note that Lemma 1.7 implies (with $g = f_{n,k}$, $l = 0$, $h = 1$, $\beta = \pi$, $p = 2$) that
\[
\int_{-\pi}^{\pi} (f_{n,k}(t))^2 \, dt = \int_A (f_{n,k}(t))^2 \, dt + \int_{[-\pi,\pi] \setminus A} (f_{n,k}(t))^2 \, dt
\]
\[
\leq m(A) \max_{t \in \mathbb{R}} (f_{n,k}(t))^2 + \int_{[-\pi,\pi] \setminus A} (f_{n,k}(t))^2 \, dt
\]
\[
\leq m(A) c_{14} (n(k+3))^{1/2} \int_{-\pi}^{\pi} (f_{n,k}(t))^2 \, dt + \int_{[-\pi,\pi] \setminus A} (f_{n,k}(t))^2 \, dt \tag{33}
\]
holds for any measurable set $A \subseteq [-\pi,\pi]$. Hence $m(A) \leq (2c_{14}^{-1})^{-1}(n(k+3))^{-1/2}$ implies that
\[
\int_{-\pi}^{\pi} (f_{n,k}(t))^2 \, dt \leq 2 \int_{[-\pi,\pi] \setminus A} (f_{n,k}(t))^2 \, dt. \tag{34}
\]
Since $f_{n,k}(t) = \alpha F_{n,k}(\cos t)$, there is an absolute constant $c_{17} > 0$ so that with the notation
\[
\delta_n := 1 - (c_{17} n(k+3))^{-1} \tag{35}
\]
we have
\[
\int_{-1}^{1} (F_{n,k}(x))^2(1-x^2)^{-1/2} \, dx \leq 2 \int_{-\delta_n}^{\delta_n} (F_{n,k}(x))^2(1-x^2)^{-1/2} \, dx. \tag{36}
\]
Now
\[
(f_{n,k}(0))^2 = (F_{n,k}(1))^2 \left( \int_{-\pi}^{\pi} (F_{n,k}(\cos t))^2 \, dt \right)^{-1}
\]
\[
\geq c_{16} n(k+3) \left( \frac{1}{4} \int_{-1}^{1} (F_{n,k}(x))^2(1-x^2)^{-1/2} \, dx \right)^{-1}
\]
\[
\geq c_{16} n(k+3) \left( \frac{\int_{-\delta_n}^{\delta_n} (F_{n,k}(x))^2(1-x^2)^{-1/2} \, dx}{\delta_n \delta_n} \right)^{-1}
\]
\[
\geq c_{16} n(k+3) (c_{17} n(k+2))^{-2} \left( \int_{-\delta_n}^{\delta_n} (F_{n,k}(x))^2 \, dx \right)^{-1}
\]
\[
\geq c_{15} (n(k+3))^{1/2} \left( \int_{-1}^{1} (F_{n,k}(x))^2 \, dx \right)^{-1} = c_{15} (n(k+2))^{1/2} \tag{37}
\]
with $c_{15} := 4^{-1} c_{16} c_{17}^{1/2}$, and the lemma is proved.

Proof of Theorem 1. Let $f$ be of the form (1), let $l := [2p^{-1}] + 1$ for $0 < p \leq 1$, and let
\[
g := f(f_{n,k})^l. \tag{38}
\]
Applying Lemma 1.7 to \( g \), and then to \( f_{n,k} \), we obtain

\[
\max_{\tau \in \mathbb{R}} |f(\tau) f_{n,k}(\tau)|^p \leq c_4(l+1)^2(n(k+3))^{1/2} \int_{-\pi}^{\pi} |f(\tau)|^p |f_{n,k}(\tau)|^2 |f_{n,k}(\tau)|^{l+p-2} d\tau
\]

\[
\leq c_4(l+1)^2(n(k+3))^{1/2} \max_{\tau \in \mathbb{R}} |f_{n,k}(\tau)|^{l+p-2} \int_{-\pi}^{\pi} |f(\tau)|^p (f_{n,k}(\tau))^2 d\tau
\]

\[
\leq c_4(l+1)^2(n(k+3))^{1/2} \left( c_4(n(k+3))^{1/2} \int_{-\pi}^{\pi} (f_{n,k}(\tau))^2 d\tau \right)^{(l+p-2)/2}
\times \int_{-\pi}^{\pi} |f(\tau)|^p (f_{n,k}(\tau))^2 d\tau
\]

\[
\leq (l+1)^2 c_{14}^{l+p/2}(n(k+3))^{l+p/4} \int_{-\pi}^{\pi} |f(\tau)|^p (f_{n,k}(\tau))^2 d\tau,
\]

(39)

where \( lp \geq 2 \) and Lemma 1.8 (i) were also used. Applying Corollary 1.6 to \( g \) defined by (38), we get

\[
|g'(\tau)|^p \leq c_{14}^p (l+1)^{2p}(n(k+3))^{p/2} \max_{\tau \in \mathbb{R}} |g(\tau)|^p
\]

(40)

for every \( \tau \in \mathbb{R} \). Putting \( \tau = 0 \) and using \( f'_{n,k}(0) = 0 \) and (39), we conclude that

\[
|f'(0)|^p |f_{n,k}(0)|^p \leq c_{14}^p (l+1)^{2p}(n(k+3))^{p/2} \max_{\tau \in \mathbb{R}} |f(\tau) (f_{n,k}(\tau))|^p
\]

\[
\leq c_{14}^p (l+1)^{2p}(n(k+3))^{p/2} (l+1)^2 c_{14}^{l+p/2}(n(k+3))^{l+p/4} \int_{-\pi}^{\pi} |f(\tau)|^p (f_{n,k}(\tau))^2 d\tau.
\]

(41)

Combining this with \((f_{n,k}(0))^2 \geq c_{14}(n(k+3))^{1/2} \) (see Lemma 1.8 (iii)) and \( l+1 \leq 2(1+p^{-1}) \), we deduce that

\[
|f'(0)|^p \leq c_{14}^p (l+1)^{2p}(n(k+3))^{p/2}(l+1)^2 c_{14}^{l+p/2} c_{15}^{-p/2} \int_{-\pi}^{\pi} |f(\tau)|^p (f_{n,k}(\tau))^2 d\tau
\]

\[
\leq c_{18} p^{-2}(n(k+3))^{p/2} \int_{-\pi}^{\pi} |f(\tau)|^p (f_{n,k}(\tau))^2 d\tau
\]

(42)

with an absolute constant \( c_{18} > 0 \). Applying (42) to \( f_1(\tau) := f(\tau + \tau) \), we obtain

\[
(c_{18} \sqrt{3})^{-1} p^2 \left( \frac{|f'(\tau)|}{\sqrt{n(k+1)}} \right)^p \leq \int_{-\pi}^{\pi} |f(\tau)|^p (f_{n,k}(\tau - \tau))^2 d\tau.
\]

(43)

Since

\[
\int_{-\pi}^{\pi} (f_{n,k}(\tau - \tau))^2 d\tau = 1 \quad \text{for} \quad \tau \in \mathbb{R}
\]

(44)

and \( \chi : [0, \infty) \rightarrow \mathbb{R} \) is a convex and nondecreasing function, (44) and an application of the Jensen inequality yield

\[
\chi \left( (c_{18} \sqrt{3})^{-1} p^2 \left( \frac{|f'(\tau)|}{\sqrt{n(k+1)}} \right)^p \right) \leq \chi \left( \int_{-\pi}^{\pi} |f(\tau)|^p (f_{n,k}(\tau - \tau))^2 d\tau \right)
\]

\[
\leq \int_{-\pi}^{\pi} \chi (|f(\tau)|^p) (f_{n,k}(\tau - \tau))^2 d\tau.
\]

(45)
Integrating both sides of (45) with respect to \( t \), and using

\[
\int_{-\pi}^{\pi} (f_{n,k}(\tau-t))^2 \, dt = 1,
\]

we get the desired inequality by the Fubini Theorem.

**Proof of Corollary 2.** This result follows from Theorem 1 with \( \bar{p} := p \) and \( \chi(x) := x \) if \( 0 < p < 1 \), and with \( \bar{p} := 1 \) and \( \chi(x) := x^p \) if \( p > 1 \).

To prove Theorem 3 we need some lemmas.

**Lemma 3.1** ([2], [6, Corollary 1.3], [9, Theorem 1], or [4, Theorem 3.4]). There is an absolute constant \( c_{19} > 0 \) such that

\[
\max_{-1 \leq x \leq 1} |f'(x)|^{c_{19}} n(k+1) \max_{-1 \leq x \leq 1} |f(x)|
\]

for all \( f \in \mathcal{D}_n \) having at most \( k \) (with \( 0 \leq k \leq n \)) zeros (counting multiplicities) in the open unit disk.

**Lemma 3.2** [3, Theorem 3.3]. There is an absolute constant \( c_{20} > 0 \) such that

\[
\max_{-1 \leq x \leq 1} |f(x)|^{p} \leq c_{20}(1 + p^2) n(k+1) \int_{-1}^{1} |f(x)|^{p} \, dx
\]

for all \( f \in \mathcal{D}_n \) having at most \( k \) (with \( 0 \leq k \leq n \)) zeros (counting multiplicities) in the open unit disk and for all \( p > 0 \).

**Lemma 3.3.** There is an absolute constant \( c_{21} > 0 \) such that

\[
\int_{-1}^{1} |f(x)|^{p} (1-x^2)^{-\alpha} \, dx \leq c_{21}(1 - \alpha)^{-1}(1 + p^2) (n(k+1))^{\alpha} \int_{-1}^{1} |f(x)|^{p} \, dx
\]

for all \( f \in \mathcal{D}_n \) having at most \( k \) (with \( 0 \leq k \leq n \)) zeros (counting multiplicities) in the open unit disk, and for all \( p > 0 \) and \( 0 < \alpha < 1 \).

**Proof.** Let \( \delta_{n,k} := 1 - (n(k+1))^{-1} \). Using Lemma 3.2, we obtain

\[
\int_{-1}^{1} |f(x)|^{p}(1-x^2)^{-\alpha} \, dx
\]

\[
\leq (1 - \delta_{n,k})^{-\alpha} \int_{-\delta_{n,k}}^{\delta_{n,k}} |f(x)|^{p} \, dx + \int_{[-1,1] \setminus [-\delta_{n,k}, \delta_{n,k}]} (1-x^2)^{-\alpha} \max_{-1 \leq y \leq 1} |f(y)|^{p}
\]

\[
\leq (n(k+1))^\alpha \int_{-1}^{1} |f(x)|^{p} \, dx + 2(1 - \alpha)^{-1}(1 - \delta_{n,k})^{1-\alpha} c_{20}(1 + p^2) (n(k+1))^{\alpha} \int_{-1}^{1} |f(x)|^{p} \, dx
\]

\[
\leq c_{21}(1 - \alpha)^{-1}(1 + p^2) (n(k+1))^{\alpha} \int_{-1}^{1} |f(x)|^{p} \, dx,
\]

where \( c_{21} := 1 + 2c_{20} \), and the lemma is proved.

**Proof of Theorem 3.** We distinguish two cases.
**Case 1: p ≥ 1.** Corollary 2 and the substitution \( x = \cos t \) yield

\[
\int_{-1}^{1} \left| f'(x) \right|^p (1-x^2)^{\left(\frac{p-1}{2}\right)} \, dx \leq c_2(p) (n(k+1))^{p/2} \int_{-1}^{1} \left| f(x) \right|^p (1-x^2)^{-1/2} \, dx \tag{47}
\]

for every \( f \in \mathcal{P}_n \) of the form (2). Now let \( \delta_{n,k} := 1 - (n(k+1))^{-1} \). Then, by (47) and Lemma 3.3, we get

\[
\int_{\delta_{n,k}}^{\delta_{n,k}} \left| f'(x) \right|^p \, dx
\leq (n(k+1))^{\left(\frac{p-1}{2}\right)} \int_{-\delta_{n,k}}^{\delta_{n,k}} \left| f'(x) \right|^p (1-x^2)^{\left(\frac{p-1}{2}\right)} \, dx
\leq (n(k+1))^{\left(\frac{p-1}{2}\right)} c_2(p) (n(k+1))^{p/2} \int_{-1}^{1} \left| f(x) \right|^p (1-x^2)^{-1/2} \, dx
\leq (n(k+1))^{\left(\frac{p-1}{2}\right)} c_2(p) (n(k+1))^{p/2} 2c_{21}(1+p^2)(n(k+1))^{1/2} \int_{-1}^{1} \left| f(x) \right|^p \, dx
\leq c_{22}(p) (n(k+1))^p \int_{-1}^{1} \left| f(x) \right|^p \, dx, \tag{48}
\]

where \( c_{22}(p) := 2c_{21}(1+p^2)c_2(p) = 2c_{21}(1+p^2)c_2(p+1+p^{-2}) \) for every \( f \in \mathcal{P}_n \) of the form (2). Using Lemmas 3.1 and 3.2 we can easily deduce that

\[
\int_{[-1,1]} \left| f'(x) \right|^p \, dx \leq 2(1-\delta_{n,k}) \max_{-1 \leq x \leq 1} \left| f'(x) \right|^p
\leq 2(n(k+1))^{-1}(c_{19} n(k+1))^p \max_{-1 \leq x \leq 1} \left| f(x) \right|^p
\leq 2(n(k+1))^{-1}(c_{19} n(k+1))^p c_{20}(1+p^2)n(k+1) \int_{-1}^{1} \left| f(x) \right|^p \, dx
\leq c_{23}(p) (n(k+1))^p \int_{-1}^{1} \left| f(x) \right|^p \, dx, \tag{49}
\]

where \( c_{23}(p) := 2c_{19}c_{20}(1+p^2) \), for every \( f \in \mathcal{P}_n \) of the form (2). Now (48) and (49) yield the theorem.

**Case 2: 0 < p < 1.** Let \( u := [p^{-1}] \). Since \( 0 < p < 1 \), we have \( |a+b|^p \leq |a|^p + |b|^p \) for any two real numbers \( a \) and \( b \). Combining this with the product rule and Corollary 2, we obtain

\[
\int_{-\pi}^{\pi} \left| f'(t) \sin^u t \right|^p \, dt
\leq \int_{-\pi}^{\pi} \left| \frac{d}{dt} (f(t) \sin^u t) \right|^p \, dt + \int_{-\pi}^{\pi} \left| f(t) u \sin^{u-1} t \cos t \right|^p \, dt
\leq c_2(p) ((n+u)(k+u+1))^{p/2} \int_{-\pi}^{\pi} \left| f(t) \sin^u t \right|^p \, dt + u \int_{-\pi}^{\pi} \left| f(t) \sin^{u-1} t \right|^p \, dt, \tag{50}
\]
for every \( f \in \mathcal{F}_n \) of the form (1). Substituting \( x = \cos t \), we get

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} |f'(x)| (1-x^2)^{(u+1)\frac{p}{2}-1/2} \, dx
\leq c_6(p) ((n+u)(k+u+1))^{p/2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^p (1-x^2)^{u\frac{p}{2}-1/2} \, dx
\]

\[
+ u \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^p (1-x^2)^{(u-1)\frac{p}{2}-1/2} \, dx
\]  

(51)

for every \( f \in \mathcal{D}_n \) of the form (2). Observe that \( 0 < p < 1 \) and \( u = \lceil p^{-1} \rceil \) imply that

\[
\frac{1}{2}(u+1)p - \frac{1}{2} \geq 0, \quad (52)
\]

\[
-\frac{1}{2} < \frac{1}{2}(u-1)p - \frac{1}{2} < \frac{1}{2}up - \frac{1}{2} \leq 0. \quad (53)
\]

Let \( \delta_{n,k} := 1 - (n(k+1))^{-1} \). Using (51), (52), (53), and Lemma 3.3, we can deduce that

\[
\int_{-\delta_{n,k}}^{\delta_{n,k}} |f'(x)|^p \, dx
\leq (n(k+1))^{(u+1)\frac{p}{2}-1/2} \int_{-\delta_{n,k}}^{\delta_{n,k}} |f'(x)|^p (1-x^2)^{(u+1)\frac{p}{2}-1/2} \, dx
\]

\[
\leq (n(k+1))^{(u+1)\frac{p}{2}-1/2} c_6(p) ((n+u)(k+u+1))^{p/2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^p (1-x^2)^{u\frac{p}{2}-1/2} \, dx
\]

\[
+ (n(k+1))^{(u+1)\frac{p}{2}-1/2} u \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^p (1-x^2)^{(u-1)\frac{p}{2}-1/2} \, dx
\]

\[
\leq c_6(p) (1+p^{-1})^{28} c_{21}(n(k+1))^{(u+1)\frac{p}{2}-1/2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^p \, dx
\]

\[
+ 4c_{21} p^{-1} (n(k+1))^{(u+1)\frac{p}{2}-1/2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^p \, dx
\]

\[
\leq c_{26}(p) (n(k+1))^p \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^p \, dx
\]  

(54)

for every \( f \in \mathcal{D}_n \) of the form (2), where \( c_{26}(p) := c_6(p) (1+p^{-1})^{28} c_{21} + 4c_{21} p^{-1} \leq c_{25} p^{-2} \) with a suitable absolute constant \( c_{25} > 0 \). Further, using Lemmas 3.1 and 3.2 we get

\[
\int_{-\delta_{n,k}}^{\delta_{n,k}} |f'(x)|^p \, dx \leq 2(1-\delta_{n,k}) \max_{-\delta_{n,k}} |f'(x)|^p
\leq 2(n(k+1))^{-1} (c_{19} n(k+1))^p \max_{-\delta_{n,k}} |f(x)|^p
\leq 2(n(k+1))^{-1} (c_{19} n(k+1))^p 2c_{20} n(k+1) \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^p \, dx
\]

\[
\leq c_{26}(n(k+1))^p \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x)|^p \, dx,
\]  

(55)

where \( c_{26} := 4c_{20}(1+c_{19}) \) for every \( f \in \mathcal{D}_n \) of the form (2). Now (54) and (55) give the theorem.
MARKOV AND BERNSTEIN TYPE INEQUALITIES IN $L_p$

We prove only Example 5, the proof of Example 4 is quite similar.

**Proof of Example 5.** From [5, Corollary 3.3] it follows that there are $G_{n,k} \in \mathcal{P}_n$ of the form

$$G_{n,k}(x) = (1+x)^n Q_{n,k}(x) \quad \text{with} \quad Q_{n,k} \in \mathcal{P}_k$$

such that

$$\int_{-1}^{1} (G_{n,k}(x))^2 \, dx = 1, \quad (57)$$

$$|G'_{n,k}(1)|^2 \geq c_{27}(n(k+1))^3, \quad (58)$$

$$|G_{n,k}(1)|^2 \geq c_{28}n(k+1) \quad (59)$$

for all integers $0 \leq k \leq n$, where $c_{27} > 0$ and $c_{28} > 0$ are absolute constants. Let $u := [2p^{-1}] + 1$, $\tilde{n} := [n/u]$ and $\tilde{k} := [k/u]$. We distinguish two cases.

**Case 1:** $\tilde{n} \geq 1$. Let $H_{n,k,p}(x) := (G_{n,k})^u$. Obviously $H_{n,k,p} \in \mathcal{P}_n$ and it is of the form (5) for all integers $0 \leq k \leq n$ and for all $p > 0$. Using Lemmas 3.2 and 3.1, we obtain

$$\int_{-1}^{1} |H'_{n,k,p}(x)|^p \, dx \geq (c_{29}(1+p^2)n(k+1))^{-1} \max_{-1 \leq x \leq 1} |H'_{n,k,p}(x)|^p$$

$$\geq (c_{29}(1+p^2))^{-1}(n(k+1))^{-1}u \max_{-1 \leq x \leq 1} |(G_{n,k}(x))^u G'_{n,k}(x)|^p$$

$$\geq (c_{29}(1+p^2))^{-1}(n(k+1))^{-1}uc_{27}^{(u-1)p}\tilde{n}(\tilde{k}+1) (\tilde{n}(\tilde{k}+1))^{(u+2)p/2} \geq c_{30}^{p+1} (p+1)^{-1}(n(k+1))^{(u+2)p/2-1} \quad (60)$$

with a suitable absolute constant $c_{29} > 0$. Further, $u \tilde{n} \geq 2$, Lemma 3.2 and (57) imply that

$$\int_{-1}^{1} |H_{n,k,p}(x)|^p \, dx = \int_{-1}^{1} |G_{n,k}(x)|^u \, dx$$

$$\leq \int_{-1}^{1} (G_{n,k}(x))^2 \, dx \max_{-1 \leq x \leq 1} |G_{n,k}(x)|^{u-2}$$

$$\leq \left( c_{29}(1+p^2) \tilde{n}(\tilde{k}+1) \int_{-1}^{1} (G_{n,k}(x))^2 \, dx \right)^{(u-2)/2}$$

$$\leq c_{30}^{p+1}(n(k+1))^{(u+2)p/2-1} \quad (61)$$

with a suitable absolute constant $c_{30} > 0$ which, together with (60), gives the desired result.

**Case 2:** $\tilde{n} = 0$. Then $n < u < 2p^{-1}+1$. Let $H_{n,k,p}(x) := 1 + x$ if $n > 0$, and $H_{n,k,p} := 1$ if $n = 0$. A simple calculation yields the desired inequality.

**References**


