The Average Norms of Polynomials

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Average Norms of Polynomials

Let $n \geq 0$ be any integer and

$$
\mathcal{F}_n := \left\{ \sum_{i=0}^{n} a_i z^i : a_i = 0, \pm 1 \right\}
$$

be the polynomials of height 1 and degree $n$. Let

$$
\beta_n(m) := \frac{1}{3^n+1} \sum_{P \in \mathcal{F}_n} \|P\|_m^m.
$$

Here $\|P\|_m^m$ is the $mth$ power of the $L_m$ norm on the boundary of the unit disc.

So $\beta_n(m)$ is the average of the $mth$ power of the $L_m$ norm over $\mathcal{F}_n$. 

Typical of the results we get is is:

**Theorem**  \(\text{For } n \geq 0, \text{ we have}\)

\[
\beta_n(2) = \frac{2}{3}(n + 1),
\]

\[
\beta_n(4) = \frac{8}{9} n^2 + \frac{14}{9} n + \frac{2}{3}
\]

and

\[
\beta_n(6) = \frac{16}{9} n^3 + 4n^2 + \frac{26}{9} n + \frac{2}{3}.
\]
The Littlewood polynomials \( \mathcal{L}_n \) are defined as follows:

\[
\mathcal{L}_n := \left\{ \sum_{i=0}^{n} a_i z^i : a_i = \pm 1 \right\}.
\]

Now let

\[
\mu_n(m) := \frac{1}{2n+1} \sum_{P \in \mathcal{L}_n} \|P\|^m
\]

be the average of the \( m \)th power of the \( L_m \) norms over \( \mathcal{L}_n \).

Here

\[
\|P\|^m := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |P(z)|^m d\theta \right\}^{\frac{1}{m}}, \quad (z = e^{i\theta})
\]
We are interested in finding exact formulae for $\mu_n(m)$.

**Theorem**  
*For $n \geq 0$, we have*

$$
\mu_n(2) = n + 1,
$$

$$
\mu_n(4) = 2n^2 + 3n + 1,
$$

$$
\mu_n(6) = 6n^3 + 9n^2 + 4n + 1
$$

*and*

$$
\mu_n(8) =
$$

$$
24n^4 + 30n^3 + 4n^2 + 5n + 4 - 3(-1)^n.
$$
That $\mu_n(2) = n+1$ is trivial since $\|P\|_2 = n + 1$ for each $P \in \mathcal{L}_n$.

The above result for $\mu_n(4)$ is due to Newman and Byrnes.

The results for $\mu_n(6)$ and $\mu_n(8)$ are new and are the tip of an iceberg.

What is striking, and perhaps surprising, is that such exact formulae exist at all.
One interesting generalization is the following. Let

\[ \mathcal{F}_n(H) := \left\{ \sum_{i=0}^{n} a_i z^i : |a_i| \leq H, a_i \in \mathbb{Z} \right\} \]

be the set of all the polynomials of height \( H \) of degree \( \leq n \). Let

\[ \beta_n(m, H) := \frac{1}{(2H + 1)^{n+1}} \sum_{P \in \mathcal{F}_n(H)} \|P\|_m^m \]

Then

\[
\beta_n(4, H) = \frac{2}{9} H^2 (H + 1)^2 n^2 \\
+ \frac{1}{45} H(H + 1)(19H^2 + 19H - 3)n \\
+ \frac{1}{15} H(H + 1)(3H^2 + 3H - 1). 
\]
There are many limiting results concerning expected norms.

The expected norms of random Littlewood polynomials, $q_n$ of degree $n$, satisfy

$$\frac{\mathbb{E}(\|q_n\|_p)}{n^{1/2}} \to (\Gamma(1 + p/2))^{1/p}$$

and for their derivatives

$$\frac{\mathbb{E}(\|q_n^{(r)}\|_p)}{n^{(2r+1)/2}} \to (2r + 1)^{-1/2}(\Gamma(1 + p/2))^{1/p}.$$
There is a considerable literature on the maximum and minimum norms of polynomials in \( \mathcal{L}_n \). In the \( L_4 \) norm this problem is often called Golay’s “Merit Factor” problem.

The specific old and difficult problem is to find the minimum possible \( L_4 \) norm of a polynomial in \( \mathcal{L}_n \). The cognate problem in the supremum norm is due to Littlewood.

Both of these problems are at least 50 years old and neither is solved.
As before let \( n \geq 0 \) be any integer and
\[
\mathcal{L}_n := \left\{ \sum_{i=0}^{n} a_i z^i : a_i = \pm 1 \right\}
\]
be the set of all Littlewood polynomials of degree \( n \). Let
\[
\mu_n(m) := \frac{1}{2^{n+1}} \sum_{P \in \mathcal{L}_n} \|P\|_m^m
\]
be the average of the \( m \)th power of the \( L_m \) norm over \( \mathcal{L}_n \). We are interested in finding exact formulae for \( \mu_n(m) \).
For any complex number $z$ on the unit circle and any real number $h$, we have

$$|z + h|^2 + |z - h|^2 = 2(|z|^2 + h^2)$$

and

$$|z + h|^4 + |z - h|^4 = 2(|z|^4 + 4h^2|z|^2 + h^4 + h^2(z^2 + \bar{z}^2))$$

With similar more complicated expressions for sixth powers and eighth powers.
Hence for any polynomial \( P(z) \),
\[
\|zP(z) + h\|_2^2 + \|zP(z) - h\|_2^2 = 2(\|P(z)\|_2^2 + h^2)
\]
and
\[
\|zP(z) + h\|_4^4 + \|zP(z) - h\|_4^4 =
2(\|P(z)\|_4^4 + 4h^2\|P(z)\|_2^2 + h^4)
\]

We also have, for induction purposes,
\[
\sum_{P \in \mathcal{L}_n} \|P\|_m^m =
\sum_{P \in \mathcal{L}_{n-1}} (\|zP(z) + 1\|_m^m + \|zP(z) - 1\|_m^m).
\]
Theorem  \( \text{For } n \geq 0, \text{ we have} \)

\[
\mu_n(2) = n + 1, \\
\mu_n(4) = 2n^2 + 3n + 1 \\
\text{and} \\
\mu_n(6) = 6n^3 + 9n^2 + 4n + 1.
\]

The first formula is trivial because every Littlewood polynomial of degree \( n \) has constant \( L_2 \) norm \( \sqrt{n+1} \). However we give a simple inductive proof because it is indicative of the basic method behind all the proofs.
Using the above formulae with $m = 2$ for the 2 norm, we have

$$
\mu_n(2) =
$$

$$
\frac{1}{2^{n+1}} \sum_{P \in \mathcal{L}_{n-1}} \left( \|zP(z) + 1\|_2^2 + \|zP(z) - 1\|_2^2 \right)
$$

$$
= \frac{1}{2^{n+1}} \sum_{P \in \mathcal{L}_{n-1}} 2(\|P\|_2^2 + 1)
$$

$$
= \mu_{n-1}(2) + 1
$$

for any $n \geq 1$. It is clear that $\mu_0(m) = 1$ for any $m$. Thus we have

$$
\mu_n(2) = \mu_0(2) + n = n + 1.
$$
With \( m = 4 \), for the 4 norm

\[
\mu_n(4) = \frac{1}{2^{n+1}} \sum_{P \in \mathcal{L}_{n-1}} 2 \left( \|P\|_4^4 + 4 \|P\|_2^2 + 1 \right)
\]

\[
= \mu_{n-1}(4) + 4\mu_{n-1}(2) + 1
\]

for any \( n \geq 1 \).

For the 6 norm we need two lemmas.

**Lemma** For \( m \neq 0 \), we have

\[
\sum_{P \in \mathcal{L}_n} \int_0^{2\pi} |P(z)|^2 z^m d\theta = 0.
\]

**Lemma** For \( m \geq 1 \), we have

\[
\sum_{P \in \mathcal{L}_n} \int_0^{2\pi} |P(z)|^2 z^m P(z)^2 d\theta = 0.
\]
Let \( n \geq 0 \) be any integer and

\[
\mathcal{F}_n := \left\{ \sum_{i=0}^{n} a_i z^i : a_i = 0, \pm 1 \right\}
\]

be the set of all the polynomials of height 1 of degree \( n \). Let

\[
\beta_n(m) := \frac{1}{3^{n+1}} \sum_{P \in \mathcal{F}_n} \|P\|_m^m.
\]

We can also obtain exact formulae for \( \beta_n(m) \). The additional details involve observing that since

\[
\|zP(z) + 0\|_m^m = \|P(z)\|_m^m
\]

The previous equations can be extended easily to allow summing over all the
height one polynomials. For example, for any polynomial \( P(z) \)

\[
\|zP(z) + h\|_4^4 + \|zP(z) - h\|_4^4 + \|zP(z) + 0\|_4^4
\]

\[
= 3\|P(z)\|_4^4 + 8h^2\|P(z)\|_2^2 + 2h^4.
\]

**Theorem**  
For \( n \geq 0 \), we have

\[
\beta_n(2) = \frac{2}{3}(n + 1),
\]

\[
\beta_n(4) = \frac{8}{9}n^2 + \frac{14}{9}n + \frac{2}{3}
\]

and

\[
\beta_n(6) = \frac{16}{9}n^3 + 4n^2 + \frac{26}{9}n + \frac{2}{3}.
\]
It is worth noting that the above technique can also be used to compute the averages of the norms of polynomials of height $H$. For example, one can show the following. Let $n \geq 0$ and $H \geq 1$ be integers and let

$$
\mathcal{F}_n(H) := \left\{ \sum_{i=0}^{n} a_i z^i : |a_i| \leq H, a_i \in \mathbb{Z} \right\}
$$

be the set of all the polynomials of height $H$ and degree $\leq n$. Let

$$
\beta_n(m, H) := \frac{1}{(2H + 1)^{n+1}} \sum_{P \in \mathcal{F}_n(H)} \|P\|_m^m.
$$
Theorem  For $n \geq 0$ and $H \geq 1$, we have

$$\beta_n(2, H) = \frac{1}{3} H(H + 1)(n + 1),$$

$$\beta_n(4, H) = \frac{2}{9} H^2(H + 1)^2 n^2$$

$$+ \frac{1}{45} H(H + 1)(19H^2 + 19H - 3)n$$

$$+ \frac{1}{15} H(H + 1)(3H^2 + 3H - 1)$$
\[ \beta_n(6, H) = \frac{2}{9}H^3(H + 1)^3n^3 \\
+ \frac{1}{5}H^2(H + 1)^2(3H^2 + 3H - 1)n^2 \\
+ \frac{1}{315}H(H + 1)(164H^4 + 328H^3 + 56H^2 - 108H + 15)n \\
+ \frac{1}{21}H(H + 1)(3H^4 + 6H^3 - 3H + 1). \]
Derivative and reciprocal polynomials

If we replace $z$ by $z/w$ in the critical identities then we have homogeneous forms like

$$|z + hw|^4 + |z - hw|^4$$

$$= 2(|z|^4 + 4h^2|z^2|w^2 + h^4|w|^4$$

$$+ h^2|w|^4\left(\left(\frac{z}{w}\right)^2 + \left(\frac{\bar{z}}{w}\right)^2\right)).$$

Let $P^{(m)}(z)$ be the $m$th derivative of $P(z)$. 

Theorem \quad For \quad n \geq 0, \quad we \quad have, \quad for \\
m \leq n \\

\[
\frac{1}{2^{n+1}} \sum_{P \in \mathcal{L}_n} \|P^{(m)}\|_2^2 \\
= m!^2 \sum_{l=m}^{n} \binom{l}{m}^2 ;
\]

and

\[
\frac{1}{2^{n+1}} \sum_{P \in \mathcal{L}_n} \|P^{(m)}\|_4^4 \\
= 2m!^4 \left( \sum_{l=m}^{n} \binom{l}{m}^2 \right)^2 - m!^4 \sum_{l=m}^{n} \binom{l}{m}^4 .
\]
A polynomial $P(z)$ of degree $n$ is rec
iprocal if $P(z) = P^*(z)$ where $P^*(z) = z^n P \left( \frac{1}{z} \right)$. Now

$$
\|P(z) + z^{n+1} P^*(z)\|_4^4 + \|P(z) - z^{n+1} P^*(z)\|_4^4
$$

$$
= 12\|P\|_4^4.
$$

This lets us prove that if $n$ is odd the average $\|P\|_4^4$ over the reciprocal Littlewood polynomials in $\mathcal{L}_n$ is

$$
3n^2 + 3n
$$

if $n$ is odd and

$$
3n^2 + 3n + 1
$$

if $n$ is even.
The Complex Case

Lemma  For $m \geq l \geq 0$, we have
\[
\min(l, m-l) \sum_{j=0}^{\min(l, m-l)} \binom{m}{2j} \binom{2j}{j} \binom{m-2j}{l-j} = \binom{m}{l}^2.
\]

Lemma  Let $1 \leq m < k$ and $\zeta_k = e^{\frac{2\pi i}{k}}$. Then for any complex number $z$, we have
\[
\sum_{j=0}^{k-1} |z + \zeta_j|^2m = k \sum_{l=0}^{m} \binom{m}{l}^2 |z|^{2l}.
\]
Let \( n \geq 0 \) and
\[
\mathcal{L}_{n,k} := \left\{ \sum_{i=0}^{n} a_i z^i : a_i^k = 1 \right\}
\]
be the set of all polynomials of degree \( \leq n \) whose coefficients are \( k \)th root of unity.

**Theorem**  For \( n \geq 0 \) and \( m < k \), we have
\[
\frac{1}{kn+1} \sum_{P \in \mathcal{L}_{n,k}} \| P \|_{2m}^{2m}
\]
\[
= \sum_{l_1=0}^{m} \sum_{l_2=0}^{l_1} \cdots \sum_{l_n=0}^{l_{n-1}} \left( \frac{m}{l_1} \right)^2 \left( \frac{l_1}{l_2} \right)^2 \cdots \left( \frac{l_{n-1}}{l_n} \right)^2.
\]