SOME HIGHLY COMPUTATIONAL PROBLEMS CONCERNING INTEGER POLYNOMIALS OF SMALL NORM.

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The PTE problem is solved by two sets of \( n \) integers satisfying any of the following:

\[
\sum_{i=1}^{n} \alpha_i^j = \sum_{i=1}^{n} \beta_i^j \quad j = 1, \ldots, n - 1
\]

\[
\prod_{i=1}^{n} (x - \alpha_i) - \prod_{i=1}^{n} (x - \beta_i) = C
\]

\[
(x - 1)^n \left| \sum_{i=1}^{n} x^{\alpha_i} - \sum_{i=1}^{n} x^{\beta_i} \right|
\]

The conjecture due to Wright (and others) is that it is always possible.

Almost equivalently (though not quite obviously) find a polynomial with coefficients \( \{0, -1, +1\} \) with

\[
\|p\|_{L^2}\{|z|=1\} = \sqrt{2n}.
\]
Partial History.

- Euler
- Prouhet (1851)
- Tarry (1910) - Small Examples
- Escott (1910) - Small Examples
- Wright, Fuchs (1935) - Easier Waring
- Letac (1941) - Size 9 and 10
- Gloden (1946) - Size 9 and 10
- Smyth (Math Comp. 1991) - Size 10
An **even ideal symmetric solution** of size $n$ is of the form

$$\{ \pm \alpha_1, \ldots, \pm \alpha_{n/2} \}, \{ \pm \beta_1, \ldots, \pm \beta_{n/2} \}$$

It satisfies

$$\sum_{i=1}^{k/2} \alpha_i^{2j} = \sum_{i=1}^{k/2} \beta_i^{2j} \quad j = 1, \ldots, \frac{k - 2}{2}$$

An **odd ideal symmetric solution** of size $n$ is of the form

$$\{ \alpha_1, \ldots, \alpha_k \}, \{- \alpha_1, \ldots, - \alpha_k \}$$

and satisfies

$$\sum_{i=1}^{k} \alpha_i^j = 0 \quad j = 1, 3, 5, \ldots, k - 2$$
Listed below are ideal symmetric solutions for sizes $2 \leq n \leq 10$, the odd symmetric solutions are all perfect. These solutions are listed in abbreviated symmetric form. For example the solution for size 6 is

$$\{\pm4, \pm9, \pm13\}, \{\pm1, \pm11, \pm12\}$$

and the solution for size 5 is

$$\{-8, -7, 1, 5, 9\}, \{8, 7, -1, -5, -9\}.$$ 

2 : $\{3\}, \{1\}$

3 : $\{-2, -1, 3\}$

4 : $\{3, 11\}, \{7, 9\}$

5 : $\{-8, -7, 1, 5, 9\}$
6 : \{4, 9, 13\}, \{1, 11, 12\}

7 : \{-51, -33, -24, 7, 13, 38, 50\}

8 : \{2, 16, 21, 25\}, \{5, 14, 23, 24\}

9 : \{-98, -82, -58, -34, 13, 16, 69, 75, 99\}

9 : \{-169, -161, -119, -63, 8, 50, 132, 148, 174\}

10 : \{436, 11857, 20449, 20667, 23750\},

\{12, 11881, 20231, 20885, 23738\}

10 : \{133225698289, 189880696822, 338027122801, 
\ 432967471212, 529393533005\},

\{87647378809, 243086774390, 308520455907, 
\ 441746154196, 527907819623\}
Size 5. The following is a one parameter example of size 5.

\[ F_5 := \]

\[(t + 2m^2) (t - 1) (t + 2m^2 - 1) \]
\[(t - 2m^2 + 1 - m) (t - 2m^2 + m + 1) \]
\[ - (t - 2m^2) (t + 1) (t - 2m^2 + 1) \]
\[(t + 2m^2 - 1 + m) (t + 2m^2 - m - 1) \]

This expands to

\[ F_5 := -4m^2 (m - 1) (2m + 1) (2m - 1) \]
\[(m + 1) (2m^2 - 1) \]
Size 6. It is possible to completely solve the even symmetric problem of size 6 in Maple. Basically one just uses “solve”. it gives the following general solution (translated with $a_6 = 0$.)

This gives as rational solution of size 6:

\[
\begin{align*}
\{a_2 &= a_2, b_1 = b_1, b_3 = b_3, \\
b_2 &= \frac{a_2^2 - a_2 b_3 + b_1 b_3 - a_2 b_1}{-b_1 - b_3 + a_2}, \\
a_1 &= 2/3 \frac{a_2^2 - b_3^2 - b_1^2 - b_1 b_3}{-b_1 - b_3 + a_2}, \\
a_3 &= \frac{-b_1^2 - b_1 b_3 + a_2 b_1 + a_2 b_3 - b_3^2}{-b_1 - b_3 + a_2}
\end{align*}
\]
The following is a simple two parameter example of size 6.

\[ F_6 := \]

\[
\left( t^2 - (2n + 2m)^2 \right) \left( t^2 - (nm + n + m - 3)^2 \right) \\
\left( t^2 - (nm - n - m - 3)^2 \right) \\
- \left( t^2 - (2n - 2m)^2 \right) \left( t^2 - (-nm + n - m - 3)^2 \right) \\
\left( t^2 - (-nm - n + m - 3)^2 \right)
\]

On expansion one sees that

\[ F_6 := -16 \, nm \, (m - 1) \, (m + 3) \, (m - 3) \, (m + 1) \]

\[
(n - 1) \, (n + 3) \, (n - 3) \, (n + 1)
\]
**Size 7. Gloden simplified.**

\[
(t - R_1) (t - R_2) (t - R_3) (t - R_4) \\
(t - R_5) (t - R_6) (t - R_7) \\
-(t + R_1) (t + R_2) (t + R_3) (t + R_4) \\
(t + R_5) (t + R_6) (t + R_7)
\]

where

\[
R_1 := - \left(-3 \, j^2 k + k^3 + j^3\right) \left(j^2 - kj + k^2\right)
\]

\[
R_2 := (j + k) (j - k) \left(j^2 - 3 k j + k^2\right) j
\]

\[
R_3 := (j - 2 k) \left(j^2 + k j - k^2\right) k j
\]

\[
R_4 := - (j - k) \left(j^2 - kj - k^2\right) \left(-k + 2 j\right) k
\]

\[
R_5 := - (j - k) \left(-2 k j^3 + j^4 - j^2 k^2 + k^4\right)
\]

\[
R_6 := \left(j^4 - 4 k j^3 + j^2 k^2 + 2 k^3 j - k^4\right) k
\]

\[
R_7 := \left(j^4 - 4 k j^3 + 5 j^2 k^2 - k^4\right) j
\]
On expansion

\[ F_7 = 2 \, j^3 \, k^3 \, (-k + 2 \, j) \, (j - 2 \, k) \, (j + k) \]

\[ (j^2 + kj - k^2) \, (j^2 - kj - k^2) \, (j^2 - 3 \, kj + k^2) \]

\[ (-3 \, j^2 \, k + k^3 + j^3) \, (j^4 - 4 \, kj^3 + 5 \, j^2 \, k^2 - k^4) \]

\[ (-2 \, kj^3 + j^4 - j^2 \, k^2 + k^4) \, (j - k)^3 \]

\[ (j^4 - 4 \, kj^3 + j^2 \, k^2 + 2 \, k^3 \, j - k^4) \, (j^2 - kj + k^2) \]

For example with \( j := 2 \) and \( k := 3 \)

\[ (t - 7) \, (t - 50) \, (t + 24) \, (t + 33) \]

\[ (t - 13) \, (t + 51) \, (t - 38) \]

\[ - (t + 7) \, (t + 50) \, (t - 24) \, (t - 33) \]

\[ (t + 13) \, (t - 51) \, (t + 38) \]

\[ = 13967553600 \]
Size 8. A (homogenous) size 8 solution due to Chernick

\[ F_8 := (t^2 - R_1^2) (t^2 - R_2^2) (t^2 - R_3^2) (t^2 - R_4^2) \]
\[ - (t^2 - R_5^2) (t^2 - R_6^2) (t^2 - R_7^2) (t^2 - R_8^2) \]

where

\[ R_1 := 5m^2 + 9mn + 10n^2 \]
\[ R_2 := m^2 - 13mn - 6n^2 \]
\[ R_3 := 7m^2 - 5mn - 8n^2 \]
\[ R_4 := 9m^2 + 7mn - 4n^2 \]
\[ R_5 := 9m^2 + 5mn + 4n^2 \]
\[ R_6 := m^2 + 15mn + 8n^2 \]
\[ R_7 := 5m^2 - 7mn - 10n^2 \]
\[ R_8 := 7m^2 + 5mn - 6n^2 \]
**Size 9.** We know no parametric solution of size 9. Indeed only two solutions are known. Both are symmetric and they are the following

$$[-98, -82, -58, -34, 13, 16, 69, 75, 99]$$

and

$$[174, 148, 132, 50, 8, -63, -119, -161, -169]$$
**Size 10.** The following size 10 example is due to Letac (and Smyth)

\[ F_{10} := \]

\[ (t^2 - R_1^2) (t^2 - R_2^2) (t^2 - R_3^2) \]
\[ (t^2 - R_4^2) (t^2 - R_5^2) \]
\[ - (t^2 - R_6^2) (t^2 - R_7^2) (t^2 - R_8^2) \]
\[ (t^2 - R_9^2) (t^2 - R_{10}^2) \]

where

\[ R_1 := (4n + 4m) \]
\[ R_2 := (mn + n + m - 11) \]
\[ R_3 := (mn - n - m - 11) \]
\[ R_4 := (mn + 3n - 3m + 11) \]
\[ R_5 := (mn - 3n + 3m + 11) \]
\[ R_6 := (4n - 4m) \]
\[ R_7 := (-mn + n - m - 11) \]
\[ R_8 := (-mn - n + m - 11) \]
\[ R_9 := (-mn + 3n + 3m + 11) \]
\[ R_{10} := (-mn - 3n - 3m + 11) \]

On expansion

\[ F_{10} := c_0 + c_2 t^2 + c_4 t^4 + c_6 t^6 \]

And each coefficient except \( c_0 \) has a factor

\[ m^2 n^2 - 13 n^2 + 121 - 13 m^2 \]

So any solution of the above biquadratic gives a size 10 solution. For example:

\[ n := 153/61 \text{ and } m = 191/79 \]

\[ n := -296313/249661 \text{ and } m = -1264969/424999 \]