An investigation of a nonlocal hyperbolic model for self-organization of biological groups

Introduction

Patterns of schooling fish, swarming insects or zigzagging flocks of birds are as mathematically complex as they are visually stimulating. This project uses a nonlocal Eulerian Hyperbolic model. The Eulerian approach uses the population density to decide which way the individuals move or turn. The hyperbolic models focus on the turning behaviour of the individuals. There are three key areas which influence the turning rates. These are attraction, repulsion, and alignment. The domain of influence is given by Figure 1. The individual in the middle will feel the strongest force to the individuals some distance in front or behind, but it will also feel a small force from all others, hence nonlocal. The odd symmetry ensures equal but opposite forces. This model is in one dimension, but it is sophisticated enough to get patterns that are observed in higher dimensions. This model also has a constant velocity, which puts a lot of importance on the turning rates. Later we will change the velocity to be dependent on attraction and repulsion so the individuals can speed up or slow down to approach neighbors or avoid collisions.

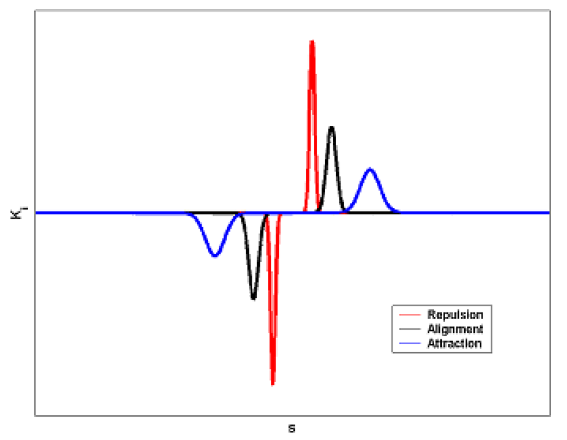


Figure 1: Three Kernels: K_r,K_{al},K_a.

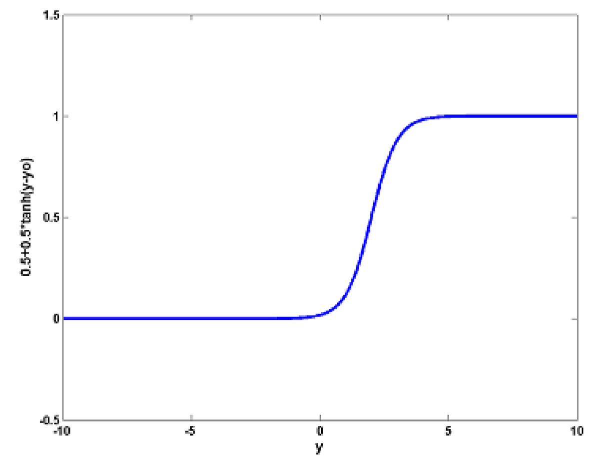


Figure 2: The turning function

Model

The model depends on the conservation laws for the densities of left moving(u^-) and right moving(u^+) individuals.

$$\partial_t u^+ + \partial_x(\gamma u^+) = -\lambda^+(u^+,u^-)u^+ + \lambda^-(u^+,u^-)u^- \quad (1)$$

$$\partial_t u^- - \partial_x(\gamma u^-) = +\lambda^+(u^+,u^-)u^+ - \lambda^-(u^+,u^-)u^- \quad (2)$$

Where γ is the constant velocity, λ^+ is the turning rate of right moving individuals that will change direction, and λ^- is the turning rate of left moving individuals that will turn to the right. These turning rates($\lambda^\pm$) are required to be bounded, positive and monotone functions of the social interations. Shown by Figure 2 and defined as:

$$\lambda^\pm(u^+,u^-) = a + b(0.5 + 0.5tanh(y^\pm(u^+,u^-) - y_0) \quad (3)$$

Where a is the random turning rate and b is the biased turning rate. And where $y^\pm$ is given by:

$$y^\pm(u^+,u^-) = \sum_i q_i \int_0^\infty K_i(z)(u^\mp(x+z,t) - u^\pm(x-z,t))dz \quad i = a, r, al \quad (4)$$

or,

$$y^\pm(u^+,u^-) = \sum_i q_i \int_0^\infty K_i(z)(u(x+z,t) - u(x-z,t))dz \quad i = a, r, al \quad (5)$$

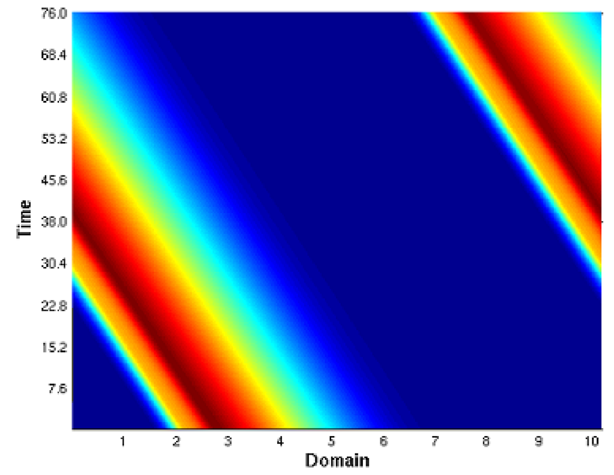
or a combination of these two as is explained in the next section. u is the total population density ($u = u^+ + u^-$). The q_i's are the strengths of the respective social interactions. The K_i is a Kernel given by the following:

$$K_i(s) = \frac{1}{\sqrt{2\pi m_i^2}} \exp(-(s-s_i)^2/(2m_i^2)), \quad s \in [0,\infty). \quad (6)$$

There are 14 parameters in this model. We will set 9 of them to be fixed. The only ones to vary are a,b,q_a,q_r, and q_{al}.The rest are set as: γ=0.1,$s_r = 0.25$,$s_al = 0.5$,$s_a = 1.0$, $m_i = s_i/8 (i = a, r, al)$,$yo = 2$, and $Amp = 2$. The s_i's are based on biological tendencies, simply $s_r < s_{al} < s_a$. The m_i's make the repulsion zone close and skinny and the attraction further away but wider, see Figure 1. yo is fixed to make the turning function $f(0) << 1$.

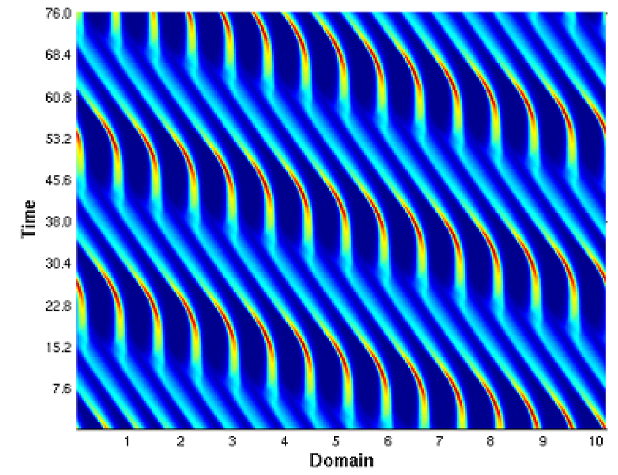
Variety of Models and Results

We use a pseudospectral method to discretize spatially, and the fourth order Runga-Kutta method to advance the solution in time. There are four different varieties of models. Model 1 will gather information from all individuals moving towards and away for attraction and repulsion, but only the individuals moving towards for alignment. The $y^\pm$ in this case is equation (5) but substituting the K_{al} from equation (4). Figure 3 is called a Traveling Pulse. The leading edge of the pulse is steep because the individuals at the front constantly turn back because of the high attraction to the clump then turn again because of the large alignment. Figure 4 is called Semi-Zigzag. This pattern has the individuals oscillating from being stationary to moving at a constant speed to one side.



$q_a = 1.6$
$q_r = 0.5$
$q_{al} = 2.0$
$a = 0.2$
$b = 0.9$

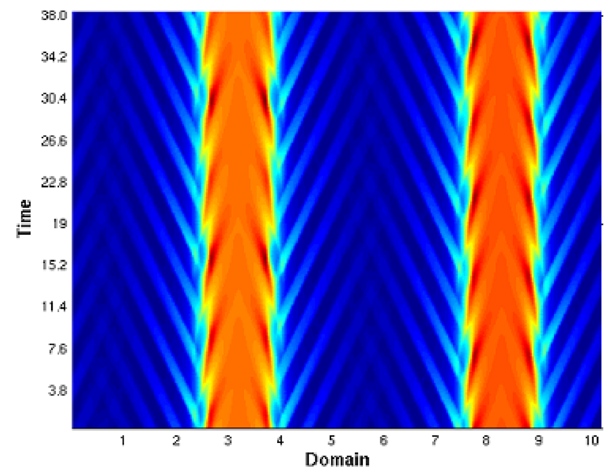
Figure 3: Traveling Pulse



$q_a = 0.0$
$q_r = 0.0$
$q_{al} = 2.7$
$a = 0.667$
$b = 3.0$

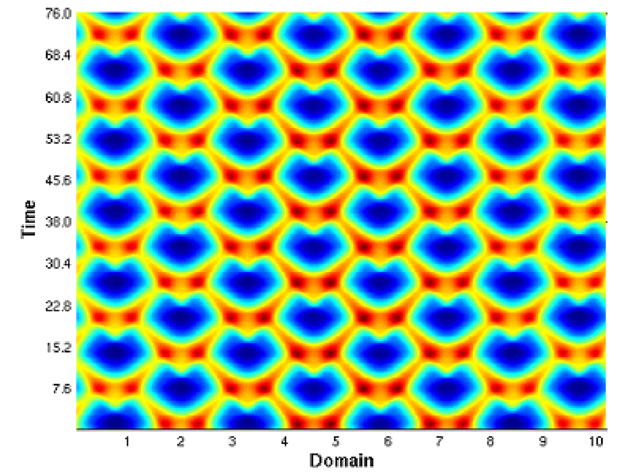
Figure 4: Semi-Zigzag

Model 2 will gather all information from individuals that are infront for all three social behaviours. The $y^\pm$ in this case is equation (5) but only taking the first term. Figure 5 shows what is called Feathers. This has a semi-constant pulse that has small pulses leaving the group left and right continually. These pulses merge with the other group, which keeps the clump at the same density. Model 3 is even more restricted. It gathers information from individuals infront and only moving towards. The $y^\pm$ is equation (4) but only taking the first term. Figure 6 shows what is called Ripples. When left moving and right moving individuals approach one another the majority of them turn around. But some go through and give an appearence of waves passing through one another.



$q_a = 6.0$
$q_r = 6.4$
$q_{al} = 0.0$
$a = 0.2$
$b = 0.9$

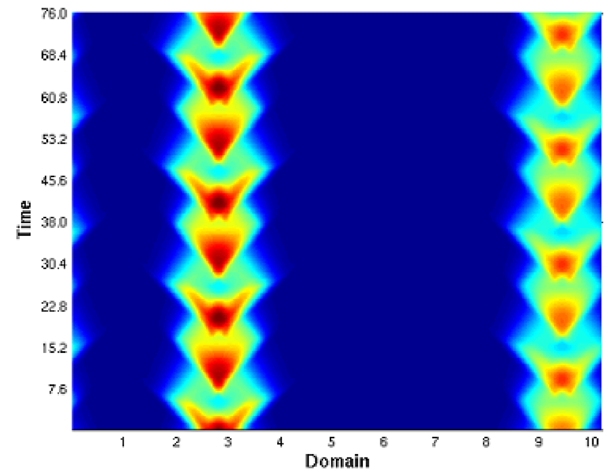
Figure 5: Feathers



$q_a = 1.5$
$q_r = 1.1$
$q_{al} = 2.0$
$a = 0.2$
$b = 0.9$

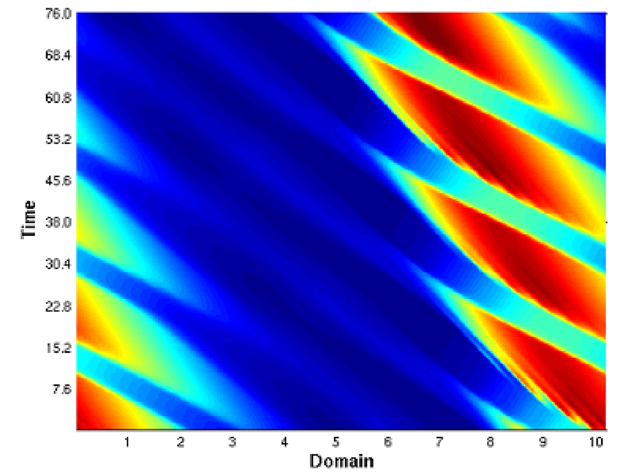
Figure 6: Ripples

Model 4 gathers information from individuals moving towards for all three social behaviours. The $y^\pm$ in this case is exactly equation (4). Figure 7 shows Breathers. This is caused by left moving and right moving individuals moving towards each other and having the forces balance out. Once they move through each other the individuals leading the group are attracted to the individuals moving in the same direction, and then they turn. Figure 8 shows Traveling Breathers. This is a combination of a Travling Pulse and a Traveling Train. There is one pulse moving slowly with large amplitude and many small pulses moving faster which are overlapping.



$q_a = 2.0$
$q_r = 1.0$
$q_{al} = 0.0$
$a = 0.2$
$b = 0.9$

Figure 7: Breathers



$q_a = 4.0$
$q_r = 4.0$
$q_{al} = 2.0$
$a = 0.2$
$b = 0.9$

Figure 8: Traveling Breathers

New Model

The New Model is very similar to the original. The only difference is the veolicity is no longer constant.

$$\partial_t u^+ + \partial_x(\Gamma^+ u^+) = -\lambda^+(u^+,u^-)u^+ + \lambda^-(u^+,u^-)u^- \quad (7)$$

$$\partial_t u^- - \partial_x(\Gamma^- u^-) = +\lambda^+(u^+,u^-)u^+ - \lambda^-(u^+,u^-)u^- \quad (8)$$

These density-dependent velocities are assumed to have the form:

$$\Gamma^\pm = \gamma(1 \pm tanh(K * u)) \quad (9)$$

where $*$ denotes convolution, and u is the total population density ($u = u^+ + u^-$). K is defined as:

$$K(x) = sgn(x)(-q_a e^{\frac{-|x|}{s_a}} + q_r e^{\frac{-|x|}{s_r}}) \quad (10)$$

Figure 9 shows a few examples of how this kernel looks like.

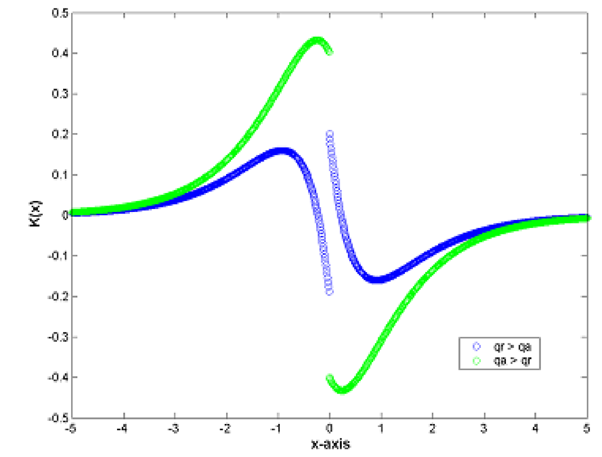


Figure 9: Two Kernels: $qr \geq$qa(q_r = 0.8,$q_a = 1.0$,$s_a = 1.0$,$s_r = 0.5$) and $qa > qr$ (q_r = 1.1,$q_a = 0.7$, $s_a = 1.0$, $s_r = 0.5$)

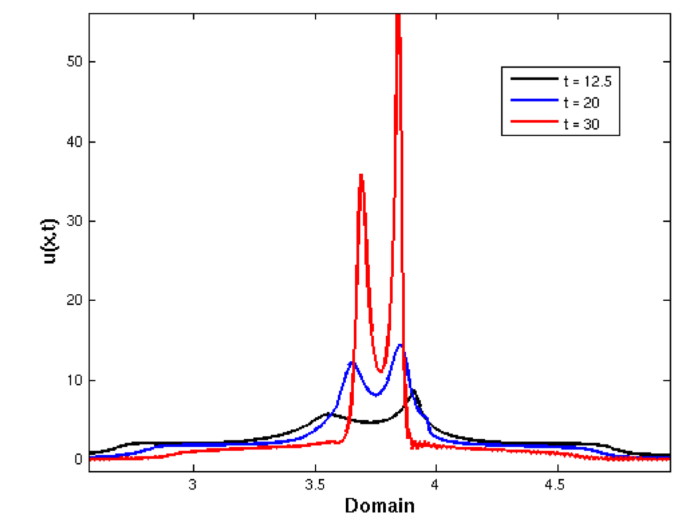
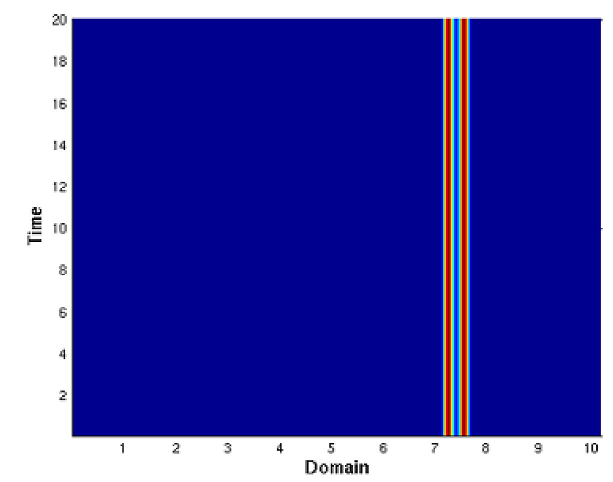


Figure 10: Blow up

There is a discontinuity at zero. This causes a lot of problems, mainly possible blow up. Much analysis has been done to find when there is blow up. Figure 10 is an example of this. We do know that if $q_r \geq q_a$ there is no blow up. One solution that has occured with the new model is the Stationary Pulse. This has been observed in the old model as well but was not shown. Figure 11 shows an example of this. This pattern shows two very dense stationary pulses. This pattern is in the category of $q_r \geq q_a$. It will not blow up because the all the individuals close by feel the force of being repelled stronger than the force of being attracted. But once it is out of the range for repulsion, it will attract and therefore reach a steady state. This addition of a density dependent velocity has added even more sophistication to the model and there shall be more new and interesting patterns emerging in the near future.



$q_a = 0.5$
$q_r = 0.6$
$q_{al} = 2.0$
$a = 0.2$
$b = 0.9$

Figure 11: Stationary Pulse

References

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Justin Meskas

Razvan Fetecau Raluca Eftimie Ralf Wittenberg

NSERC CRSNG