Two new Radial Basis Functions

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\textbf{Abstract}

The 3 most common radial basis functions for two dimensional interpolation are:

1. the Gaussian $\exp\left(\frac{1}{2} \cdot r^2\right)$
2. the Multiquadric $\sqrt{1 + r^2}$
3. the Thin Plate Spline $r^2 \cdot \ln(r)$

We have discovered 2 others, which are:

4. the tanh–rule weight function $\text{sech}(r)^2$
5. the tanh–sinh–rule weight function $\text{sech}\left(\frac{\pi}{2} \cdot \sinh(r)\right)^2 \cdot \cosh(r)$
Introduction

The most common radial basis function is the Gaussian. One can view the Gaussian as the weight function for the erf-rule quadrature formula. The weight function is the derivative of the variable transformation function. We noticed that the weight functions for the tanh-rule and the tanh-sinh-rule quadrature formulas also look like Gaussian curves. We may introduce a scale parameter $R$ by replacing $r$ with $\frac{r}{R}$ in the above formulas. We have chosen scale parameters so all 3 functions have the same definite integral. The chosen values are: for the Gaussian $R = (2\cdot\pi)^{\left(\frac{-1}{2}\right)}$, for the tanh-rule $R = \frac{1}{2}$, for the tanh-sinh-rule $R = \frac{\pi}{4}$.
Interpolation Conditions

Given \( N \) distinct data points \((x[1], y[1]), (x[2], y[2]), \ldots, (x[N], y[N])\) in the plane and corresponding heights \( z[1], z[2], \ldots, z[N] \). Choose a radial basis function \( u(r) \). The form of the radial basis interpolator function is

\[
g(x, y) = \sum_{j=1}^{N} c[j] \cdot u\left( \left( \frac{(x-x[j])^2 + (y-y[j])^2}{2} \right)^{1/2} \right)
\]

The \( N \) interpolation conditions are:

\[
z[i] = \sum_{j=1}^{N} c[j] \cdot u\left( \left( \frac{(x[i]-x[j])^2 + (y[i]-y[j])^2}{2} \right)^{1/2} \right)
\]

for \( i \) from 1 to \( N \). We need solve a dense \( N \) by \( N \) symmetric linear system of equations.

Radial basis function procedures

We choose the global variable \( R \) as our scale parameter, and define global variables \( R1:=1/R; \) and \( R2:=R1^2; \)

\[
> \text{rbf[1]} := \text{proc}(r) \exp(-1/2*R2*r^2) \text{ end proc:\n}
> \text{rbf[2]} := \text{proc}(r) (1+R2*r^2)^{(1/2)} \text{ end proc:\n}
> \text{rbf[3]} := \text{proc}(r) \text{ local } R1r; \text{ if } r=0 \text{ then } 0 \text{ else } R1r := R1*r; R1r^2*ln(R1r) \text{ end if end proc:\n}
> \text{rbf[4]} := \text{proc}(r) \text{ sech}(R1*r)^2 \text{ end proc:\n}
> \text{rbf[5]} := \text{proc}(r) \text{ local } R1r; R1r := R1*r; \cosh(R1r)*\text{sech}(evalf(Pi)/2*\text{sinh}(R1r))^2 \text{ end proc:\n}
\]

Exact function

\[
> \text{exactf} := \text{proc}(x,y) \text{ local } x1,y1; x1:=x+1/4; y1:=y-1/6; \exp(-2*x1^2+3*x1*y1-3*y1^2)*(1+x1+x1^2+2*x1*y1+5*y1^2)^{(-3/4)} \text{ end proc:\n}
\]
exact function
512 uniformly random data points
Gaussian radial basis function, $R=0.125$
multiquadric radial basis function, $R=0.5$
thin plate spline radial basis function, $R=1$
tanh-rule radial basis function, $R=0.5$
tanh-sinh-rule radial basis function, $R=0.5$
tanh-sinh-rule RBF interpolation
rbf[1] interpolation error
rbf[4] interpolation error
rbf[5] interpolation error
Conclusions

Here is a table of the 2-norm of the interpolation error for each of the 5 choices of radial basis function.

\[
\begin{array}{c|c}
RBF & ||error|| \\
1 & 0.00184363 \\
2 & 0.00001448 \\
3 & 0.00218664 \\
4 & 0.00001266 \\
5 & 0.00001088 \\
\end{array}
\]

We can see from the above table that our two new radial basis functions had the least error.
References


Bengt Fornberg and Natasha Flyer, *Accuracy of radial basis function interpolation and derivative approximations on 1-D infinite grids*, [amath.colorado.edu/faculty/fornberg/Docs/RBF.pdf](http://amath.colorado.edu/faculty/fornberg/Docs/RBF.pdf)