MATH445/745, Fall 2007, Homework 1.

This homework is due for Thursday September 20th, by 12:00PM. You can hand it in in class or drop it in the homework box.

This homework is individual. You can discuss it with other students, but you are not allowed to write it together. You should also not copy an answer from any documentation source (web, book, previous homework, . . .). In case of any doubt, indicate you documentation sources on your homework.

1. Show that $\chi(G) \leq |V(G)| - \alpha(G) + 1$.

2. For every $n > 1$ find a bipartite graph on $2^n$ vertices and a vertex ordering of this graph for which the greedy algorithm requires $n$ colors.

3. Show that every graph $G$ has a vertex ordering for which the greedy algorithm uses only $\chi(G)$ colors.

4. Let $G$ be a graph with the property that $V(C_1) \cup V(C_2) \neq \emptyset$ whenever $C_1$ and $C_2$ are odd cycles of $G$. Show that $\chi(G) \leq 5$.

5. Show that $\chi(G) + \chi(\overline{G}) \leq |V(G)| + 1$, where $\overline{G}$ is the complement of $G$. Hint: use induction on $|V(G)|$.

6. Let $G$ be a graph such that $\chi(G - x - y) = \chi(G) - 2$ for all pairs $x, y$ of distinct vertices. Prove that $G$ is a complete graph.

7. Given a proper $k$-coloring of a graph $G$ with $\chi(G) = k$, prove that for every color $i$ there is a vertex of color $i$ which is adjacent to at least one vertex of every other color.

8. Prove that if $G$ is color-critical, then the graph $G'$ obtained from it by Mycielski’s construction is also color-critical.

9. Prove that if $G$ has no induced subgraph isomorphic to $2K_2$, then $\chi(G) \leq \left(\frac{\omega(G)+1}{2}\right)$. Hint: choose a maximum clique to define a collection of $\left(\frac{\omega(G)}{2}\right) + \omega(G)$ independent sets.

10. Prove that $\chi(G) = \omega(G)$ if $\overline{G}$ is bipartite.

11. Let $G$ be the unit-distance graph in the plane: $V(G) = \mathbb{R}^2$, and $(x, y) \in E(G)$ if their Euclidean distance is equal to 1 (this graph is infinite). Prove that $4 \leq \chi(G) \leq 7$. 