1. Let $G$ be a $k$-chromatic graph on $n$ vertices with no cycle of length 6 or greater. Define $G'$ as follows: (1) let $T$ be an independent set of $kn$ vertices in $G$ and $p = \binom{kn}{n}$, (2) take pairwise disjoint copies of $G$, denoted $G_1, \ldots, G_p$, (3) associate to each $G_i$ a distinct $n$ elements subset $S_i$ of $T$, (4) for every pair $i \neq j$ add matching edges between the identical vertices of $S_i$ and $S_j$. Prove that $\chi(G') = k + 1$.

2. Prove that $\chi(C_n; k) = (k - 1)n + (-1)^n(k - 1)$, where $C_n$ is the cycle with $n$ vertices.

3. Let $G$ be a connected graph that is not a tree and $k \geq 3$. Prove that $G$ has at most $k(k - 1)^{n-1}$ proper $k$-colorings.

4. Let $G$ be a connected graph. The distance between two vertices $x$ and $y$ of $G$ is the number of edges on the shortest path between $x$ and $y$. For an integer $r \geq 0$ and a vertex $x$ of $G$, let $G_{x,r}$ be the subgraph of $G$ induced by the vertices at distance exactly $r$ from $x$. Prove that there exists $r \geq 0$ such that $\chi(G)$ is at most $\chi(G_{x,r}) + \chi(G_{x,r+1})$.

5. Using Tutte’s 1-factor Theorem (Theorem 3.3.3, page 137 of the textbook), prove that every connected line graph of even order has a perfect matching, and then that the edges of a simple connected graph of even size can be partitioned into paths of length 2.

6. Let $G$ be a regular graph with a cut-vertex. Prove that $\chi'(G) > \Delta(G)$.

For exercises 7 and 8, we will study a new notion related to colorings: the choosability of a graph. Let $G$ be a graph and $L$ a family of finite sets of positive integers, indexed by $V(G)$: $L(v)$ denotes the set associated to $v$, for $v \in V(G)$. $G$ is said to be $L$-choosable if there is a proper coloring $f$ of $G$ such that $f(v) \in L(v)$ for every vertex $v$ of $G$ (if every $L(v) = \{1, \ldots, |V(G)|\}$, then this is equivalent to a proper coloring; otherwise not all colorings are allowed). For an integer $k$, $G$ is said to be $k$-choosable if it is $L$-choosable for every $L$ such that $|L(v)| \geq k$. The choice number of $G$, denoted $ch(G)$, is the smallest integer $k$ such that $G$ is $k$-choosable: obviously, $\chi(G) \leq ch(G)$.

7. For each of the following statement, is it true (and then prove it) or false (and then give a counter-example) ?
   (1) $ch(G) \geq \omega(G)$; (2) $ch(G) \geq \frac{|V(G)|}{\alpha(G)}$; (3) $ch(G) \leq \Delta(G) + 1$; (4) $ch(G) \leq 1 + \max_{H \subseteq G} \delta(H)$;

8. For every integer $k$ prove that there exists a graph $G$ with $\chi(G) = 2$ and $ch(G) \geq k$. 