Properties of the Robinson-Schensted correspondence for oscillating and skew oscillating tableaux

Cedric Chauve, Serge Dulucq

LaBRI, Université de Bordeaux I

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PROPERTIES OF THE ROBINSON-SCHENSTED CORRESPONDENCE FOR OSCILLATING AND SKEW OSCILLATING TABLEAUX

CEDRIC CHAUVE AND SERGE DULUCQ

Abstract. In this paper, we consider the Robinson-Schensted correspondence for oscillating tableaux and skew oscillating tableaux defined in [15] and [3]. First we give an analogue, for the oscillating tableaux, of the classical geometric construction of Viennot for standard tableaux ([16]). Then, we extend a construction of Sagan and Stanley ([10]), dealing with standard tableaux and skew tableaux, to deduce a property about the number of odd height columns of a skew oscillating tableau. Finally, we give analogues for skew oscillating tableaux of two classical constructions about this correspondence, Knuth classes ([7]) and Beissinger algorithm ([1]).

1. Introduction

The Robinson-Schensted correspondence is a classical bijection between permutations and pairs of standard tableaux of the same shape. It was defined in [11], and followed by numerous papers dealing with the combinatorial properties of this correspondence, like [12], [7], [5], [13], [16] or [1]. More recently, this correspondence was extended to various kinds of tableaux that are generalizations, in the Young lattice, of the standard tableaux: semi-standard tableaux ([7]), skew tableaux ([10]), oscillating tableaux (first by Sundaram in [14, 15], then independtly Delest, Dulucq and Favreau in [2]) and skew oscillating tableaux ([3]).
In this article, we extend classical properties and constructions related to the Robinson-Schensted correspondence to the correspondences for oscillating tableaux and skew oscillating tableaux. In the sections 2 and 3, we give basic definitions on biwords and tableaux and we present the correspondence for skew oscillating tableaux. Then we extend a geometric version of the Robinson-Schensted correspondence due to Viennot ([16]) to the case of oscillating tableaux, and, following [10], we define a construction allowing extension of properties of oscillating tableaux to skew oscillating tableaux. Finally, in sections 6 and 7, we extend to skew oscillating tableaux a result of Knuth ([7]) and an algorithm of Beissinger ([1]).

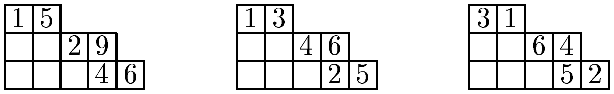
2. Definitions and notations

We assume that the reader is familiar with the combinatorics of the Young tableaux in general and, in particular, with the Robinson-Schensted correspondence (see [11], or [9] for a survey).

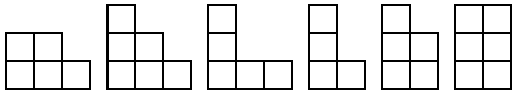
We use the notation $\lambda = (\lambda_1, \dots, \lambda_k)$ for both a partition and the corresponding Ferrers diagram displayed in “French” notation (the smallest part λ_k in the top row). The conjugate of λ is the partition $\lambda' = (\lambda'_1, \dots, \lambda'_m)$, where $\lambda'_j = Card\{i \ \ \lambda_i \geq j\}$. If $\mu \subseteq \lambda$ then the corresponding skew shape λ/μ is the set $\{c \mid c \in \lambda,\ c \notin \mu\}$. If $|\lambda/\mu| = n$ then we write $\lambda/\mu \vdash n$ and say that λ/μ is a skew partition of n.

A Young tableau T of shape λ/μ is a labeling of the cells of λ/μ with an ordered alphabet (of positive integers here) so that the rows and columns are weakly increasing. We denote by $T(i,j)$ the label of the cell in the i^{th} row and j^{th} column, so that $k \in T$ means $k = T(i,j)$ for some i, j. A Young tableau is partial if its elements are distinct. Further it is standard if it is partial and the labels are 1 through $n = |\lambda/\mu|$. The sets of partial and (skew) standard tableaux of shape λ/μ will respectively be denoted by $PT(\lambda/\mu)$ and $ST(\lambda/\mu)$. Analogously, the set of tableaux of shape λ/μ with rows and columns strictly decreasing will be denoted by $\overline{PT}(\lambda/\mu)$. For example, when

$\lambda = (5,4,2)$ and $\mu = (3,2)$, the three following tableaux belong respectively to $PT(\lambda/\mu)$, $ST(\lambda/\mu)$ and $\overline{PT}(\lambda/\mu)$.



A skew oscillating tableau of length n, initial shape α, and final shape β is a sequence of Ferrers diagrams $T = (\alpha = \lambda_0, \lambda_1, \ldots, \lambda_n = \beta)$ where λ_k is obtained from λ_{k-1} by insertion or deletion of a cell. If $\alpha = \emptyset$, T is an oscillating tableau. We denote by $SOT_n(\alpha \to \beta)$ the set of skew oscillating tableaux of length n, initial shape α and final shape β and by $OT_n(\beta)$ the set of oscillating tableaux of length n and final shape β. For example, if $\alpha = (3,2)$ and $\beta = (2,2,2)$, the following tableau belongs to $SOT_5(\alpha \to \beta)$.

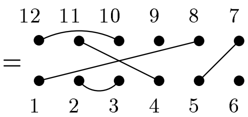


Of course, a skew oscillating tableau of $SOT_n(\mu \to \lambda)$ having only insertion steps, is a (skew) standard tableau of $ST(\lambda/\mu)$, the label of a cell being given by the step of creation of this cell.

Classicaly, we denote by S_n the set of the permutations of $[n]$ and by INV_n the set of the involutions of $[n]$, and, for a permutation σ, we denote by σ^{-1} its inverse. Sometimes, we will write an involution as a product of cycles in increasing order of their greatest element and with $a < b$ for each cycle (a,b), as for example, $\sigma = (1)(2,3)(5)(4,6)$. A sequence of at most n integers, pairwise distinct and less than or equal to n, will be called a partial permutation of $[n]$. For example, 4 1 5 is a partial permutation on $[n]$ for $n \geq 5$.

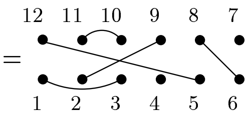
A biword π on $[n] = \{1, 2, \ldots, n\}$ is a sequence of vertical pairs of positive integers of $[n]$, pairwise distinct, $\pi = \binom{i_1\ i_2\ \cdots\ i_k}{j_1\ j_2\ \cdots\ j_k}$. In this paper we consider biwords such that all the i_l and j_l are distinct in π and $i_l > j_l$, for $l = 1, \ldots k$. We denote by $\hat{\pi}$ the top row of π and by $\check{\pi}$ its bottom row, and we write such biwords with the convention $i_1 > i_2 > \ldots > i_k$. We denote the set of these biwords on $[n]$ by BW_n. Moreover, we often represent an element π of BW_{2n} with a graph $G(\pi)$, as in the following example.

$$\pi = \begin{pmatrix} 12 & 11 & 8 & 7 & 3 \\ 10 & 4 & 1 & 5 & 2 \end{pmatrix} \qquad G(\pi) =$$



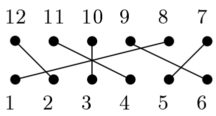
In such a graph, we call diagonal edges the edges between the two rows of vertices. If $\pi \in BW_n$, its inverse π^{-1} is obtained by changing every pair (i_l, j_l) to $(n+1-j_l, n+1-i_l)$. It follows that, for $\pi \in BW_{2n}$, $G(\pi^{-1})$ is obtained from $G(\pi)$ by an horizontal symmetry.

$$\pi^{-1} = \begin{pmatrix} 12 & 11 & 9 & 8 & 3 \\ 5 & 10 & 2 & 6 & 1 \end{pmatrix} \qquad G(\pi^{-1}) =$$



Finally, there is an immediate bijection between elements of BW_{2n} whose graph has exactly n diagonal edges and permutations of S_n.

$$\pi = \begin{pmatrix} 12 & 11 & 10 & 9 & 8 & 7 \\ 2 & 4 & 3 & 6 & 1 & 5 \end{pmatrix} \; G(\pi) =$$



$\longleftrightarrow$ 2 4 3 6 1 5

Hence, if π corresponds to a permutation σ, then π^{-1} corresponds to σ^{-1}.

3. Robinson-Schensted correspondence

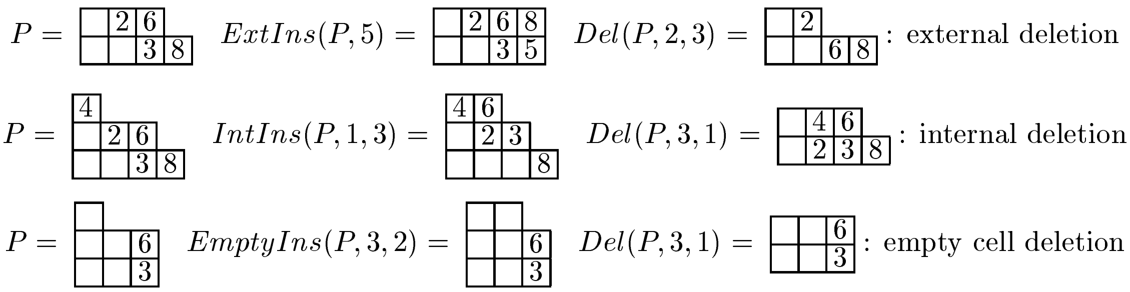
In [11], Schensted gives an algorithm which associates to a permutation σ of S_n a pair of standard tableaux (P, Q) with the same shape λ such that $\lambda \vdash n$. The tableau P is called the *insertion tableau* of σ and the tableau Q its *recording tableau*. This construction is the limiting case of one for partial permutations and partial tableaux.

In [3], Dulucq and Sagan extend this algorithm to deal with skew oscillating tableaux. We recall their result, keeping their notations. This algorithm relies on three kinds of insertion in a skew tableau (see Example 3.1).
Let P be a skew tableau of shape λ/μ.

(1) The *external insertion* is the insertion defined by Schensted ([11]) and inserts an integer x in P. We denote the new tableau obtained after this insertion by $ExtIns(P, x)$.
(2) The *internal insertion* was introduced in [10]. Let (u, v) be a cell of P such that $(u, v) \notin \mu$ or $v = 1$, $(u-1, v) \in \mu$ or $u = 1$, and $(u, v-1) \in \mu$. The internal insertion of the cell (u, v) inserts the integer x contained in $P(u, v)$ from the row $(u+1)$ using the external insertion algorithm. This process doesn't add a label in the tableau P. We denote the new tableau by $IntIns(P, u, v)$.
(3) The *empty insertion* adds an empty cell in $P(u, v)$, where $(u-1, v) \in \mu$ or $u = 1$, $(u, v-1) \in \mu$ or $v = 1$, and $(u, v) \notin \lambda$. We denote the new tableau obtained after this insertion by $EmptyIns(P, u, v)$.

Conversely, the deletion of the cell $P(u, v)$, denoted by $Del(P, u, v)$, can be an *empty deletion* if the cell is an empty cell, an *internal deletion* if the process (this is the classical process of deletion defined by Schensted) ends in filling a cell of μ, or an *external deletion* if the process ends with the expulsion of an integer out of P.

Example 3.1.



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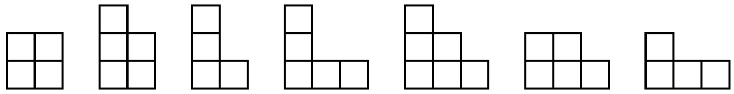
Remark 3.1. We call the elements displaced during an insertion process the "bumped" elements.

We now give a description and a detailed example (Example 3.2) of this algorithm and of its inverse (cf [3]). The first, denoted by Φ_{SO}, has for input a triple (π, T, U) of $BW_n \times \bigcup_{\mu \subseteq \alpha\cap\beta} [PT(\beta/\mu) \times \overline{PT}(\alpha/\mu)]$ and for output a tableau P of $SOT_n(\alpha \to \beta)$.

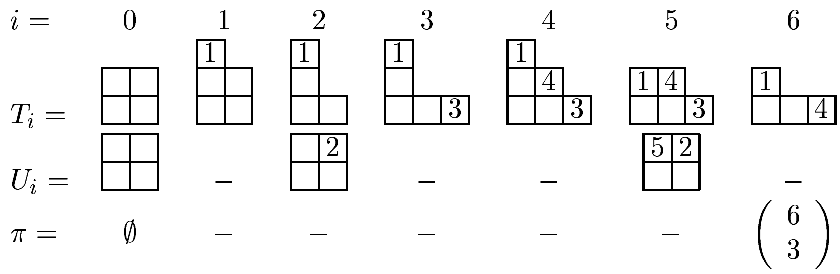
Algorithm 1. $\Phi_{SO}(\pi, T, U)$ *- The output is a tableau P.*
Let $P_n = T$.
For i from n to 1:
(a) if there is a cell $P_i(u, v) = i$, then erase this cell to obtain P_{i-1},
(b) else if the pair (i, x) belongs to π, then $P_{i-1} = ExtIns(P_i, x)$,
(c) else if $U(u, v) = i$ and $P_i(u, v)$ exists (with label x), then $P_{i-1} = IntIns(P_i, u, v)$,
(d) else $P_{i-1} = EmptyIns(P_i, u, v)$.
Finally, the tableaux P_i have respective shapes λ_i/μ_i and $P = (\lambda_0, \ldots, \lambda_n)$.

Algorithm 2. $\Phi_{SO}^{-1}(P)$ - *The output is a triple* (π, T, U).
Suppose $P = (\alpha = \lambda_0, \dots, \lambda_n = \beta)$.
Let $\pi = \emptyset$, $T_0 = \alpha$ *and* $U_0 = \alpha$.
For i *from* 1 *to* n:
(a) *if* $\lambda_i = \lambda_{i-1} + (u, v)$, *then add in* T_{i-1} *a cell in position* (u, v) *with label* i *to obtain* T_i, $U_i = U_{i-1}$,
(b) *else* $(\lambda_i = \lambda_{i-1} - (u, v))$ $T_i = Del(T_{i-1}, u, v)$:
$(b.1)$ *if this deletion is external* (x *ejected out of* T_{i-1}), *then add the pair* (i, x) *to* π, $U_i = U_{i-1}$.
$(b.2)$ *else if it is internal (the cell* $T_{i-1}(u', v')$ *is filled), then label the cell* $U_{i-1}(u', v')$ *with* i *to obtain* U_i,
$(b.3)$ *else label the cell* $U_{i-1}(u, v)$ *with* i *to obtain* U_i.
Finally, $T = T_n$ *and* $U = U_n$.

Example 3.2. Let P be the following skew oscillating tableau.

$P =$ 

Then:



P is in bijection with (π, T_6, U_6). $\diamond$

Theorem 3.1. ([3]) *Let* α *and* β *be fixed partitions and* n *a fixed integer.* Φ_{SO} *is a bijection from triples* (π, T, U) *of* $BW_n \times \bigcup_{\mu \subseteq \alpha \cap \beta} [PT(\beta/\mu) \times \overline{PT}(\alpha/\mu)]$, *such that* $\pi \dot{\cup} T \dot{\cup} U = [n]$, *to tableaux* P *of* $SOT_n(\alpha \to \beta)$.

Remark 3.2. If the skew oscillating tableau P has only insertion steps (it is a skew standard tableau P' of shape β/α), the bijection Φ_{SO} is such that $\Phi_{SO}^{-1}(P) = (\emptyset, P', \alpha)$.

Theorem 3.2. ([3]) *Let* α *be a fixed partition and* n *a fixed integer. There is a bijection* RS_{SO} *from triples* (π, T, U) *of* $BW_{2n} \times \bigcup_{\mu \subseteq \alpha \cap \beta} [PT(\alpha/\mu) \times \overline{PT}(\alpha/\mu)]$ *such that* $\pi \dot{\cup} T \dot{\cup} U = [2n]$ *to pairs of tableaux* (P, Q) *of* $\bigcup_\beta [SOT_n(\alpha \to \beta) \times SOT_n(\alpha \to \beta)]$.

This result follows immediately from the previous theorem and:

$$(\lambda_0 (= \alpha), \dots, \lambda_n, \dots, \lambda_{2n} (= \alpha)) \longleftrightarrow (P = (\lambda_0, \lambda_1, \dots, \lambda_n (= \beta)),\ Q = (\lambda_{2n}, \lambda_{2n-1}, \dots, \lambda_n (= \beta))).$$

Using skew oscillating tableaux with empty initial and final shapes, we have a Robinson-Schensted correspondence for oscillating tableaux, as stated in the following results.

Theorem 3.3. ([15]) *Let* β *be a fixed partition and* n *a fixed integer. There is a bijection* Φ_O *from pairs* (π, T) *of* $BW_n \times PT(\beta)$ *such that* $\pi \dot{\cup} T = [n]$ *to tableaux* P *of* $OT_n(\beta)$.

Theorem 3.4. ([15]) *Let* n *be a fixed integer. There is a bijection* RS_O *from biwords* π *of* BW_{2n} *of size* $2n$ *to pairs of tableaux* (P, Q) *of* $\bigcup_\beta [OT_n(\beta) \times OT_n(\beta)]$.

It is straightforward to verify that if P and Q have only insertion steps (i.e. P and Q are standard tableaux), $G(\pi)$ has exactly n diagonal edges (then π corresponds to a permutation) and we have the original Robinson-Schensted correspondence for standard tableaux.

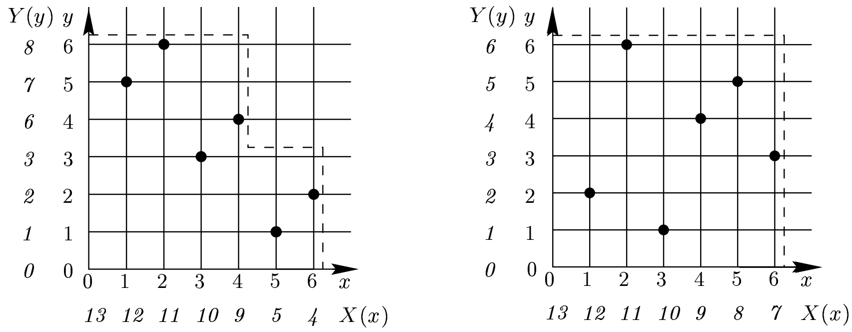
4. Analogue of the geometric construction of Viennot

In this section, we give an analogue for oscillating tableaux of a beautiful construction of Viennot for standard tableaux ([16]). We follow the presentation of the construction of Viennot given in [9].

First, we explain how we represent a biword on $[2n]$, $\pi = \binom{i_1\ i_2\ \dots\ i_n}{j_1\ j_2\ \dots\ j_n}$ of $[n] \times [n]$ in the part $\{0, 1, \dots, n\} \times \{0, 1, \dots, n\}$ of the Cartesian plane (see Example 4.1).

(1) We define a map X from abscissas x ($x = 0, 1, \dots, n$) into $\{2n+1\} \cup \hat{\pi}$ such that $X(0) = 2n+1$ and $X(x)$ is the x^{th} greatest element of $\hat{\pi}$, for $x > 0$.
(2) We define a map Y from ordinates y ($y = 0, 1, \dots, n$) into $\{0\} \cup \check{\pi}$ such that $Y(0) = 0$ and $Y(y)$ is the y^{th} lowest element of $\check{\pi}$, for $y > 0$.
(3) We define a *valid domain* which is the set of points (x, y) such that $X(x) > Y(y)$.
(4) For each pair (i_l, j_l) of π, we set up the point $(X^{-1}(i_l), Y^{-1}(j_l))$ (which is in the valid domain).

Example 4.1. Here we give a biword π of BW_{12} and its representation (for readability, the limit of the valid domain, the dashed line, is slightly extended on the figure).



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We introduce definitions intuitively related to a lighting of the representation of a biword from $(0, 0)$ (illustrated in Example 4.2).

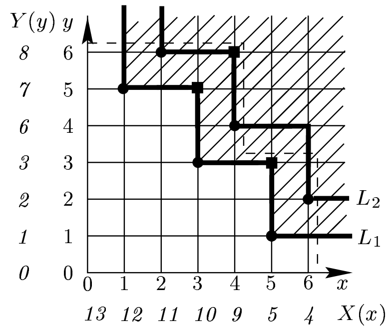
Definition 4.1. *The shadow $S(\pi)$ of a biword π is the set of points (x, y) such that there is a point (x', y') of the representation of π with $x' \le x$ and $y' \le y$.*
Shadow lines of π are defined recursively. The first shadow line L_1 of π is the boundary of $S(\pi)$. To construct the shadow line L_{i+1} of π, remove the points of the representation of π lying on L_i and construct the shadow line of the remaining points. This procedure ends when there is no remaining point on the plane.
The SW-corners of a shadow line are the points of the representation of π located on this line.
The NE-corners of a shadow line are the points (x, y) of the shadow line such that (x, y) is in the valid domain and $(x+1, y)$ and $(x, y+1)$ don't belong to this shadow line.

Definition 4.2. *The k^{th} skeleton $\pi^{(k)}$ of a biword π is a biword defined recursively by*

(1) $\pi^{(1)} = \pi$
(2) $\pi^{(k+1)} = \binom{X(i_1)\ X(i_2)\ \dots\ X(i_m)}{Y(j_1)\ Y(j_2)\ \dots\ Y(j_m)}$ *where $(i_1, j_1), \dots, (i_m, j_m)$ are the NE-corners of $\pi^{(k)}$.*

The shadow diagram of π is the set of shadow lines of all the skeletons $\pi^{(k)}$ of π. The shadow lines of $\pi^{(k)}$ are denoted $L_i^{(k)}$.

Example 4.2. Let π be the biword of Example 4.1. The figure on the left gives the shadow of π (which has two shadow lines). The points marked with a circle (i.e. the points of the biword) are the SW-corners, and the points marked with a square are the NE-corners. The figure on the right gives the shadow diagram of π.



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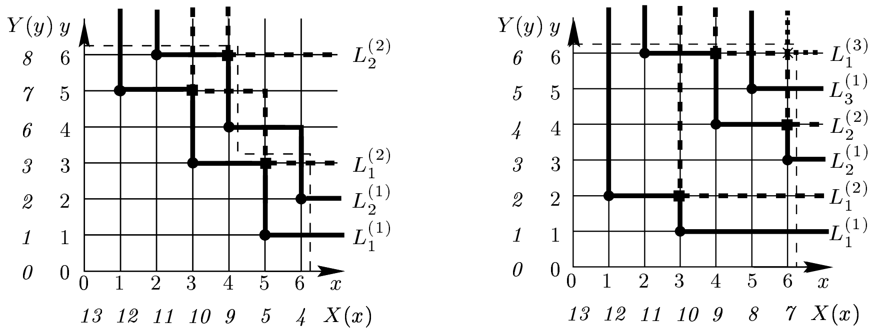
In the following result, we show that, for a biword π, the behaviour of the algorithm $\Phi_O(\pi,\emptyset)$, can be described by the shadow diagram of π.

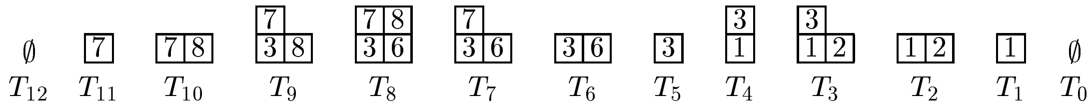
Theorem 4.1. *Let π be a biword of BW_{2n} and T an oscillating tableau of OT_{2n} such that $\Phi_O(\pi,\emptyset)=T$. Following, from the left to the right, the shadow line $L_j^{(i)}$ describes the behaviour of the j^{th} cell of the i^{th} row of the tableaux $T_{2n},\dots,T_0$ in the following way:*

1. *a SW-corner (x,y) indicates that during step $X(x)$, the label $Y(y)$ fills this cell, which is created if not present,*
2. *when the line leaves the valid domain at (x,y), this cell is deleted during the step $Y(y)$,*
3. *otherwise, the cell remains unchanged.*

Example 4.3. Let π be the biword of BW_{12} defined above (see Example 4.1). We recall its shadow diagram.

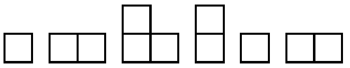
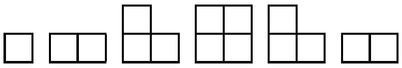


The execution of $\Phi_O(\pi,\emptyset)$ produces the following partial tableaux $T_{12},\dots,T_0$:



Now, we observe, for example, the second cell of the first row and the shadow line $L_2^{(1)}$.
The line has a first SW-corner at $(2,6)$ with $X(2)=11$ and $Y(6)=8$, followed by the SW-corner $(4,4)$ with $X(4)=9$ and $Y(4)=6$, leaves the valid domain at the same point $(4,4)$, enters the valid domain again to go to the SW-corner $(6,2)$ with $X(6)=4$ and $Y(2)=2$ and finally leaves the valid domain at the same point $(6,2)$.
On the other hand, during the execution of Φ_O, (the tableaux are constructed from T_{12} to T_0), the second cell of the first row is created during step 11 (passing from T_{11} to T_{10}) with label 8, this label is replaced during step 9 by the label 6, which remains unchanged until the first deletion of the cell during step 6. The cell is created again during step 4 with label 2 and is finally deleted during step 2.

Finally we can say that π is in bijection with the following pair (P, Q) of oscillating tableaux.

$P = \emptyset$  , $Q = \emptyset$ 

$\diamond$

Remark 4.1. If π is a biword of BW_{2n} such that $G(\pi)$ has exactly n diagonal edges, our construction is equivalent to the original construction of Viennot (in particular, the valid domain is $\{0, 1, \ldots, n\} \times \{0, 1, \ldots, n\}$ and there is no need of the maps X and Y).

The previous result is "too strong" if we want only to describe the shape of the oscillating tableau T in bijection with the biword π (we don't need the labeling of the Ferrers diagrams). Moreover, by definition of Φ_O, it constructs T from final shape to initial shape (it constructs T^R). The following result solves these problems.

Corollary 4.1. *Let π and $T = (\lambda_0(=\emptyset), \lambda_1, \ldots, \lambda_{2n}(=\emptyset))$ be a biword and an oscillating tableau such that $\Phi_O(T) = (\pi, \emptyset)$. Following the shadow line $L_j^{(i)}$, from right to left, gives the behaviour of the j^{th} cell of the i^{th} row of $\lambda_0, \ldots \lambda_{2n}$:*

- *when the line enters the valid domain at (x, y), we pass from $\lambda_{Y(y)-1}$ to $\lambda_{Y(y)}$ by creating the j^{th} cell of the i^{th} row.*
- *when the line leaves the valid domain at (x, y), we pass from $\lambda_{X(x)-1}$ to $\lambda_{X(x)}$ by deleting the j^{th} cell of the i^{th} row.*

Proof. Immediate consequence of Theorem 4.1. $\square$

We now give the proof of the Theorem 4.1, following the proof of the result of Viennot given in [9]. In a first time, we focuse on the shadow lines $L_j^{(1)}$.

Lemma 4.1. *Consider the shadow diagram of a biword $\pi \in BW_{2n}$ such that $\Phi_O(\pi, \emptyset) = T$, and suppose the vertical line $x = k$ $(n \geq k \geq 0)$ intersects in the valid domain i shadow lines $L_j^{(1)}$. Let y_j be the ordinate of the lowest point of the intersection between $x = k$ and $L_j^{(1)}$. Then, the behaviour of the first row of $T_{2n}, T_{2n-1}, \ldots, T_{X(k)-1}$ during the execution of $\Phi_O(\pi, \emptyset)$ is given by the rules of Theorem 4.1 with shadow lines $L_j^{(1)}$ followed up to the meeting with the line $x = k$. In particular, the first line of $T_{X(k)-1}$ is $Y(y_1) \ldots Y(y_i)$.*

Proof. We reason by induction on k. If $k = 0$, $X(k) = 2n + 1$ and $T_{2n} = \emptyset$ by definition. Hence we have the basis of our induction.
Assume the result holds for $x = k$ $(k \geq 1)$ and consider $x = k + 1$, with $X(k + 1) = X(k) - j$, $(j \geq 1)$. We have then to look at the first row of the partial tableaux $T_{X(k)-2}, \ldots T_{X(k)-j-1}$.
First, we can remark some useful facts :

(1) the minimal ordinate (height) of the valid domain at the left of the vertical line decreases of $j - 1$ between $x = k$ and $x = k + 1$ (if $j = 1$ it doesn't decrease),
(2) the line $x = k$ intersects only one vertical segment of the shadow lines $L_j^{(1)}$, all the other intersections are with horizontal segments,
(3) if $x = k$ intersects the line $L_j^{(1)}$ in a vertical segment, then $x = k+1$ doesn't, but intersects another shadow line $L_{j'}^{(1)}$ in a vertical segment,
(4) two shadow lines of π leaving the valid domain don't cross outside this domain,
(5) if $x = k$ and $x = k + 1$ intersect the line $L_j^{(1)}$ in a horizontal segment, then this is the same horizontal segment.

During the execution of Φ_O, the steps $X(k)-2,\ldots,X(k)-j$ are steps of deletion of cells with labels y such that $X(k) > y > X(k)-j-1$ (by definition of Φ_O). By (1) and by induction, the cells of the first row having a such label y correspond to shadow lines $L_j^{(1)}$ leaving the valid domain at a point $(Y^{-1}(y), x)$. Hence the lemma holds for the tableaux $T_{X(k)-2},\ldots,T_{X(k)-j}$.
With the possible decreasing of the valid domain, $x = k+1$ intersects in horizontal segments in the valid domain i' shadow lines $L_j^{(1)}$ $(1 \leq i' \leq i+1)$ at ordinates $y'_1,\ldots y'_{i'}$, and following the previous exam of steps $X(k)-2,\ldots,X(k)-j$, we can say that the first row of $T_{X(k)-j}$ is given by $Y(y'_1)\ldots Y(y'_{i'})$.
Now, we examine the step between $T_{X(k)-j}$ and $T_{X(k)-j-1}$, i.e. the insertion in $T_{X(k)-j}$ of the image of $X(k)-j-1$ in π, denoted $y_{i'+1}$. By definition of the representation of π, there is a point (x,y) such that $X(x) = X(k)-j-1$ and $Y(y) = y_{i'+1}$. There are two cases.
(a) If $y_{i'+1} > y_{i'}$. Φ_O inserts it at the end of the first row of $T_{X(k)-j}$. On the representation, we can see that (x,y) is the beginning of the vertical segment of the shadow lines that are intersected by $x = k+1$ and that if $y_{i'+1} > y_{i'}$, this segment enters the valid domain and, by (4), belongs the the shadow line $L_{i'+1}^{(1)}$. Hence the lemma holds in this case.
(b) If $y'_1 < \ldots < y'_{j-1} < y'_{i'+1} < y'_j < \ldots < y'_{i'}$. Φ_O inserts $y'_{i'+1}$ in the j^{th} cell of the first row of $T_{X(k)-j}$, "bumping" y'_j in the second row. As previously, by definition of the representation of π, we can say that (x,y) is the beginning of the vertical segment of the shadow lines intersected by $x = k+1$. But now, this vertical segment is entirely in the valid domain, belongs to $L_j^{(1)}$, and has for high extremity $(k+1, y'_j)$ and for low extremity $(k+1, y'_{i'+1})$ (from facts (2), (3), (4) and (5)). Hence, following rules of Theorem 4.1, the lemma holds.
We can remark the following fact :

(6) the point $(k+1, y'_j)$, corresponding to a label "bumped" from row 1 to row 2, is a NE-corner of the shadow lines of π.

$\square$

Proof of Theorem 4.1. We reason by induction on the indice i of the shadow lines $L_j^{(i)}$.
For $i = 1$, the Lemma 4.1 needs only to be completed for the last tableaux $T_{X(n)-1},\ldots,T_0$, i.e. the steps $X(n),\ldots,1$ of Φ_O. These steps erase the cells of $T_{X(n)-1}$ in increasing order of the labels of these cells. On the shadow diagram of π, the shadow line $L_j^{(1)}$, that was in the valid domain, leaves this domain by a point (x,y) such that its last SW-corner has y for ordinate. Hence, the induction holds for $i = 1$.
We assume it holds for $i \geq 1$ and we examine the case $(i+1)$. We work then on the shadow diagram of the $(i+1)^{th}$ skeleton $\pi^{(i+1)}$ of π. The points of $\pi^{(i+1)}$ are the NE-corners of the shadow diagram of $\pi^{(i)}$. By proof of the Lemma 4.1 (fact (6)), these NE-corners correspond to elements "bumped" from row i to row $(i+1)$. Thus, with a proof similar to the proof of Lemma 4.1, with complement for the last steps of deletion, our induction holds. $\square$

Now, we show how we can use the shadow diagram of a biword to deduce combinatorial properties of the Robinson-Schensted correspondence for oscillating tableaux.

Theorem 4.2. *Let π be a biword of BW_{2n} and P, Q two oscillating tableaux of $OT_n(\beta)$ such that $RS_O(\pi) = (P,Q)$. Then, the i^{th} row of β has k cells if and only if k shadow lines $L_j^{(i)}$ intersect the line $x = y$ in the valid domain.*

Example 4.4. The biword π of Example 4.1 is in bijection with a couple (P,Q) of oscillating tableaux of $OT_6(\beta)$ such that $\beta = \square\square$. On the shadow diagram of π (see Example 4.3), the line $x = y$ intersects in the valid domain only $L_1^{(1)}$ and $L_2^{(1)}$ and we verify the result. $\diamond$

Proof of Theorem 4.2. By definition, to have β, it suffices to have the shape of T_n, where $T_0, \dots T_{2n}$ are the tableaux constructed by $\Phi_O(\pi, \emptyset)$. There are two cases.
(a) First, if the n^{th} step of $\Phi_O(\pi, \emptyset)$ is a deletion (i.e. there is $1 \le x_{n'} \le n$ such that $X(x_{n'}) = n$). Assume the highest intersection points between the line $x = x_{n'}$ and the shadow lines $L_j^{(i)}$, in the valid domain, have for ordinates $y_1 < \dots < y_{k'}$. Then, following the Theorem 4.1, T_n has k cells in the i^{th} row if and only if $k' = k$. Let y_{VD} the height of the valid domain at $x = x_{n'}$. If $X(x_{n'}) = n$, then there are $x_{n'} - 1$ ordinates y'_i such that $Y(y'_i) < n$. It follows that $x = x_{n'}$ intersects $x = y$ at the point $(x_{n'}, y_{VD}+1)$. Thus, every shadow line crossing $x = x_{n'}$ in the valid domain (we consider the high intersection) crosses the line $x = y$ in the valid domain and conversely.
(b) Else, the n^{th} step of $\Phi_O(\pi, \emptyset)$ is an insertion. Then there is $1 \le x_{n'} \le n$ such that $X(x_{n'}) = n+l$ and $x_{n'} = n$ or $X(x_{n'}+1) = n - l'$ (l and l' strictly positive integers), and there is $y_{n'}$ such that $Y(y_{n'}) = n$ (this is the case of Example 4.2 where $x_{n'} = 5$ and $y_{n'} = 4$). Assume the lowest intersection points between the line $x = x_{n'}$ and the shadow lines $L_j^{(i)}$, in the valid domain and with ordinates lower or equal to $y_{n'}$, have for ordinates $y_1 < \dots < y_{k'}$. Then, following the Theorem 4.1, T_n has k cells in the i^{th} row if and only if $k' = k$. With a similar argument than previously, we can show that $x = x_{n'}$ intersects $x = y$ on a vertical segment of the limit of the valid domain. Thus every shadow line crossing $x = x_{n'}$ in the valid domain with an ordinate lower or equal to $y_{n'}$ (we consider the low intersection) crosses the line $x = y$ in the valid domain and conversely. $\square$

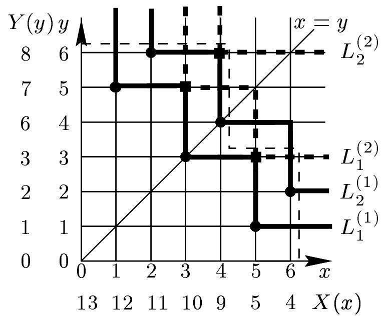
One of the beautiful properties of the Robinson-Schensted correspondence relies on the inversion of the standard tableaux P and Q. Schützenberger showed, in [12], that if (P, Q) is in bijection with a permutation σ, then (Q, P) is in bijection with σ^{-1}, and so, the Robinson-Schensted correspondence is a bijection between involutions and standard tableaux. Later, these results were again demonstrated by Viennot in [16] using symmetry properties of its geometric construction. In the same way, with our construction we can prove similar results for oscillating tableaux.

Property 4.1. *If π is a biword of BW_{2n}, then the valid domain and the shadow lines of π^{-1} are the reflection in the line $x = y$ of those of the representation of π.*

Definition 4.3. *We denote INV_n^c the set of involutions on $[n]$ such that every cycle (a, b) $(a \neq b)$ and every fixed point (a) can be of two types (colors), called bold or normal.*

Proposition 4.1. *There is a bijection ζ: $\{\pi \in BW_{2n} | \pi = \pi^{-1}\} \to INV_n^c$.*

Example 4.5.



We have then $\zeta(\pi) = (\mathbf{3})(\mathbf{2}, \mathbf{4})(1, 5)(6)$: the cycle $(\mathbf{2}, \mathbf{4})$ and the fixed point $(\mathbf{3})$ are bold, the cycle $(1, 5)$ and the fixed point (6) are normal.

Using Property 4.1 and the previous Proposition, we have a geometric proof of the following results.

Proof. We recall the relation between π and π^{-1} : if the pair (a, b) belongs to π, then the pair $(2n+1-b, 2n+1-a)$ belongs to π^{-1}. Hence we have immediately that, if $X(x) = a$ and $Y(y) = b$

in the representation of π, then $X(y) = 2n+1-b$ and $Y(x) = 2n+1-a$ in the representation of π^{-1}. Furthermore, the valid domain of π^{-1} is obtained by reflecting by the line $x = y$ the valid domain of π. Hence we can say that the representation of π^{-1} is obtained by reflecting by $x = y$ the representation of π. The result follows from these observations in a straightforward way. $\square$

Theorem 4.3. ([2]) *Let π be a biword of BW_{2n} and P, Q oscillating tableaux such that $RS_O(\pi) = (P,Q)$. Then $RS_O(\pi^{-1}) = (Q,P)$.*

Corollary 4.2. ([2]) *RS_O induces a bijection between $OT_n(\beta)$ and the involutions of INV_n^c having no bold fixed point.*

Proof. Immediate consequence of Theorem 4.3 and Proposition 4.1. $\square$

Furthermore, with the bijection between involutions and standard tableaux, Schützenberger ([12]) showed that the number of columns of this standard tableau with odd height is the number of fixed points of the involution, a result which has a direct geometric proof using the construction of Viennot. Similarly, using our construction, we have a geometric proof of an analogous result for oscillating tableaux.

Theorem 4.4. *Let σ be an involution of INV_n^c without bold fixed point and P an oscillating tableau of $OT_n(\beta)$ such that σ is in bijection with P as in Corollary 4.2. Then $odd(\beta) = fix_N(\sigma)$, where $odd(\beta)$ is the number of columns in β with odd height and $fix_N(\sigma)$ is the number of normal fixed points of σ.*

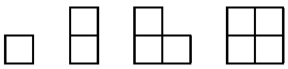
Proof. Let π be a biword of BW_{2n} such that $\zeta(\pi) = \sigma$. We have then $\pi = \pi^{-1}$. With the Theorem 4.2, we know that the shape β is given by the shadow lines of the shadow diagram of π intersecting the line $x = y$ in the valid domain. Furthermore, a normal fixed point (a) of π corresponds to a pair $(2n+1-a, a)$ in π, and, seeing that $\pi = \pi^{-1}$, we can see that $X^{-1}(2n+1-a) = Y^{-1}(a)$. Thus, the point corresponding to $(2n+1-a, a)$ lies on the line $x = y$, in the valid domain. Now, it is not difficult to see that the odd columns of β are induced by SW-corners of $\pi^{(1)}$ lying on the line $x = y$ in the valid domain, i.e. normal fixed points of σ, which proves that $odd(\beta) = fix_N(\pi)$. $\square$

5. Properties of skew oscillating tableaux

In this section, we extend some of the previous properties of the Robinson-Schensted correspondence for oscillating tableaux to the correspondence for skew oscillating tableaux. We define an extension of a construction of Sagan and Stanley (see [10]), in order to treat a pair of skew oscillating tableaux as a pair of oscillating tableaux.

Definition 5.1. *The column word of a partial tableau P is the sequence of integers corresponding to the columns of P read from left to right, each column being read from top to bottom.*
Let α be a Ferrers diagram. We denote C_α the oscillating tableau corresponding to the construction of α column by column, from left to right, each column being constructed from top to bottom. P_α is the standard tableau corresponding to C_α (which has only insertion steps), and π_α the column word of P_α.

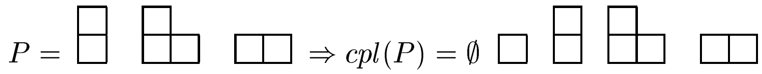
Example 5.1. $\alpha = (2,2)$

$C_\alpha = \emptyset$  , $P_\alpha = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}$ and $\pi_\alpha = 2\ 1\ 4\ 3$

$\diamond$

Now we introduce the notion of completion of a skew oscillating tableau. Let α be a partition such that $\alpha \vdash m$. We define a map *cpl* from $SOT_n(\alpha \to \beta)$ into $OT_{n+m}(\beta)$, that associates to a skew oscillating tableau P the oscillating tableau *cpl*(P) that is the concatenation of C_α and P. We say that C_α is the completion of P.

Example 5.2.



$\diamond$

Furthermore, we use a new alphabet for the labels of partial tableaux, $\mathbb{N} \cup \{\underline{j^k} : j,k \in \mathbb{N}\} \cup \{j^k : j,k \in \mathbb{N}\}$, such that, for x, j, k and k' positive integers we have

$$j^k < j^{k+1} < (j+1)^{k'} < x < \underline{j^k} < \underline{j^{k+1}} < \underline{(j-1)^{k'}}.$$

Next, let α, μ and λ be partitions such that $\mu \subseteq \alpha$ and $\mu \subseteq \lambda$. We use a map $[.]_\lambda$ (defined in [10]) from $PT(\alpha/\mu)$ into $PT(\alpha)$, that associates to a partial tableau P the tableau $[P]_\lambda$, obtained in labeling (from top to bottom) the empty cells of the j^{th} column of P with $j^{\lambda'_j}, j^{\lambda'_j - 1}, \dots, j^{\lambda'_j - \mu'_j + 1}$. If $\lambda = \mu$, we write $[P]$ for $[P]_\lambda$. For example, let $\lambda = (4,3,1)$ ($\lambda' = (3,2,2,1)$).

$$P = \begin{array}{|c|c|c|c|c|} \hline 1 \\ \hline & & 2 & 7 \\ \hline & & & 5 & 8 \\ \hline \end{array}\;, \; [P]_\lambda = \begin{array}{|c|c|c|c|c|} \hline 1 \\ \hline 1^3 & 2^2 & 2 & 7 \\ \hline 1^2 & 2^1 & 3^2 & 5 & 8 \\ \hline \end{array}\;, \; [P] = \begin{array}{|c|c|c|c|c|} \hline 1 \\ \hline 1^2 & 2^2 & 2 & 7 \\ \hline 1^1 & 2^1 & 3^1 & 5 & 8 \\ \hline \end{array}$$

Now, following [10], we define a transformation of a triple (π, T, U) of Theorem 3.2 ($(\pi, T, U) \in BW_{2n} \times PT(\alpha/\mu) \times \overline{PT}(\alpha/\mu)$ with $\alpha \vdash m$ and α' has l rows) into a biword of $BW_{2n'}$ with $n' = n+m$. We denote this biword $[\pi, T, U]$ or, when there is no possible confusion, $[\pi]$.

Algorithm 3. $[.](\pi, T, U)$ - *The output is a biword* $[\pi]$ *of* $BW_{2n'}$.
Let $\hat{\overline{[\pi]}} = \hat{\pi} \cup U \cup \underline{l^1} \dots \underline{l^{\alpha'_l}}\ \underline{(l-1)^1} \dots \underline{(l-1)^{\alpha'_{l-1}}} \dots \underline{1^1} \dots \underline{1^{\alpha'_1}}$.
For each element a *of* $\hat{\overline{[\pi]}}$:
(*a*) *if* $a \in \hat{\pi}$, *then* $\overline{[\pi]}(a) = \pi(a)$,
(*b*) *else if* $a = \underline{j^k}$, *then* $\overline{[\pi]}(a) = [T]_\alpha(k,j)$,
(*c*) *else* ($a = U(k,j)$), $\overline{[\pi]}(a) = j^{k'}$, *with* $k' = \alpha'_j - k + 1$.
Finally, $[\pi]$ *is given by the normalization of* $\overline{[\pi]}$ *on* $[2n']$.

Example 5.3. With the following triple (π, T, U)

$$\pi = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \; T = \begin{array}{|c|c|} \hline 3 & 6 \\ \hline & \\ \hline \end{array} \; U = \begin{array}{|c|c|} \hline 5 & 2 \\ \hline & \\ \hline \end{array}$$

we have

$$\overline{[\pi]} = \begin{pmatrix} \underline{1^2} & \underline{1^1} & \underline{2^2} & \underline{2^1} & 5 & 4 & 2 \\ 3 & 1^2 & 6 & 2^2 & 1^1 & 1 & 2^1 \end{pmatrix}, \; [\pi] = \begin{pmatrix} 14 & 13 & 12 & 11 & 9 & 8 & 6 \\ 7 & 2 & 10 & 4 & 1 & 5 & 3 \end{pmatrix}.$$

$\diamond$

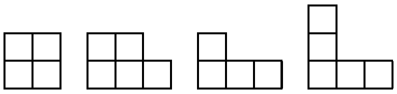
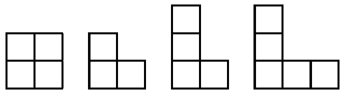
Property 5.1. *The first part of* $\overline{[\pi]}$ *(pairs* $(\underline{j^k}, b)$*) is the column word of* $[T]_\alpha$.

Property 5.2. $\forall j \in [m]$ *and* $k \in [\alpha'_j]$ *the sequence* $j^k, j^{k-1}, \dots j^1$ *is a decreasing subsequence of* $\check{\overline{[\pi]}}$.

We now give the result that relates the Robinson-Schensted correspondence for skew oscillating tableaux to the Robinson-Schensted correspondence for oscillating tableaux. This is an extension of a similar result of [10].

Lemma 5.1. *Let* $(\pi, T, U) \in BW_{2n} \times PT(\alpha/\mu) \times \overline{PT}(\alpha/\mu)$, P *and* Q *be tableaux of* $SOT_n(\alpha \to \beta)$. *If* $RS_{SO}(\pi, T, U) = (P, Q)$ *then* $RS_O([\pi]) = (cpl(P), cpl(Q))$.

Example 5.4. The triple (π, T, U) of the Example 5.3 is in bijection by RS_{SO} with the couple of skew oscillating tableaux (P, Q).

$P =$  , $Q =$  ,

$[\pi]$ is in bijection by RS_O with the oscillating tableaux $(P', Q') = (cpl(P), cpl(Q))$. $\diamond$

Proof of Lemma 5.1. Let R be a skew oscillating tableau such that $\Phi_{SO}(\pi, T, U) = R$ and R' an oscillating tableau such that $\Phi_O(\overline{[\pi]}, \emptyset) = R'$ (we can clearly use $\overline{[\pi]}$ in place of $[\pi]$). Then $R \in SOT_{2n}(\alpha \to \alpha)$ and $R' \in OT_{2n'}(\emptyset)$, with $\alpha \vdash m$ and $n' = n + m$. We denote $T_0, T_1, \ldots, T_{2n}$ the tableaux constructed by $\Phi_{SO}(\pi, T, U)$, having for respective shapes $\lambda_0/\mu_0, \lambda_1/\mu_1, \ldots, \lambda_{2n}/\mu_{2n}$, and $T'_0, T'_1, \ldots, T'_{2n'}$ the tableaux constructed by $\Phi_O(\overline{[\pi]}, \emptyset)$, having for respective shapes λ'_0/μ'_0, λ'_1/μ'_1, $\ldots, \lambda'_{2n'}/\mu_{2n'}$. Then, we can say that $R = \lambda_0, \lambda_1, \ldots, \lambda_{2n}$ and $R' = \lambda'_0, \lambda'_1, \ldots, \lambda'_{2n'}$.
In a first time, we prove that $\lambda'_{2n'}, \lambda'_{2n'-1}, \ldots, \lambda'_{2n'-m} = C_\alpha$. We study the m first steps of $\Phi_O(\overline{[\pi]}, \emptyset)$, corresponding to the tableaux T_i with $i = 2n', \ldots, 2n' - m$. $T'_{2n'} = \emptyset$ and by Property 5.1, these m steps correspond to the insertion of the column word of $[T]_\alpha$. And so, by definition of the column word, we have $\lambda'_{2n'}, \lambda'_{2n'-1}, \ldots, \lambda'_{2n'-m} = C_\alpha$. (1)
Now, we prove that $\forall r \in [0, 2n]$, $[T_{2n-r}]_\alpha = T'_{2n+m-r}$. So, we consider steps $i = 2n + m (= 2n' - m), \ldots, m$ of $\Phi_O(\overline{[\pi]}, \emptyset)$ and $i = 2n, \ldots, 0$ of $\Phi_{SO}(\pi, T, U)$ and we use an induction on r. If $r = 0$, by (1) and as $T = T_{2n}$, $T'_{2n+m} = [T_{2n}]_\alpha$. Let r such that $2n \geq r > 0$.
(a) If the $(r+m)^{th}$ step of $\Phi_O(\overline{[\pi]}, \emptyset)$ is a deletion (of a cell with label r), then $\overline{[\pi]}(r) = a$ with $a > r$ (r was inserted during a previous step). By induction, $T'_{2n+m-r+1} = [T_{2n-r+1}]_\alpha$. Then, we can say that r is in the same cell in $T'_{2n+m-r+1}$ and T_{2n-r+1}. In $\Phi_O(\overline{[\pi]}, \emptyset)$ and $\Phi_{SO}(\pi, T, U)$, we delete this cell and we have then $[T_{2n-r}]_\alpha = T'_{2n+m-r}$.
(b) Else, the $(r+m)^{th}$ step of $\Phi_O(\overline{[\pi]}, \emptyset)$ is an insertion. Let $b = \overline{[\pi]}(r)$ be the inserted element.
$(b.1)$ If $b \in [2n]$. Then $(r, b) \in \pi$, $T_{2n-r} = ExtIns(T_{2n-r+1}, b)$, $T'_{2n+m-r} = ExtIns(T'_{2n+m-r+1}, b)$, and during these external insertions, as b is greater than every element of the form j^k, these elements are not moved. By induction $T'_{2n+m-r+1} = [T_{2n-r+1}]_\alpha$. Then b is inserted in the same location in these two tableaux, causes the same sequence of "bumpings" and we have $[T_{2n-r}]_\alpha = T'_{2n+m-r}$.
$(b.2)$ If $b = j^k$. Then, there is $r = U(\alpha'_j - k + 1, j)$. By Property 5.2 and definition of Φ_O, we can deduce that the elements of $\check{\overline{[\pi]}}$ $j^{\alpha'_j}, \ldots, j^{k+1}$ have already been inserted in $T'_{2n+m-r+1}$. By induction $T'_{2n+m-r+1} = [T_{2n-r+1}]_\alpha$, and so the $\alpha'_j - k$ first cells of the j^{th} column of $T'_{2n+m-r+1}$ are labeled, from bottom to top, with $j^{k+1}, \ldots, j^{\alpha'_j}$. Then, the external insertion of $T'_{2n+m-r+1}$ begins with a "bumping" of this column. We must examine two cases.
$(b.2.1)$ If the cell $(\alpha'_j - k + 1, j)$ exists in $T'_{2n+m-r+1}$, by induction, it exists too in T_{2n-r+1}, with the same label x. It means that in $T'_{2n+m-r+1}$ we continue the insertion process with x from the row $\alpha'_j - k + 2$, while, in the same time, $T_{2n-r} = IntIns(T_{2n-r+1}, \alpha'_j - k + 1, j)$. By induction we have then $[T_{2n-r}]_\alpha = T'_{2n+m-r}$.
$(b.2.2)$ Else, the cell $(\alpha'_j - k + 1, j)$ doesn't exist in $T'_{2n+m-r+1}$ and so in T_{2n-r+1}. Then $\Phi_O(\overline{[\pi]}, \emptyset)$ stops the insertion process after the bumping of the j^{th} column of $T'_{2n+m-r+1}$ (that has only labels of the form j^k), while in the same time, $T_{2n-r} = EmptyIns(T_{2n-r+1}, \alpha'_j - k + 1, j)$ and we can say, by induction, that $[T_{2n-r}]_\alpha = T'_{2n+m-r}$.
We just prove that $\forall r \in [2n]$, $[T_{2n-r}]_\alpha = T'_{2n+m-r}$. It follows that $\lambda'_{2n'-m}, \lambda_{2n'-m-1}, \ldots, \lambda'_m = \lambda_{2n}, \lambda_{2n-1}, \ldots, \lambda_0 = R$. (2)
From (2), we have $T'_m = [T_0]_\alpha$. By definition of Φ_{SO}, T_0 is the Ferrers diagram α. It follows immediately that $\lambda'_m, \lambda'_{m-1}, \ldots, \lambda'_0 = C^R_\alpha$. (3)
With (1), (2) and (3), we have $RS_O([\pi]) = (cpl(P), cpl(Q))$. $\square$

Using this construction, we have new proofs of properties of the Robinson-Schensted correspondence for skew oscillating tableaux.

Definition 5.2. *Let n be an integer and P a tableau of $PT(\lambda/\mu)$ (respectively $\overline{PT}(\lambda/\mu)$), such that all these labels are less than or equal to n. We define the tableau P^c of $\overline{PT}(\lambda/\mu)$ (respectively $PT(\lambda/\mu)$) by $P^c(u,v) = n+1-P(u,v)$ for each cell $(u,v) \in \lambda/\mu$.*

Example 5.5. Let $n = 8$.

$$P = \begin{array}{|c|c|c|c|} \hline 6 \\ \hline \square & 2 \\ \hline \square & 1 & 3 & 5 \\ \hline \end{array} \Rightarrow P^c = \begin{array}{|c|c|c|c|} \hline 3 \\ \hline \square & 7 \\ \hline \square & 8 & 6 & 4 \\ \hline \end{array}$$

◇

Theorem 5.1. ([3]) *Let $(\pi, T, U) \in BW_{2n} \times PT(\alpha/\mu) \times \overline{PT}(\alpha/\mu)$ and P, Q two skew oscillating tableaux of $SOT_n(\alpha \to \beta)$, such that $RS_{SO}(\pi, T, U) = (P, Q)$. Then $RS_{SO}(\pi^{-1}, U^c, T^c) = (Q, P)$.*

Proof. By Theorem 4.3 and Lemma 5.1, we have only to prove that $[\pi^{-1}, U^c, T^c] = [\pi, T, U]^{-1}$. Let (π', T', U') such that $[\pi', T', U'] = [\pi, T, U]^{-1}$.

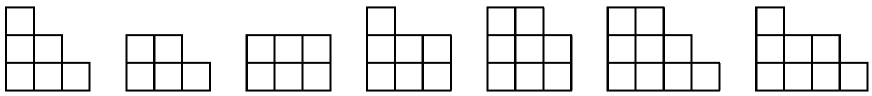
- If $(a, b) \in \overline{[\pi]}$, it follows from the definition of algorithm [.] that $(2n+1-b, 2n+1-a) \in \overline{[\pi]}^{-1}$. (1)
- With little extra work, we can see that if $(\underline{j^k}, b) \in \overline{[\pi]}$, then $(2n+1-b, j^{\alpha'_j-k+1}) \in \overline{[\pi]}^{-1}$. (2)
- Likewise, if $(a, j^k) \in \overline{[\pi]}$, then $(\underline{j^{\alpha'_j-k+1}}, 2n+1-a) \in \overline{[\pi]}^{-1}$. (3)

Therefore, with (1), we have $\pi = \pi^{-1}$. With (2), we can say that if $T(k,j) = b$, then $U'(k,j) = 2n+1-b$. With (3), we can say that if $U(\alpha'_j - k + 1, j) = a$, then $T'(\alpha'_j - k + 1, j) = 2n+1-a$. And we have then $\pi = \pi^{-1}$, $U' = T^c$ and $T' = U^c$. □

Corollary 5.1. ([3]) *RS_{SO} induces a bijection between $SOT_n(\alpha \to \beta)$ and the pairs (σ, T) such that $\sigma \in INV_n^c$, $T \in PT(\alpha/\mu)$, $\zeta^{-1}(\sigma)\dot{\cup}T\dot{\cup}T^c = [2n]$.*

Proof. Immediate consequence of Theorem 5.1 and Proposition 4.1. □

Example 5.6. Let P be the following skew oscillating tableau.

$P =$ 

The pair (P, P) is in bijection with (π, T, U):

$$\pi = \begin{pmatrix} 11 & 9 & 8 \\ 4 & 2 & 5 \end{pmatrix} = \pi^{-1},\ T = \begin{array}{|c|c|c|} \hline 12 \\ \hline \square & 3 \\ \hline \square & \square & 7 \\ \hline \end{array},\ U = \begin{array}{|c|c|c|} \hline 1 \\ \hline \square & 10 \\ \hline \square & \square & 6 \\ \hline \end{array} = T^c$$

and, as $\pi = \pi^{-1}$ and $U = T^c$, we have P in bijection with (σ, T):

$$\sigma = (\mathbf{1})(\mathbf{3})(2,4)(5)(\mathbf{6}),\ T = \begin{array}{|c|c|c|} \hline 12 \\ \hline \square & 3 \\ \hline \square & \square & 7 \\ \hline \end{array}.$$

◇

Moreover, using Theorem 4.4 and Lemma 5.1 we can prove a new result on the number of odd height columns of the final shape of a skew oscillating tableau.

Theorem 5.2. *Let σ be an involution of INV_n^c, $T \in PT(\alpha/\mu)$ and $P \in SOT_n(\alpha \to \beta)$, such that (σ, T) is in bijection with P following Corollary 5.1. Then $odd(\beta) = odd(\mu) + fix_N(\sigma)$.*

Proof. We can prove this result using Lemma 5.1 and Theorem 4.4, showing that every odd column of μ and every fixed point of σ induces a fixed point of $(\overline{[\pi]})$, with $\pi = \zeta^{-1}(\sigma)$. We don't here, but we will give another proof in Section 7. □

We can verify this result on the previous example (Example 5.6), where the final shape β of P has two odd columns, μ has one odd column and σ has one normal fixed point.

6. Knuth classes

Another classical property of the Robinson-Schensted correspondence is due to Knuth ([7]). He gives a characterization of permutations having same insertion tableau (this result was later extended to give a characterization of permutations having same recording tableau).

Definition 6.1. *Two partial permutations on $[n]$, σ and σ', differ by a Knuth relation if and only if there are three integers a, b and c $(1 \le a < b < c \le n)$ such that*

$$\sigma = j_1 \dots j_k\ b\ a\ c\ j_{k+4} \dots j_p \text{ and } \sigma' = j_1 \dots j_k\ b\ c\ a\ j_{k+4} \dots j_p, \text{ or}$$
$$\sigma = j_1 \dots j_k\ a\ c\ b\ j_{k+4} \dots j_p \text{ and } \sigma' = j_1 \dots j_k\ c\ a\ b\ j_{k+4} \dots j_p.$$

Definition 6.2. *Two partial permutations on $[n]$, σ and σ', such that the elements of σ are $j_1 < j_2 < \dots < j_p$ and those of σ' are $j'_1 < j'_2 < \dots < j'_p$, differ by a dual Knuth relation if and only if there is an integer k such that*

$$\sigma = j_{i_1} \dots j_{k+1} \dots j_k \dots j_{k+2} \dots j_{i_p} \text{ and } \sigma' = j'_{i_1} \dots j'_{k+2} \dots j'_k \dots j'_{k+1} \dots j'_{i_p}, \text{ or}$$
$$\sigma = j_{i_1} \dots j_{k+1} \dots j_{k+2} \dots j_k \dots j_{i_p} \text{ and } \sigma' = j'_{i_1} \dots j'_k \dots j'_{k+2} \dots j'_{k+1} \dots j'_{i_p}.$$

Example 6.1. 5 3 4 1 7 6 and 5 3 4 7 1 6 differ by a Knuth relation of the form $(b\ a\ c \to b\ c\ a)$. $\underline{3}$ 10 $\underline{1}$ 5 $\underline{2}$ and $\underline{4}$ 9 $\underline{2}$ 8 $\underline{7}$ differ by a dual Knuth relation of the form $(j_{k+2} \dots j_k \dots j_{k+1} \to j'_{k+1} \dots j'_k \dots j'_{k+2})$ (the elements involved by this transformation are underlined). ◇

Definition 6.3. *Two partial permutations on $[n]$, σ and σ', are K-equivalent, $\sigma \simeq_K \sigma'$ (respectively K'-equivalent, $\sigma \simeq_{K'} \sigma'$), if and only if there are $\sigma_1, \dots, \sigma_p$, partial permutations on $[n]$, such that $\sigma_1 = \sigma$, $\sigma_p = \sigma'$, σ_i and σ_{i+1} differ only by a Knuth relation (respectively a dual Knuth relation).*

The result of Knuth is the following.

Theorem 6.1. ([7]) *Let σ and σ' be two permutations of S_n, respectively in bijection, by the Robinson-Schensted correspondence, with (P, Q) and (P', Q').*

(1) *$P = P'$ if and only if $\sigma \simeq_K \sigma'$.*
(2) *$Q = Q'$ if and only if $\sigma \simeq_{K'} \sigma'$.*

We extend naturally the previous definitions to the biwords.

Definition 6.4. *Two biwords π, π' of BW_n are K-equivalent, $\pi \simeq_K \pi'$ (respectively K'-equivalent, $\pi \simeq_{K'} \pi'$), if and only if there are $\pi_1, \dots, \pi_p$, biwords of BW_n, such that $\pi_1 = \pi$, $\pi_p = \pi'$, $\check{\pi}_i$ and $\check{\pi}_{i+1}$ differ only by a Knuth relation (respectively $\pi_1 = \pi$, $\pi_p = \pi'$, $\hat{\pi}_i = \hat{\pi}_{i+1}$, $\check{\pi}_i$ and $\check{\pi}_{i+1}$ differ only by a dual Knuth relation).*

Example 6.2.

$$\pi = \pi_1 = \begin{pmatrix} 10 & 8 & 5 & 3 \\ 6 & \underline{4} & \underline{1} & \underline{2} \end{pmatrix} \to \pi_2 = \begin{pmatrix} 8 & 7 & 5 & 3 \\ \underline{6} & \underline{1} & \underline{4} & 2 \end{pmatrix} \to \pi_3 = \pi' = \begin{pmatrix} 10 & 8 & 7 & 5 \\ 1 & 6 & 4 & 2 \end{pmatrix} \ :\ \pi \simeq_K \pi'$$

$$\pi = \pi_1 = \begin{pmatrix} 14 & 12 & 11 & 9 & 6 \\ \underline{5} & 10 & \underline{3} & 7 & \underline{4} \end{pmatrix} \to \pi_2 = \pi' = \begin{pmatrix} 14 & 12 & 11 & 9 & 6 \\ 3 & 10 & 2 & 8 & 5 \end{pmatrix} \ :\ \pi \simeq_{K'} \pi'$$

◇

In terms of oscillating tableaux, a pair of standard tableaux having same shape λ (of size n) is an oscillating tableau $\lambda_0 = \emptyset, \ldots, \lambda_{2n} = \emptyset$ such that the first n steps are insertion steps, and the last n are deletion steps and $\lambda_n = \lambda$. The Knuth classes are then the classes (for this kind of tableaux) of tableaux having same $\lambda_0, \ldots, \lambda_n$, and the dual Knuth classes, the classes of tableaux having same $\lambda_n, \ldots, \lambda_{2n}$. We propose now an extension of these results and give a characterization of classes of oscillating tableaux of $OT_{2n}(\mu)$ having same $\lambda_k, \ldots, \lambda_{k'}$ (with $k' \geq k$). This characterization needs preliminary definitions.

Definition 6.5. *Two biwords of BW_n, π and π', are lK' – equivalent ($\pi \simeq_{lK'} \pi'$) if and only if there are $\pi_1, \ \ldots \ , \pi_p$, biwords of BW_n, such that $\pi_1 = \pi$, $\pi_p = \pi'$, the l last elements of $\hat{\pi}_i$ are the same as the l last elements of $\hat{\pi}_{i+1}$ and $\check{\pi}_i$ and $\check{\pi}_{i+1}$ differ by a dual Knuth relation.*

Example 6.3.

$$\pi = \begin{pmatrix} 12 & 11 & 9 & 8 & 7 & 3 \\ 10 & 6 & 2 & 4 & 5 & 1 \end{pmatrix} \simeq_{3K'} \pi' = \begin{pmatrix} 12 & 11 & 10 & 8 & 7 & 3 \\ 9 & 5 & 2 & 4 & 6 & 1 \end{pmatrix}$$

We have the same three elements for $\hat{\pi}$ and $\hat{\pi}'$, i.e. $(8\ 7\ 3)$. Moreover, we pass from $\check{\pi}$ to $\check{\pi}'$ by a dual Knuth relation of the form $(j_{k+2} \ldots j_k \ldots j_{k+1} \rightarrow j'_{k+1} \ldots j'_k \ldots j'_{k+2})$: $(6 \ldots 4 \ldots 5 \rightarrow 5 \ldots 4 \ldots 6)$. $\diamond$

Remark 6.1. If $l = |\hat{\pi}| = |\hat{\pi}'|$, then $\pi \simeq_{K'} \pi'$.

Definition 6.6. *Let π be a biword and k a fixed integer.*
We denote $\pi_{<k}$ (respectively $\pi_{>k}$, $\pi_{\leq k}$, $\pi_{\geq k}$) the restriction of π to pairs (i_l, j_l) such that $j_l < k$ (respectively $j_l > k$, $j_l \leq k$, $j_l \geq k$).
We denote $\pi^{<k}$ (respectively $\pi^{>k}$, $\pi^{\leq k}$, $\pi^{\geq k}$) the restriction of π to pairs (i_l, j_l) such that $i_l < k$ (respectively $i_l > k$, $i_l \leq k$, $i_l \geq k$).

Example 6.4.

$$\pi = \begin{pmatrix} 12 & 11 & 9 & 8 & 7 & 3 \\ 10 & 6 & 2 & 4 & 5 & 1 \end{pmatrix} \Rightarrow \pi_{<6} = \begin{pmatrix} 9 & 8 & 7 & 3 \\ 2 & 4 & 5 & 1 \end{pmatrix}, \ \pi^{<9} = \begin{pmatrix} 8 & 7 & 3 \\ 4 & 5 & 1 \end{pmatrix}$$

$\diamond$

Definition 6.7. *Let P be a tableau of $PT(\lambda/\mu)$ or $\overline{PT}(\lambda/\mu)$ and k a fixed integer.*
We denote $P_{>k}$ (respectively $P_{\geq k}$) the restriction of P to cells $P(i,j) > k$ (respectively $P(i,j) \geq k$), obtained by emptying (deleting if $P \in \overline{PT}(\lambda/\mu)$) the cells $P(i,j) \leq k$ (respectively $P(i,j) < k$).
We denote $P_{<k}$ (respectively $P_{\leq k}$), the restriction of P to cells $P(i,j) < k$ (respectively $P(i,j) \leq k$) obtained by deleting (emptying if $P \in \overline{PT}(\lambda/\mu)$) of the cells $P(i,j) \geq k$ (respectively $P(i,j) > k$).

Example 6.5.

$$P = \begin{array}{|c|c|c|c|} \cline{1-1} 9 \\ \cline{1-2} 7 & 11 \\ \cline{1-3} 4 & 6 & 13 \\ \hline 1 & 2 & 3 & 8 \\ \hline \end{array} \Rightarrow P_{>8} = \begin{array}{|c|c|c|c|} \cline{1-1} 9 \\ \cline{1-2} & 11 \\ \cline{1-3} & & 13 \\ \hline & & & \\ \hline \end{array}, \ P_{<8} = \begin{array}{|c|c|c|} \cline{1-1} 7 \\ \cline{1-2} 4 & 6 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$$

$\diamond$

Now, we introduce some notations used in the main result of this section. We illustrate these notations step by step. Let π be a biword of BW_{2n}, k and k' two fixed integers such that $2n \geq k' \geq k \geq 0$, and $\pi_{k,k'} = (\pi^{>k'})_{\leq k}$.

$$\pi = \begin{pmatrix} 18 & 15 & 12 & 11 & 9 & 5 \\ 3 & 6 & 10 & 5 & 8 & 1 \end{pmatrix}, \ k = 7, \ k' = 9, \ 2n = 18, \ \pi'_{k,k'} = \pi^{>k'}_{\leq k} = \begin{pmatrix} 18 & 15 & 11 \\ 3 & 6 & 5 \end{pmatrix}$$

Let P be the insertion tableau of $\check{\pi}_{k,k'}$ (considered as a partial permutation). We denote $\sigma_{\pi,k,k'}$ the column word of P and $[\sigma_{\pi,k,k'}]$ the column word of $[P_{>2n}]$ ($P_{>2n}$ is obtained in emptying all the cells of P).

$$P = \begin{array}{|c|c}\hline 6 & \\ \hline 3 & \multicolumn{1}{|c|}{5} \\ \hline\end{array}\,,\ [P_{>2n}] = \begin{array}{|c|c}\hline 1^2 & \\ \hline 1^1 & \multicolumn{1}{|c|}{2^1} \\ \hline\end{array}\,,\ \sigma_{\pi,k,k'} = 6\ 3\ 5,\ [\sigma_{\pi,k,k'}] = 1^2 1^1 2^1,$$

Finally, we denote $[\pi]_{k,k'}$ the biword obtained from π by replacing in $\check{\pi}$ the subsequence of elements of $\check{\pi}_{k,k'}$ by $[\sigma_{\pi,k,k'}]$, and $|\pi|_{k,k'}$ the biword obtained from π by replacing in $\check{\pi}$ the subsequence of elements of $\check{\pi}_{k,k'}$ by $\sigma_{\pi,k,k'}$.

$$[\pi]_{k,k'} = \left(\begin{array}{cccccc} 18 & 15 & 12 & 11 & 9 & 5 \\ 1^2 & 1^1 & 10 & 2^1 & 8 & 1 \end{array}\right),\ |\pi|_{k,k'} = \left(\begin{array}{cccccc} 18 & 15 & 12 & 11 & 9 & 5 \\ 6 & 3 & 10 & 5 & 8 & 1 \end{array}\right).$$

In the following theorem, we use two biwords π and δ and we write π^1, π^2, δ^1 and δ^2 respectively for $(\pi^{>k})^{\leq k'}$, $\pi^{>k'}$, $(\delta^{>k})^{\leq k'}$, and $\delta^{>k'}$. Finally, we denote l the number of pairs (i,j) of $(\pi^1)_{\leq k}$ (or $(\delta^1)_{\leq k}$, this is the same).

Theorem 6.2. *Let T and T' be two oscillating tableaux of empty final shape and respective length $2n$ and $2m$ ($T = \lambda_0, \ldots, \lambda_{2n}$ and $T' = \lambda'_0, \ldots, \lambda'_{2m}$), $\pi \in BW_{2n}$ and $\delta \in BW_{2m}$ two biwords such that $RS_O(\pi) = T$ and $RS_O(\delta) = T'$. Let k and k' be two fixed integers such that $0 \leq k \leq k' \leq min\{2n, 2m\}$. Then $\lambda_k, \ldots, \lambda_{k'} = \lambda'_k, \ldots, \lambda'_{k'}$ if and only if*

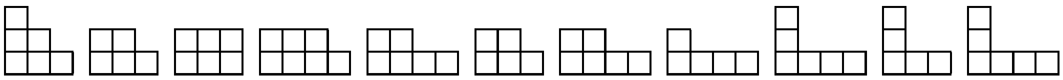
1. $\hat{\pi}^1 = \hat{\delta}^1$
2. $(\pi^1)_{>k} = (\delta^1)_{>k}$
3. $[(\pi^2)_{\leq k'}]_{k,k'} \simeq_K [(\delta^2)_{\leq k'}]_{k,k'}$
4. $(\pi^1)_{\leq k}.|(\pi^2)_{\leq k}|_{k,k'} \simeq_{lK'} (\delta^1)_{\leq k}.|(\delta^2)_{\leq k}|_{k,k'}$.

Example 6.6. Let π and δ be the following biwords

$$\pi = \left(\begin{array}{cccccccccccccccc} 31 & 32 & 30 & 29 & 27 & 26 & 24 & 23 & 21 & 19 & 17 & 16 & 13 & 8 & 6 & 3 \\ 28 & 9 & 20 & 11 & 14 & 25 & 5 & 22 & 18 & 10 & 15 & 12 & 4 & 7 & 1 & 2 \end{array}\right)$$

$$\delta = \left(\begin{array}{cccccccccccccccc} 32 & 31 & 30 & 28 & 27 & 25 & 24 & 23 & 21 & 19 & 17 & 16 & 13 & 9 & 4 & 2 \\ 7 & 29 & 12 & 20 & 26 & 22 & 14 & 6 & 18 & 10 & 15 & 11 & 5 & 8 & 3 & 1 \end{array}\right)$$

They are in bijection with oscillating tableaux T and T' of same length 32 such that $\lambda_{12}, \ldots, \lambda_{22} = \lambda'_{12}, \ldots, \lambda'_{22}$



We can the verify the conditions of Theorem 6.2 on π and δ ($k = 12$ and $k' = 22$).

$$\pi^1 = \left(\begin{array}{ccccc} 21 & 19 & 17 & 16 & 13 \\ 18 & 10 & 15 & 12 & 4 \end{array}\right),\ \pi^2 = \left(\begin{array}{cccccccc} 32 & 31 & 30 & 29 & 27 & 26 & 24 & 23 \\ 28 & 9 & 20 & 11 & 14 & 25 & 5 & 22 \end{array}\right)$$

$$\delta^1 = \left(\begin{array}{ccccc} 21 & 19 & 17 & 16 & 13 \\ 18 & 10 & 15 & 11 & 5 \end{array}\right),\ \delta^2 = \left(\begin{array}{cccccccc} 32 & 31 & 30 & 28 & 27 & 25 & 24 & 23 \\ 7 & 29 & 12 & 20 & 26 & 22 & 14 & 6 \end{array}\right)$$

$$l = 3$$

Conditions *1* and *2* :

$$\check{\pi}^1 = 21\ 19\ 17\ 16\ 13 = \check{\delta}^1,\ (\pi^1)_{>k} = \left(\begin{array}{cc} 21 & 17 \\ 18 & 15 \end{array}\right) = (\delta^1)_{>k}.$$

Condition *3* :

$$[(\pi^2)_{\leq k'}]_{k,k'} = \left(\begin{array}{cccccc} 31 & 30 & 29 & 27 & 24 & 23 \\ 1^2 & 20 & 1^1 & 14 & 2^1 & 22 \end{array}\right) \simeq_K [(\delta^2)_{\leq k'}]_{k,k'} = \left(\begin{array}{cccccc} 32 & 30 & 28 & 25 & 24 & 23 \\ 1^2 & 1^1 & 20 & 22 & 14 & 2^1 \end{array}\right).$$

Condition *4* :

$$(\pi^1)_{\leq k}.|(\pi^2)_{\leq k}|_{k,k'} = \begin{pmatrix} 31 & 29 & 24 & 19 & 16 & 13 \\ 9 & 5 & 11 & 10 & 12 & 4 \end{pmatrix}$$

$$(\delta^1)_{\leq k}.|(\delta^2)_{\leq k}|_{k,k'} = \begin{pmatrix} 32 & 30 & 23 & 19 & 16 & 13 \\ 7 & 6 & 12 & 10 & 11 & 5 \end{pmatrix}.$$

$$(\pi^1)_{\leq k}.|(\pi^2)_{\leq k}|_{k,k'} \simeq_{3K'} (\delta^1)_{\leq k}.|(\delta^2)_{\leq k}|_{k,k'}$$

◇

We give now a sketch of proof of Theorem 6.2, after a preliminary lemma, that is a direct consequence of [2, proof of Theorem 4] (see also [4, Theorem2.8])..

Lemma 6.1. *Let* $\pi \in BW_{2n}$ *and* $T = \Phi_O(\pi, \emptyset)$. *We denote* $T_0, \dots T_{2n}$ *the standard tableaux constructed by* $\Phi_O(\pi, \emptyset)$. *Then, for all* k *such that* $0 \leq k \leq 2n$, *the tableau* T_k *is the insertion tableau of the partial permutation* $(\pi^{>k})_{\leq k}$.

Example 6.7.

$$\pi = \begin{pmatrix} 14 & 13 & 12 & 11 & 9 & 8 & 6 \\ 7 & 2 & 10 & 4 & 1 & 5 & 3 \end{pmatrix},\ k = 8 \Rightarrow (\pi^{>k})_{\leq k} = \begin{pmatrix} 14 & 13 & 11 & 9 \\ 7 & 2 & 4 & 1 \end{pmatrix},\ T_k = \begin{array}{|c|c} \hline 7 \\ \hline 2 \\ \hline 1 & \multicolumn{1}{|c|}{4} \\ \hline \end{array}.$$

◇

Proof of Theorem 6.2. We denote $T_0, \dots, T_{2n}$, $T'_0, \dots, T'_{2m}$ the standard tableaux constructed by $\Phi_O(\pi, \emptyset)$ and $\Phi_O(\delta, \emptyset)$.
(a) We can see that if $\lambda_k, \dots, \lambda_{k'} = \lambda'_k, \dots, \lambda'_{k'}$, then, for $l = 0, \dots, k' - k$, $(T_{k+l})_{>k} = (T'_{k+l})_{>k}$.
(b) Now, we suppose that $\lambda_k, \dots, \lambda_{k'} = \lambda'_k, \dots, \lambda'_{k'}$. The condition *1* follows immediately (the deletions occur at the same steps). The condition *2* follows immediately from *(a)*. If we consider $T_{k'}$ and $T'_{k'}$ (by *(a)*, $(T_{k'})_{>k} = (T'_{k'})_{>k}$), the condition *3* follows from Lemma 6.1. The most difficult part of the proof is to see that the condition *4* holds. With the conditions *1* and *2*, we can say that the insertions in the tableaux T_{k+l} and T'_{k+l} ($k' - k \geq l \geq 0$) of elements lower or equal to k occur at the same steps (although the element a inserted in T_{k+l} is not necessarily the same as the element a' inserted in T'_{k+l}). Moreover, *(a)* allows us to say that the, possible, first element greater than k displaced in T_{k+l} during this external insertion is the same as those displaced in T'_{k+l}, and then stays in the same cells $T_{k+l}(i,j)$ and $T'_{k+l}(i,j)$. It follows that during the insertion of a in T_{k+l} (a' in T'_{k+l}), the sequence of bumpings of elements lower or equal than k must end in the cell $T_{k+l}(i,j)$ ($T'_{k+l}(i,j)$). With little extra work, using the definitions of the $lK'-$equivalence and of the K'-equivalence, we have the condition *4*.
(c) Now, we suppose that π and δ verify the conditions *(1)* to *(4)* and we show that for $k' - k \geq l \geq 0$, we have $(T_{k+l})_{>k} = (T'_{k+l})_{>k}$, using a decreasing induction on l. If $l = k' - k$, the condition *3* allows us to say that $(T_{k'})_{>k} = (T'_{k'})_{>k}$ and we have the basis of our induction. Suppose that for $k' - k \geq l > 0$, $(T_{k+l})_{>k} = (T'_{k+l})_{>k}$. If the step $k + l$ is a deletion of the cell (i,j), we have then immediately (by induction hypothesis) $(T_{k+l-1})_{>k} = (T'_{k+l-1})_{>k}$. Else it is an insertion (a in T_{k+l} and a' in T'_{k+l}). By condition *(2)* $a > k$ if and only if $a = a'$ and we have then $(T_{k+l-1})_{>k} = (T'_{k+l-1})_{>k}$. If $a \leq k$ then a' too and by the condition *4*, the bumping sequences of elements lower or equal to k in T_{k+l} and T'_{k+l} ends at the same cell (i,j) and by induction we can say that $(T_{k+l-1})_{>k} = (T'_{k+l-1})_{>k}$. □

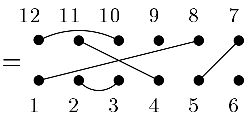
We can now define the analogue of the Knuth classes and of the dual Knuth classes for the Robinson-Schensted correspondence for oscillating tableaux.

Definition 6.8. *Let π be a biword of BW_{2n}. We define the three following biwords π_h, π_l and π_c by*

$$\pi_h = \{(i_l, j_l) | i_l > n,\ j_l > n\},\ \pi_l = \{(i_l, j_l) | i_l \leq n,\ j_l \leq n\},\ \pi_c = \{(i_l, j_l) | i_l > n,\ j_l \leq n\}.$$

Example 6.8. With the following biword

$$\pi = \begin{pmatrix} 12 & 11 & 8 & 7 & 3 \\ 10 & 4 & 1 & 5 & 2 \end{pmatrix} \qquad G(\pi) =$$



we have

$$\pi_h = \begin{pmatrix} 12 \\ 10 \end{pmatrix},\ \pi_l = \begin{pmatrix} 3 \\ 2 \end{pmatrix},\ \pi_c = \begin{pmatrix} 11 & 8 & 7 \\ 4 & 1 & 5 \end{pmatrix}.$$

◇

Corollary 6.1. *Let π and π' be two involutions of BW_{2n} such that $RS_O(\pi) = (P, Q)$ and $RS_O(\pi') = (P', Q')$.*

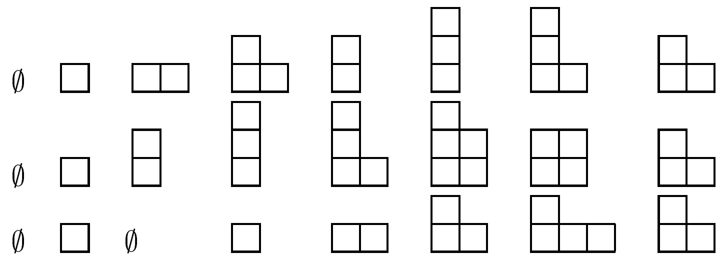
(1) *$P = P'$ if and only if $\pi_l = \pi'_l$ and $\pi_c \simeq_K \pi'_c$.*
(2) *$Q = Q'$ if and only if $\pi_h = \pi'_h$ and $\pi_c \simeq_{K'} \pi'_c$.*

Example 6.9. Let P, P', Q and Q' be the following oscillating tableaux

$P = P' =$

$Q =$

$Q' =$



and π and π' two biwords such that $RS_O(\pi) = (P, Q)$ and $RS_O(\pi') = (P', Q')$:

$$\pi = \begin{pmatrix} 14 & 13 & 12 & 11 & 10 & 7 & 4 \\ 9 & 5 & 3 & 8 & 6 & 1 & 2 \end{pmatrix}\ \pi = \begin{pmatrix} 14 & 12 & 11 & 10 & 9 & 7 & 4 \\ 13 & 5 & 6 & 8 & 3 & 1 & 2 \end{pmatrix}.$$

We have

$$\pi_l = \pi'_l = \begin{pmatrix} 7 & 4 \\ 1 & 2 \end{pmatrix} \text{ and } \pi_c = \begin{pmatrix} 13 & 12 & 10 \\ 5 & 3 & 6 \end{pmatrix}\ \pi'_c = \begin{pmatrix} 12 & 11 & 9 \\ 5 & 6 & 3 \end{pmatrix} \Rightarrow \check{\pi}_c = 5\ 3\ 6 \simeq_K 5\ 6\ 3 = \check{\pi}'_c.$$

◇

Proof of Corollary 6.1. To prove (1), we apply the Theorem 6.2 with $k = 0$ and $k' = n$. The condition *2* is equivalent to $\pi_l = \pi'_l$. Like $k = 0$ we have then $|\pi^2_{\leq k'}| = \pi_c$ and $|\pi'^2_{\leq k'}| = \pi'_c$ and the condition is equivalent to $\pi_c \simeq_K \pi'_c$. The conditions *1* and *3* have no further implication.
Similarly, to prove (2), we apply the Theorem 6.2 with $k = n$ and $k' = 2n$. The condition *2* is equivalent to $\pi_h = \pi'_h$. With the Remark 6.1, the condition *4* is equivalent to $\pi_c \simeq_{K'} \pi'_c$. The two other conditions have no further implication. □

Now, using this result and Lemma 5.1, we have a characterization of Knuth classes and dual Knuth classes for skew oscillating tableaux.

Corollary 6.2. *Let (π, T, U) and (π', T', U') with $RS_{SO}(\pi, T, U) = (P, Q)$ and $RS_{SO}(\pi', T', U') = (P', Q')$.*

(1) *$P = P'$ if and only if $[\pi]_l = [\pi']_l$ and $[\pi]_c \simeq_K [\pi']_c$.*

(2) $Q = Q'$ *if and only if* $[\pi]_h = [\pi']_h$ *and* $[\pi]_c \simeq_{K'} [\pi']_c$.

Proof. This result follows immediately from Corollary 6.1 and Lemma 5.1. □

There are no simple necessary and sufficient conditions on the triples (π, T, U) of Theorem 3.2 to express these notions of Knuth classes and dual Knuth classes, but there are simple necessary conditions.

Corollary 6.3. *Let* (π, T, U) *and* (π', T', U') *with* $RS_{SO}(\pi, T, U) = (P, Q)$ *and* $RS_{SO}(\pi', T', U') = (P', Q')$.

(1) *If* $P = P'$, *then* $\pi_l = \pi'_l$ *and* $U_{\leq n} = U'_{\leq n}$.
(2) *If* $Q = Q'$ *then* $\pi_h = \pi'_h$ *and* $T_{\geq n} = T'_{\geq n}$.

Proof. These are direct consequences of the conditions $[\pi]_l = [\pi']_l$ and $[\pi]_h = [\pi']_h$ of Corollary 6.2, and of the construction used in Lemma 5.1. □

7. Beissinger algorithm

The Robinson-Schensted correspondence implies a bijection between involutions and standard tableaux (an involution σ is in bijection with a couple of standard tableaux (P, P)). In [1], Beissinger gives a simple algorithm to directly obtain the standard tableau P in bijection with σ without using the Robinson-Schensted correspondence. In this section, we extend this result to the case skew oscillating tableaux (see Corollary 5.1).

First, we recall briefly the algorithm of Beissinger. Let σ be an involution written as in Section 2.

Algorithm 4. *Beiss*(σ) *- The output is a tableau* P.
Let $P_0 = \emptyset$.
For i from 1 *to* k, *the number of cycles of* σ:
(a) if the i^{th} *cycle* c_i *of* σ *is a fixed point* (a), *then* $P_i = ExtIns(P_{i-1}, a)$.
(b) else $(c_i = (a, b))$, $P'_i = ExtIns(P_{i-1}, a)$, *this insertion ending in row* u, *and add* b *at the end of row* $u + 1$ *of* P'_i *to obtain* P_i.
Finally, $P = P_k$.

Example 7.1. Let $\sigma = (3)(1, 5)(4, 6)(2, 7)(8)(9)$. The execution of $Beiss(\sigma)$ gives the following result.

$$
\begin{array}{cccccccc}
i & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
P_i & \emptyset & \begin{array}{|c|}\hline 3\\\hline\end{array} & \begin{array}{|c|}\hline 5\\\hline 3\\\hline 1\\\hline\end{array} & \begin{array}{|c|c|}\cline{1-1} 5\\\hline 3&6\\\hline 1&4\\\hline\end{array} & \begin{array}{|c|c|}\cline{1-1} 7\\\hline 5&6\\\hline 3&4\\\hline 1&2\\\hline\end{array} & \begin{array}{|c|c|c|}\cline{1-1} 7\\\cline{1-2} 5&6\\\cline{1-2} 3&4\\\hline 1&2&8\\\hline\end{array} & \begin{array}{|c|c|c|c|}\cline{1-1} 7\\\cline{1-2} 5&6\\\cline{1-2} 3&4\\\hline 1&2&8&9\\\hline\end{array} = P
\end{array}
$$

◇

Theorem 7.1. ([1]) *Let* $\sigma \in INV_n$ *and* $P = Beiss(\pi)$. *Then* σ *is in bijection, by the Robinson-Schensted correspondence for standard tableaux, with* (P, P).

Now, we propose a similar algorithm related to the Robinson-Schensted correspondence for skew oscillating tableaux. This algorithm, denoted $Beiss_{SO}$, has for input a couple (σ, T) satisfying the conditions of Corollary 5.1, for output a triple (π', T', U') where π is a biword, T and P two partial tableaux and has the following property, that is a generalization of Theorem 7.1.

Theorem 7.2. *Let* $P \in SOT_n(\alpha \to \beta)$, $\sigma \in INV_n^c$, $T \in PT(\alpha/\mu)$ *and* $\pi = \zeta^{-1}(\sigma)$ *such that* $RS_{SO}^{-1}(P, P) = (\pi, T, T^c)$ *and* $\Phi_{SO}^{-1}(P) = (\pi', T', U')$. *Then* $Beiss_{SO}(\sigma, T) = (\pi', T', U')$.

Algorithm 5. *$Beiss_{SO}(\sigma, T)$ - The output is a triple (π', T', U').*
$(\sigma \in INV_n^c$, $T \in PT(\alpha/\mu))$ $\pi' = \emptyset$, $T'_0 = \mu$, $U' = T^c_{<n+1}$.
For i from 1 to the number of cycles of σ:
(a) if the i^{th} cycle c_i of σ is such that $c_i = (\mathbf{a}, \mathbf{b})$, then add (a, b) to π', $T'_i = T'_{i-1}$.
(b) else if $c_i = (a, b)$, then $T''_i = ExtIns(T'_{i-1}, a)$, this insertion ending in row q, and adding b at the end of row $q + 1$ of T''_i gives T'_i.
(c) else if $c_i = (a)$, then $T'_i = ExtIns(T'_{i-1}, a)$.
(d) else $(c_i = (\mathbf{a}))$,
(d.1) if $a = T_{<n+1}(u, v)$ and $T'_{i-1}(u, v)$ exists, then $T''_i = IntIns(T'_{i-1}, u, v)$, this insertion ending in row q, and adding a at the end of row $q + 1$ of T''_i gives T'_i.
(d.2) else if $a = T_{<n+1}(u, v)$ and the cell $T'_{i-1}(u, v)$ doesn't exist, then $T''_i = EmptyIns(T'_{i-1}, u, v)$, and adding a at the end of row $u + 1$ of T''_i gives T'_i.
(d.3) else $a = T^c_{<n+1}(u, v)$, $T'_i = T'_{i-1}$.
Finally, $T' = T_k$.

Example 7.2. Let (σ, T) be the following couple,

$$\left(\sigma = (\mathbf{1})(3)(2,5)(\mathbf{6})(\mathbf{4},\mathbf{7})(\mathbf{8}),\ T = \begin{array}{|c|c|cc}\hline & 16 & \multicolumn{2}{c}{} \\ \hline & & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{8} \\ \hline\end{array}\right) \Rightarrow T_{<9} = \begin{array}{|c|ccc}\hline & \multicolumn{3}{c}{} \\ \hline \multicolumn{1}{|c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{6} & \multicolumn{1}{c|}{8} \\ \hline\end{array},\ T^c_{<9} = \begin{array}{|c|c|cc}\hline & 1 & \multicolumn{2}{c}{} \\ \hline & & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} \\ \hline\end{array}$$

$Beiss_{SO}(\sigma, T)$ has the following behaviour

i	0	1	2	3	4	5	6
T'_i	$\begin{array}{\|c\|c\|}\hline & \\ \hline & \\ \hline\end{array}$ (rows: 1; 2)	–	rows: [ ]; [, , 3]	rows: [5]; [, 3]; [, , 2]	rows: [6]; [5]; [3]; [, 2]; [, ,]	–	rows: [6]; [5]; [3]; [, 2, 8]; [, , ,]
π'	$\emptyset$	–	–	–	–	$\begin{pmatrix} 7 \\ 4 \end{pmatrix}$	–

$$\pi' = \begin{pmatrix} 7 \\ 4 \end{pmatrix},\ T' = \begin{array}{|c|c|c|c|}\hline 6 \\ \cline{1-1} 5 \\ \cline{1-1} 3 \\ \hline & 2 & 8 \\ \hline & & & \\ \hline\end{array},\ U' = \begin{array}{|c|c|c|c|}\hline & 1 \\ \hline & & & \\ \hline\end{array}.$$

◇

Remark 7.1. During steps $(d.1)$, if the cell $T'(u, v)$ exists, it can't be empty, because it doesn't belong to the shape of μ ($|T|_{n+1}(u, v)$ is not empty) and there couldn't be an insertion of empty cell or an internal insertion concerning it before these steps.

Remark 7.2. Likewise, during steps $(d.2)$, we have cells $T'(u, v-1)$ and $T'(u-1, v)$ empty. Indeed, for each of these cells, either it belongs to the shape μ (and was not filled during previous steps), or it corresponds to a cell of $|T|_{n+1}$ whose label has been inserted (because labels of $|T|_{n+1}$ are increasing in rows and column and the cycles of σ are inserted in the order of their greatest element) and so is empty (after an internal insertion or empty cell insertion). Therefore, an empty cell insertion in $T'(u, v)$ is possible.

To prove that this algorithm is an extension of the Beissinger algorithm, we first restrict to oscillating tableaux. Hence, $T = \emptyset$ and σ has no bold fixed point (see Corollary 4.2). So, the restriction of algorithm $Beiss_{SO}$ to this case doesn't use steps (d) and contructs a tableau $U' = \emptyset$: we denote it $Beiss_O(\sigma)$ and say that its output is a couple (π', T').

Lemma 7.1. *Let $P \in OT_n(\beta)$, $\sigma \in INV_n^c$ without bold fixed point and $\pi = \zeta^{-1}(\sigma)$ such that $RS_O(\pi) = (P, P)$. If $Beiss_O(\sigma) = (\pi', T')$ and $\Phi_O^{-1}(P) = (\pi'', T'')$ then $T' = T''$ and $\pi' = \pi''$.*

Proof. By definition of σ, we can say that the biword π_c is in bijection with a involution σ_c of INV_n^c having no bold cycle (Proposition 4.1). By the application of $Beiss$ on the normal fixed points and cycles of σ_c, we obtain a partial tableau P_c. We have furthermore $\Phi_O(\pi, \emptyset) = (P, P)$ and with Lemma 6.1, definition of Φ_O, and Theorem 7.1, we can say that $T'' = P_c$ and $\pi'' = \pi_l$. Now, we remark that σ is in fact composed of normal cycles and fixed points due to π_c, i.e. normal cycles and fixed points of σ_c, and bold cycles corresponding to π_l. Then, by definition of $Beiss_O$, we can say that $T' = P_c = T''$. Finally, π' is composed of the bold cycles of σ, i.e. π_l, so $\pi'' = \pi'$. $\square$

Lemma 7.2. *Let $\sigma \in INV_n^c$, $T \in PT(\alpha/\mu)$, $P \in SOT_n(\alpha \to \beta)$ and $\pi = \zeta^{-1}(\sigma)$ such that $RS_{SO}(\pi, T, T^c) = (P, P)$. If $Beiss_O(\zeta(\overline{[\pi]})) = (\pi', T')$ and $Beiss_{SO}(\sigma, T) = (\pi'', T'', U'')$ then $T' = [T'']_\alpha$.*

Proof. We use an induction on the number of cycles of σ. If $\sigma = \emptyset$, then $n = 0$, $P = \alpha = \beta$ and $\mu = \alpha$. So, we have $T'' = \alpha$ and hence, $[T'']_\alpha = [T''] = [\alpha]$. As $\sigma = \emptyset$, $\pi = \emptyset$, and, by Property 5.2 $\overline{[\pi]}$ is the column word of $[\alpha]$. By Lemma 7.1, we have $T' = [\alpha]$ and thus $T' = [T'']_\alpha$.
If π has k cycles ($k > 0$), then $\sigma = c_1.c_2 \dots c_k$, where c_i is either a pair, or a fixed point, and the cycles are in increasing order of their greatest element. So, it is clear that n is the greatest element of c_k. We denote $\sigma_1 = c_1.c_2 \dots c_{k-1}$, $\pi_1 = \zeta^{-1}(\sigma_1)$, $Beiss_O(\zeta(\overline{[\pi_1]})) = (\pi_1', T_1')$ and $Beiss_{SO}(\sigma_1, T_1) = (\pi_1'', T_1'', U_1'')$. By induction, we have $T_1' = [T_1'']_\alpha$. Now, we consider the insertion of the cycle c_k.
(a) If c_k is a bold cycle ($c_k = (\mathbf{a}, \mathbf{n})$), then $(n, a) \in \pi$ and $(2n+1-a, n+1) \in \pi$ (because $\pi = \pi^{-1}$). So, $(n, a) \in \overline{[\pi]}$ and $(2n+1-a, n+1) \in \overline{[\pi]}$ and thus, $\zeta(\overline{[\pi]}) = \zeta(\overline{[\pi_1]}).(\mathbf{a}, \mathbf{n})$. We have then $T' = T_1'$, $T'' = T_1''$, and by induction, $T' = [T'']_\alpha$.
(b) If c_k is a normal cycle ($c_k = (a, n)$), then $(n+1, a) \in \pi$ and $(2n+1-a, n) \in \pi$. Thus, we have $\zeta(\overline{[\pi]}) = \zeta(\overline{[\pi_1]}).(a, n)$. The insertion of elements a and n in T_1'' doesn't modify the internal empty shape of this tableau. Likewise, a and n are greatest than elements of the form j^k, and the insertion of c_k in T_1' doesn't displace an element j^k. As, by induction, $T_1' = [T_1'']_\alpha$, the integers a and n are inserted in the same place in T_1' and T_1'' and cause the same "bumpings" sequence , and thus, we have $T_1' = [T_1'']_\alpha$.
(c) If c_k is a normal fixed point, ($c_k = (n)$), then $(n+1, n) \in \pi$ and $(n) \in \zeta(\overline{[\pi]})$, and for the same reasons than in the case (b), we have $T_1' = [T_1'']_\alpha$.
(d) Finally, if c_k is a bold fixed point ($c_k = (\mathbf{n})$), there are two cases.
$(d.1)$ If $n \notin |T|_{n+1}$, then $n \in U(= T^c)$, and $\overline{[\pi]}$ has a cycle (n, j^k). Thus, by symmetry of $\overline{[\pi]}$, $\zeta(\overline{[\pi]}) = \zeta(\overline{[\pi_1]}).(\mathbf{j^k}, \mathbf{n})$. So, we have $T' = T_1'$ and $T'' = T_1''$, and by induction, $T' = [T'']_\alpha$.
$(d.2)$ If $n \in |T|_{n+1}$, then $n = T(j, k)$. So $(j^k, n) \in \overline{[\pi]}$ and $(n+1, j^{k'}) \in \overline{[\pi]}$, with $k' = \alpha_j' - k + 1$ (by symmetry of $\overline{[\pi]}$). Thus, $\zeta(\overline{[\pi]}) = \zeta(\overline{[\pi_1]}).(j^{k'}, n)$. By induction, $T_1'' = [T_1']_\alpha$, and there is l ($l \geq 0$) cells of the j^{th} column of T_1'' labeled, from bottom to top, with $j^{\alpha_j'-l+1}, \dots, j^{\alpha_j'}$. We can remark that $k' = \alpha_j' - l$. So, the insertion of $j^{k'}$ in T_1' begins in "bumping" the l cells of the form j^k of the j^{th} column, inserting in this column the element $j^{\alpha_j'-m}$.
$(d.2.1)$ If the cell (j, k) of T_1'' doesn't exist, by induction, it doesn't exist in T_1' and $l = k - 1$. Thus, the insertion of $j^{k'}$ in T_1' stops after it has "bumped" the j^{th} column of this tableau, which corresponds clearly to an enpty cell insertion in T_1''. For the two algorithms, the insertion of c_k ends by the adding n at the end of row $k+1$ and we have $T' = [T'']_\alpha$.
$(d.2.2)$ Else, the cell (j, k) of T_1'' exists. It clearly can't be empty (because $T(k, j) = n$), and so, it is labeled with an integer x. Furthermore, by induction, it exists too in T_1', with the same label, and we have $k = m + 1$ (because, we can't make an internal insertion in T_1'' and $T_1' = [T_1'']_\alpha$). So, the end of the insertion of $j^{k'}$ in T_1' corresponds clearly to the internal insertion of the cell (j, k) in T_1''. For the two algorithms, the insertion of the cycle c_k ends by adding n at the end of a same row and we have $T' = [T'']_\alpha$. $\square$

We can now prove that $Beiss_{SO}$ is an extension of $Beiss$.

Proof of Theorem 7.2. Let $Beiss_{SO}(\sigma, T) = (\pi'', T'', U'')$. First, we show that $\pi' = \pi''$. By definition of $Beiss_{SO}$ and by symmetry of π ($\pi = \pi^{-1}$), $\pi'' = \pi_l$, which is clearly equal to π', by definition of $\Phi_{SO}^{-1}(P)$ and $RS_{SO}^{-1}(P, P)$.
Now, we show that $T' = T''$. Let $(T_0, U_0), \ldots, (T_{2n}, U_{2n})$ the tableaux constructed by $RS_{SO}^{-1}(P, P)$. By definition of Φ_{SO}, we have $T' = T_n$, and, if $Beiss_O(\zeta(\overline{[\pi]})) = (\pi_1, P_1)$, by Lemma 7.2, we have $P_1 = [T'']_\alpha$. Now, we have only to show that $P_1 = [T_n]_\alpha$. By Lemma 5.1, $RS_O(\zeta(\overline{[\pi]})) = (cpl(P), cpl(P))$, and by Lemma 7.1 $\Phi_O^{-1}(cpl(P)) = (\pi_1, P_1)$. Let $n' = n + m$ with $\alpha \vdash m$, and $T'_0, T'_1, \ldots, T'_{2n'}$ the tableaux constructed by $RS_O(\overline{[\pi]})$. We have $P_1 = T'_{n'}$ and by the point (2) of the proof of Lemma 5.1, $T'_{n'} = [T_n]_\alpha$. So, $P_1 = [T_n]_\alpha$.
Finally, to show that $U' = U''$, it suffices to remark that $U'' = |T^c|_{n+1}$, that is equal to U_n, and so to U'. □

Finally, we give a proof of Theorem 5.2. Let (σ, T) such that $T \in PT(\alpha/\mu)$ and $Beiss_{SO}(\sigma, T) = P$, with P of final shape β. We want to show that $obb(\beta) = odd(\mu) + fix_N(\sigma)$.

Proof of Theorem 5.2. By Theorems 3.1 and 7.2, we know that T' has for shape β/μ. Therefore, we look at the number of odd columns of T'. Let $T'_0, \ldots, T'_k$ the tableaux constructed by $Beiss_{SO}(\sigma, T)$: we have $T' = T'_k$. We proceed by induction on k. If $k = 0$, then $\sigma = 0$, $fix_N(\sigma) = 0$ and $odd(T') = odd(T'_0) = odd(\mu) + fix_N(\sigma)$ and $odd(\beta) = odd(\mu) + fix_N(\sigma)$. Else ($k > 0$), let σ_1 be σ without its last cycle. By induction, $odd(T'_{k-1}) = odd(\mu) + fix_N(\sigma_1)$. Now, we consider the last cycle of σ.
(a) If it is a bold cycle, $fix_N(\sigma_1) = fix_N(\sigma)$, $T'_k = T'_{k-1}$ and we have $odd(\beta) = odd(T'_{k-1}) = odd(\mu) + fix_N(\sigma)$.
(b) If it is a normal cycle, $fix_N(\sigma_1) = fix_N(\sigma)$, its insertion increments the height of two columns of consecutive height and the number of odd columns doesn't change. So $odd(\beta) = odd(T'_{k-1}) = odd(\mu) + fix_N(\sigma)$.
(c) If it is a bold fixed point, $fix_N(\sigma_1) = fix_N(\sigma)$ and as we perform either no operation or two additions of cells, for the same reason than above, $odd(\beta) = odd(T'_{k-1}) = odd(\mu) + fix_N(\sigma)$.
(d) If it is a normal fixed point (x), $fix_N(\sigma_1) = fix_N(\sigma) + 1$. The cycles being inserted in order of their greatest element, x is inserted at the end of the first row and add an odd column and we have $odd(\beta) = odd(\mu) + fix_N(\sigma)$. □

8. Conclusion

As stated in [3], the Robinson-Schensted correspondence for the skew oscillating tableaux defined by Dulucq and Sagan proves to be a very natural extension of the Robinson-Schensted correspondence, and, following the ideas of Sagan and Stanley in [10], it seems reasonable to hope find another properties of this correspondence, in particular, with the notions of generalized insertion and generalized suppression (see [6]). On the other hand, it could be interesting to study the class of shifted skew oscillating tableaux (see [8, 17]). Finally, there are few other classical properties of the Robinson-Schensted correspondence we didn't extend, like the "Jeu de taquin" of Schützenberger ([13]) or the "vidage-remplissage" ([12, 5], extended in the case of oscillating tableaux in [2]).

References

[1] J. S. Beissinger. Similar constructions for Young tableaux and involutions, and their application to shiftable tableaux. *Discrete Math.*, 67(2):149–163, 1987.

[2] M. Delest, S. Dulucq, and L. Favreau. An analogue to the Robinson-Schensted correspondence for oscillating tableaux. In *Séminaire Lotharingien de Combinatoire (Alghero, 1988).* Univ. Louis Pasteur, Strasbourg, 1988.

[3] S. Dulucq and B. E. Sagan. La correspondance de Robinson-Schensted pour les tableaux oscillants gauches. *Discrete Math.*, 139(1-3):129–142, 1995. Formal power series and algebraic combinatorics (Montreal, PQ, 1992).

[4] L. Favreau. *Combinatoire des tableaux oscillants et des polynômes de Bessel*. PhD thesis, LaBRI, 1991.
[5] D. Foata. Une propriété du vidage-remplissage des tableaux de Young. In *Combinatoire et représentation du groupe symétrique*, pages 121–135. Lecture Notes in Math., Vol. 579. Springer, Berlin, 1977.
[6] S. Fomin. Schur operators and Knuth correspondences. *J. Combin. Theory Ser. A*, 72(2):277–292, 1995.
[7] D. E. Knuth. Permutations, matrices, and generalized Young tableaux. *Pacific J. Math.*, 34:709–727, 1970.
[8] B. E. Sagan. Shifted tableaux, Schur Q-functions, and a conjecture of R. Stanley. *J. Combin. Theory Ser. A*, 45(1):62–103, 1987.
[9] B. E. Sagan. *The symmetric group*. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1991. Representations, combinatorial algorithms, and symmetric functions.
[10] B. E. Sagan and R. P. Stanley. Robinson-Schensted algorithms for skew tableaux. *J. Combin. Theory Ser. A*, 55(2):161–193, 1990.
[11] C. Schensted. Longest increasing and decreasing subsequences. *Canad. J. Math.*, 13:179–191, 1961.
[12] M.-P. Schützenberger. Quelques remarques sur une construction de Schensted. *Math. Scand.*, 12:117–128, 1963.
[13] M.-P. Schützenberger. La correspondance de Robinson. In *Combinatoire et représentation du groupe symétrique*, pages 59–113. Lecture Notes in Math., Vol. 579. Springer, Berlin, 1977.
[14] S Sundaram. *On the combinatorics of representations of $Sp(2n, \mathbb{C})$*. PhD thesis, M.I.T, 1986.
[15] S. Sundaram. The Cauchy identity for $Sp(2n)$. *J. Combin. Theory Ser. A*, 53(2):209–238, 1990.
[16] G. Viennot. Une forme géométrique de la correspondance de Robinson-Schensted. In *Combinatoire et représentation du groupe symétrique*, pages 29–58. Lecture Notes in Math., Vol. 579. Springer, Berlin, 1977.
[17] D. R. Worley. *A theory of shifted Young tableaux*. PhD thesis, M.I.T, 1984.

E-mail address: `[chauve,dulucq]@labri.u-bordeaux.fr`

LaBRI, Université Bordeaux I, 351 Cours de la Libération, 33405 Talence, France