Enumeration of some labelled trees

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ENUMERATION OF SOME LABELLED TREES

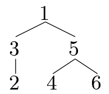
CEDRIC CHAUVE, SERGE DULUCQ AND OLIVIER GUIBERT

Abstract. In this paper we are interesting in the enumeration of rooted labelled trees according to the relationship between the root and its sons. Let $\mathcal{T}_{n,k}$ be the family of Cayley trees on $[n]$ such that the root has exactly k smaller sons. In a first time we give a bijective proof of the fact that $|\mathcal{T}_{n+1,k}| = \binom{n}{k}n^{n-k}$. Moreover, we use the family $\mathcal{T}_{n+1,0}$ of Cayley trees for which the root is smaller than all its sons to give combinatorial explanations of various identities involving n^n. We rely this family to the enumeration of minimal factorization of the n-cycle $(1, 2, \ldots, n)$ as a product of transpositions. Finally, we use the fact that $|\mathcal{T}_{n+1,0}| = n^n$ to prove bijectively that there are $2n^n$ ordered alternating trees on $[n + 1]$.

Résumé. Dans cet article nous nous intéressons à l'énumération d'arbres étiquetés enracinés, en considérant un nouveau paramètre relatif à l'ordre existant entre la racine et ses fils. Soit donc $\mathcal{T}_{n,k}$ la famille des arbres de Cayley sur $[n]$ tels que la racine ait exactement k fils qui lui soient inférieurs. Dans un premier temps, nous donnons une preuve bijective du fait que $|\mathcal{T}_{n+1,k}| = \binom{n}{k}n^{n-k}$. Ensuite, nous donnons des interprétations combinatoires de plusieurs identités relatives à n^n en utilisant la famille $\mathcal{T}_{n+1,0}$ des arbres de Cayley pour lesquels la racine est inférieure à tous ses fils. Nous lions également cette famille à l'énumération des factorisations minimales transitives du n-cycle $(1, 2, \ldots, n)$ comme produit de transpositions. Finalement, nous utilisons le fait que $|\mathcal{T}_{n+1,0}| = n^n$ pour démontrer combinatoirement qu'il y a $2n^n$ arbres ordonnés alternants sur $[n + 1]$.

1. Introduction

A labelled rooted Cayley tree on $[n]$ is an acyclic connected graph with n vertices, labelled $1, 2, \ldots, n$, such that one vertex, called the root, is distinguished and gives an orientation to the graph (we can speak for the sons of a vertex). Trees were first extensively studied by Cayley who states [4] the well known formula n^{n-1} for the number of labelled rooted trees on n vertices. Afterwards, we call these trees *Cayley trees*. The first explicit combinatorial proof of the formula n^{n-1} for the number of Cayley trees on $[n]$ is due to Prüfer [22, 18]. This proof is based on a beautiful correspondence between a Cayley tree on $[n]$ and a word of length $n-1$ on the alphabet $[n]$, so called the Prüfer sequence of the Cayley tree. For a given Cayley tree T on $[n]$, the Prüfer sequence is obtained recursively by removing the largest leaf y (nonroot endpoint) of T (and its incident edge) and adding to the Prüfer sequence the only vertex x adjacent to y in T (the father of y). The following example shows a Cayley tree and its corresponding Prüfer sequence.



A number of proofs of Cayley's formula are known [17, 18, 23], and various formulas for the number of Cayley trees verifying certain properties have been established (see [18] for a survey on the enumeration of labelled trees). However, if we consider the $(n + 1)^n$ Cayley trees on $[n + 1]$, there is no direct combinatorial explanation on these trees of the following distribution derived from the binomial formula

$$(n+1)^n = \sum_{k=0}^{n} \binom{n}{k} n^{n-k}.$$

Note that there is an explanation of the binomial formula for $(n + 1)^{(n-1)}$ in the case of labelled (unrooted) trees on $[n + 1]$ in which the degree of a given vertex is specified.

Let $\mathcal{T}_{n,k}$ be the family of Cayley trees on $[n]$ such that among the set of all the sons of the root, exactly k of them are smaller than the root. Considering this parameter on Cayley trees, we give a combinatorial proof of the required distribution

$$|\mathcal{T}_{n+1,k}| = \binom{n}{k} n^{n-k}$$

first for the particular case $k = 0$ in Section 2, and next for the general case by extending the previous bijection. Then, we give formulas for the number of rodered, plane and cyclic trees such that the root has no smaller son.

In the following section, we study from a combinatorial point of view the family of trees $\mathcal{T}_{n+1,0}$ (Cayley trees in which the root is smaller than its sons) and we show that these trees have interesting enumerative properties. In particular, we give combinatorial interpretations of four distributions of n^n which are, as far as we know, not present in the litterature or have no combinatorial explanation. The most interesting is

$$n^n = (n+1)^{n-1} + \sum_{k=1}^{n-1} (n-k)(n+1)^{n-k-1} n^{k-1}$$

which will be proved with an extsension ofthe Prüfer encoding. We also obtain a distribution of Cayley trees according to the number of decreasing edges, which leads to a new identity about Stirling numbers of the first kind. Always considering the Cayley trees of $\mathcal{T}_{n+1,0}$, we complete this section giving the first bijective proof of the following special case of an identity introduced in [11] (see also [24])in the study of minimal transitive factorizations of permutations as a product of transpositions

$$n^n = n(n-1)^{n-1} + \sum_{k=1}^{n-1} \binom{n}{k} k^k \; (n-1-k)^{n-1-k}.$$

In the next section, we rely these results to the enumeration of minimal factorizations of the n-cycle $(1, 2, \ldots, n)$ as a product of transpositions. It is well known that the number of such factorizations is n^{n-2}. This result was first given in [7] and the first direct bijective proof is due to Moszkowski [19]. Here we give a new "algorithmic" description of the bijection of Moszkowski which allows to give an interpretation of n^n in terms of such factorizations.

Finally, in the last section, we derive from the bijection used to prove $|\mathcal{T}_{n+1,0}| = n^n$ (given in Section 2) a bijective proof of the fact that the number of ordered alternating trees (rooted trees with an order on the sons of each vertex and every three consecutive vertices a, b, c verifying $a < b > c$ or $a > b < c$) on $[n+1]$ is given by $2n^n$ (see [5]).

2. Number of sons greater than the root in a Cayley tree

Theorem 2.1. *Let $\mathcal{T}_{n,k}$ be the set of Cayley trees on $[n]$ such that the root has exactly k smaller sons. Then*

$$|\mathcal{T}_{n+1,k}| = \binom{n}{k} n^{n-k}.$$

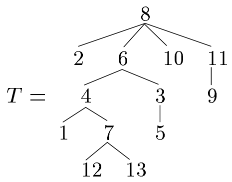
In order to prove this result, we proceed in two stages. First, we focuse on the case $k = 0$ and we show, using a bijection ϕ, that the number of Cayley trees on $[n+1]$ such that the root is smaller than all its sons is n^n. Next, we prove the general result with a bijection Φ which generalizes ϕ.

Let $\mathcal{A}_n$ be the family of rooted Cayley trees on $[n]$ such that the vertex n is a leaf. It is an immediate consequence of the Cayley's formula (by considering the Prüfer sequence) that $|\mathcal{A}_{n+1}| = n^n$.

Lemma 2.1. *There is a bijection ϕ from $\mathcal{A}_n$ (the family of rooted Cayley trees on $[n]$ such that the vertex n is a leaf) to $\mathcal{T}_{n,0}$.*

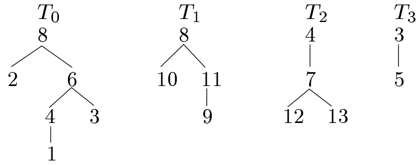
Proof. First, we introduce the notions of decreasing edge and decreasing subtree. Let x and y be two vertices of a tree T such that x is the father of y. The edge (x, y) is said to be a *decreasing edge* if $x > y$. A subtree of T is said to be a *decreasing subtree* if all its edges are decreasing.

Now, we describe the bijection ϕ and illustrate it step by step with an example. Let T be a tree of $\mathcal{A}_n$ with root r. In order to construct a tree $T' = \phi(T)$ belonging to $\mathcal{T}_{n,0}$ we proceed in three steps.

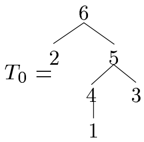


Step 1: First we decompose T into a set $T_0, T_1, \ldots, T_k$ of trees such that:

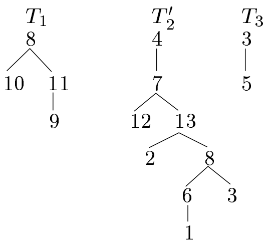
- T_0 is the maximal (considering the number of vertices) decreasing subtree of T having r for root,
- $T_1, T_2, \ldots, T_k$ are the trees not reduced to a single vertex which are obtained from T by deleting all the edges belonging to T_0.



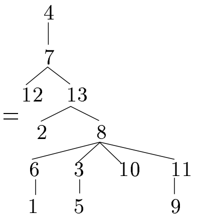
Then, by definition, each tree T_i with $1 \leq i \leq k$ has a root r_i smaller than its sons. Moreover, if the decreasing tree T_0 has m vertices, it can be described by a couple (T_0', S) where T_0' is a decreasing tree on $[m]$ and S the subset of size m of $[n]$ containing the labels of the vertices of T_0. In our example, we have $m = 6$, $S = \{1, 2, 3, 4, 6, 8\}$ and T_0' is the following tree.



Step 2: Next, let T_j be the tree containing the vertex n and r_j its root. We define $S' = \{n\} \cup S \setminus \{r_j\}$. In our example, T_j is the tree T_2 with root $r_j = 4$ and $S' = \{1, 2, 3, 6, 8, 13\}$. Now we graft on the vertex n of T_j the decreasing tree described by the couple (T_0', S') to obtain the new tree T_j'. Hence, we have a set of k trees $T_1, \ldots, T_{j-1}, T_j', T_{j+1}, \ldots, T_k$.



Step 3: Finally, each of the remaining trees T_i (with $1 \leq i \leq k$ and $i \neq j$) is grafted on the vertex r_i in T_j'. So, we obtain a tree $T' = \phi(T)$ belonging to $\mathcal{T}_{n,0}$.

$T' = \phi(T) =$ 

Remark 2.1. Clearly the number of decreasing edges in $\phi(T)$ is the same than in T.

Now, we give a quick description of the inverse of ϕ. Let T' be a tree of $\mathcal{T}_{n,0}$ having r for root.

- In the tree T', let U be the maximal decreasing subtree rooted in n. U can be described by the couple (U', S'). Moreover, we obtain the trees $T_1, T_2, \ldots, T_k$ as before by deleting in T' all the edges belonging to U.
- Now, let T_0 be the tree described by the couple (U', S) where $S = \{r\} \cup S' \setminus \{n\}$.
- Finally, we graft the root of each tree T_i for all $1 \leq i \leq k$ on the same vertex of T_0 to obtain the tree T.

□

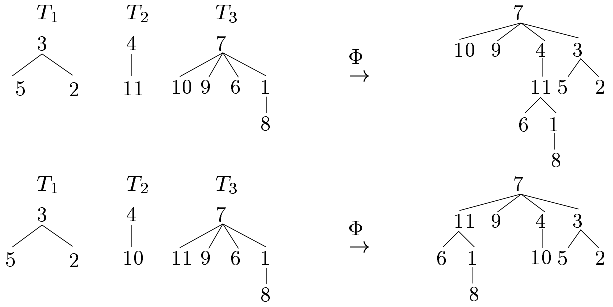
Now, we can complete the proof of Theorem 2.1. First, we denote $\mathcal{F}_{n,k}$ the set of forests of k Cayley trees having n vertices and such that n is a leaf. There is a direct bijection from these forests to the Cayley trees on $[n+1]$ having 1 for root and $n+1$ as a leaf, and such that 1 has k sons. From the Prüfer encoding of trees, we can say that these trees are encoded by words of length $n-1$ on the alphabet $[n+1]$ having $k-1$ occurrences of 1 and no occurrence of $n+1$. Hence, we have $|\mathcal{F}_{n,k}| = \binom{n-1}{k-1}(n-1)^{n-k}$.

The proof of Theorem 2.1 can be immediately deduced from the following lemma.

Lemma 2.2. *There is a bijection Φ from $\mathcal{F}_{n,k+1}$ (the set of forests of $k+1$ Cayley trees having n vertices and such that n is a leaf) to $\mathcal{T}_{n,k}$.*

Proof. Let F be a forest of $\mathcal{F}_{n,k+1}$, that is a forest of $k+1$ trees $T_1, T_2, \ldots, T_{k+1}$ of respective roots $r_1, r_2, \ldots, r_{k+1}$ with $r_1 < r_2 < \cdots < r_{k+1}$. Let T_j be the tree of F containing n (as a leaf). We distinguish two cases.

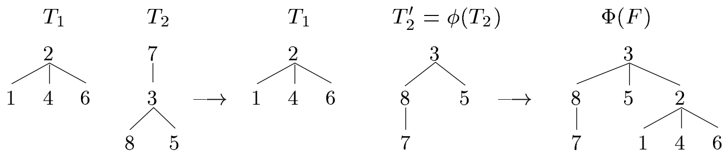
Case 1: If $r_{k+1} \neq r_j$, or if $r_{k+1} = r_j$ and the path from r_{k+1} to n starts with an increasing edge, then $\Phi(F)$ is obtained from F by grafting on n all the sons of r_{k+1} smaller than r_{k+1} and by grafting on r_{k+1} the k trees $T_1, T_2, \ldots, T_k$.



Remark 2.2. Let r be the root of $\Phi(F)$. Then all the sons of n in $\Phi(F)$ are smaller than r.

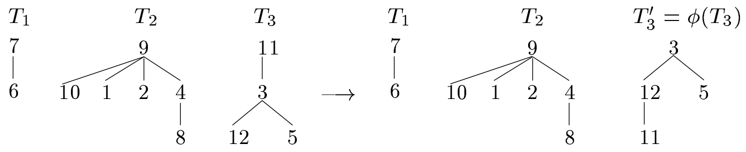
Case 2: Now, we suppose that $r_{k+1} = r_j$ and the path from r_{k+1} to n starts with a decreasing edge. In a first time, we apply to the tree T_{k+1} the bijection ϕ previously defined and we denote by T'_{k+1} the new tree obtained and r'_{k+1} its root (r'_{k+1} replaces r_{k+1}) which has no smaller son by definition of ϕ. We distinguish two subcases.

<u>Subcase 2a</u>: If $r'_{k+1} > r_k$ then $\Phi(F)$ is obtained from F by grafting on r'_{k+1} the k trees $T_1, T_2, \dots, T_k$.

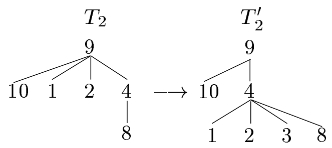


Remark 2.3. Let r be the root of $\Phi(F)$. At least one son of n is greater than r. Moreover, the path from r to n starts with an increasing edge.

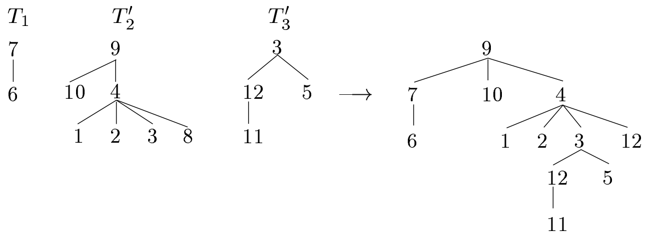
<u>Subcase 2b</u>: Otherwise $r'_{k+1} < r_k$ and so the greatest root of the forest $T_1, T_2, \dots, T_k, T'_{k+1}$ is r_k.



First let $T_k^{\searrow}$ be the maximal decreasing subtree of T_k rooted in r_k. As in the proof of Lemma 2.1, it can be described by a couple $(T'^{\searrow}_k, S)$. Then, we define $S' = \{r'_{k+1}\} \cup S \setminus \{r_k\}$ and T'_k is the tree T_k where the edges of $T_k^{\searrow}$ are deleted and the decreasing tree described by $(T'^{\searrow}_k, S')$ is grafted on r_k, the other edges being unchanged.



Next, we graft in T'_k on r'_{k+1} the tree T'_{k+1} and on r_k the remaining trees $T_1, T_2, \dots, T_{k-1}$. So, we obtain a tree on $[n]$ such that the root has exactly k smaller sons.



Remark 2.4. Let r be the root of $\Phi(F)$. At least one son of n is greater than r. Moreover, the path from r to n starts with a decreasing edge.

Note that each Cayley tree verifies exactly one of the remarks 2.2, 2.3 and 2.4. So, in order to define the inverse of this map Φ that is to build a forest of $\mathcal{F}_{n,k+1}$ from a Cayley tree on $[n]$ such that the root has exactly k smaller sons, it suffices to find which remark is verified by this tree and then to perform the corresponding case or subcase in reverse order. □

3. Ordered, plane and cyclic trees

In this section, we extend the result of Lemma 2.1 to three others families of trees, the ordered, plane and cyclic trees, which have the following definitions (see [1], for example, for their basic properties).

Definition 3.1. *A tree on $[n]$ is* ordered *if there is an order on the sons of each vertex. A* plane *tree is an equivalence class of ordered trees embedded in an oriented plane. A* cyclic *tree is an equivalence class of ordered trees where two trees are equivalent if one can transform one into the other by circular permutations of the subtrees of each vertex.*

Let $\mathcal{B}_n$ be a family of rooted labelled trees on $[n]$: we denote by b_n the cardinality of $\mathcal{B}_n$ and by $B(x)$ the exponential generating function of these trees. We suppose that $B(x)$ satisfies the functional equation $B(x) = xR(B(x))$, where $R(x) = \sum_{k\geq 0} r_k x^k/k!$ is a formal power series such that $R(0) \neq 0$. Moreover, we denote by $\mathcal{B}_{n,k}$ the elements of $\mathcal{B}_n$ such that the root has exactly k smaller sons (we denote $b_{n,k} = |\mathcal{B}_{n,k}|$ and $B_k(x)$ the exponential generating function of these trees). Finally, we denote by $\tilde{\mathcal{B}_n}$ the elements of $\mathcal{B}_n$ such that the root is 1 (hence $b_n = n\tilde{b}_n$).

Lemma 3.1. *If*

$$S(x) = \int_0^x R(t)dt = \sum_{k\geq 1} r_{k-1}\frac{x^k}{k!}$$

then

$$B_0(x) = \frac{x}{B(x)} S(B(x)).$$

Proof. We can express $b_{n,0}$ with the following formula:

$$b_{n,0} = \sum_{r=1}^{n} \left\{ \sum_{k=0}^{n-r} \binom{n-r}{k} \left\{ \sum_{n_1+\ldots+n_k=n-1} \binom{n-1-k}{n_1-1,\ldots,n_k-1} \prod_{i=1}^{k} r_k \tilde{b}_{n_i} \right\} \right\},$$

where r is the label of the root, k the number of sons of this root, n_i the sizes of the subtrees having these sons for roots and r_k the number of ways to graft these subtrees on the root r. Now we can rewrite this formula in the following way:

$$b_{n,0} = \sum_{k=0}^{n-1} \left\{ \sum_{n_1+\ldots+n_k=n-1} \binom{n-1-k}{n_1-1,\ldots,n_k-1} \left\{ \prod_{i=1}^{k} r_k \tilde{b}_{n_i} \right\} \left\{ \sum_{r=1}^{n} \binom{n-r}{k} \right\} \right\}.$$

As we have clearly that

$$\sum_{r=1}^{n} \binom{n-r}{k} = \binom{n}{k+1},$$

we deduce that

$$b_{n,0} = \sum_{k=0}^{n-1} \left\{ \sum_{n_1+\ldots+n_k=n-1} \left\{ \prod_{i=1}^{k} r_k \tilde{b}_{n_i} \right\} \binom{n-1-k}{n_1-1,\ldots,n_k-1} \binom{n}{k+1} \right\},$$

which implies that

$$\frac{b_{n,0}}{n!} = \sum_{k=0}^{n-1} \frac{1}{(k+1)!} r_k \left\{ \sum_{n_1+\ldots+n_k=n-1} \left\{ \prod_{i=1}^{k} \frac{\tilde{b}_{n_i}}{(n_i-1)!} \right\} \right\}.$$

For the generating function we have

$$B_0(x) = \sum_{k\geq 0} \frac{x}{(k+1)!} r_k \left\{ \sum_{(n_1,\ldots,n_k)} \left\{ \prod_{i=1}^{k} \frac{b_{n_i}}{n_i!} x^{n_i} \right\} \right\}$$

which is equivalent to

$$B_0(x) = \sum_{k\geq 0} \frac{x}{(k+1)!} r_k (B(x))^k$$

Finally we apply a classical result (see [13] for example): if $G(x)$, $F(x)$, $R(x)$ and $S(x)$ are formal power series such that

$$G(x) = \sum_{k\geq 0} \frac{1}{(k+1)!} r_k (F(x))^k,\ R(x) = \sum_{k\geq 0} r_k \frac{x^k}{k!},\ S(x) = \int_0^x R(t)dt = \sum_{k\geq 1} r_{k-1} \frac{x^k}{k!},$$

then

$$G(x) = \frac{x}{F(x)} S(F(x)).$$

□

We use this lemma together with the following version of the Lagrange inversion formula (see [9] for example).

Theorem 3.1. *Let $f(x) = \sum_{k\geq 0} f_k x^k$ be a formal power series such that $f_0 \neq 0$ sand let $Y(x)$ be the unique formal power series solution of the equation $Y = x\ f(Y)$. The coefficients of $g(Y)$ (for an arbitrary formal power series $g(x)$) are given by*

$$[x^n]g(Y(x)) = \frac{1}{n}[x^{n-1}](f(x))^n \frac{\partial g}{\partial x}(x).$$

Then we have a formal proof of Lemma 2.1 and of its analogous for the family of ordered, cyclic and plane trees.

Proof of Lemma 2.1. It suffices to remark than $r_k = 1$ for each k. By applying Lemma 3.1, we have

$$R(x) = e^x \Rightarrow S(x) = e^x - 1.$$

We deduce that

$$T_0(x) = \frac{x}{T(x)}\left(e^{T(x)} - 1\right).$$

Then we can apply Theorem 3.1 with $f(x) = e^x$ and $g(x) = -1/e^x$. □

Proposition 3.1. *The number of ordered trees on $[n]$ such that the root is smaller than its sons is*

$$\frac{(2n-2)!}{(n-1)!} - \sum_{k=0}^{n-2} \frac{(n+k-1)!}{(n-k-1)k!}.$$

Proof. We use the notations $O(x)$, $O_0(x)$ and $o_{n,0}$ for this family of trees. Here we have clearly $r_k = k!$ and $S(x) = -\ln(1-x)$. If we apply Lemma 3.1 we have that

$$O_0(x) = \frac{-x}{O(x)} \ln(1 - O(x))$$

Moreover, we know (see [9]) that $O(x)$ satisfies the functional equation

$$O(x) = \frac{x}{1 - O(x)}$$

So

$$O_0(x) = -(1 - O(x)) \ln(1 - O(x)).$$

We can apply Theorem 3.1 with the following functions

$$f(x) = \frac{1}{1-x},$$

$$g(x) = -(1-x)\ln(1-x) \Rightarrow \frac{\partial g}{\partial x}(x) = 1 + \ln(1-x).$$

Then we have

$$\begin{aligned} o_{n,0} &= n!\frac{1}{n}[x^{n-1}]\left(\frac{1}{1-x}\right)^n (1+\ln(1-x)) \quad n \geq 1 \\ \Rightarrow o_{n,0} &= (n-1)!\left([x^{n-1}](1-x)^{-n} + [x^{n-1}](1-x)^{-n}\ln(1-x)\right) \\ \Rightarrow o_{n,0} &= (n-1)!\left(\binom{2n-2}{n-1} - \sum_{k=0}^{n-2}\frac{1}{n-1-k}\binom{n+k-1}{k}\right) \\ \Rightarrow o_{n,0} &= \frac{(2n-2)!}{(n-1)!} - \sum_{k=0}^{n-2}\frac{(n+k-1)!}{(n-k-1)k!}. \end{aligned}$$

□

Let $s(n,k)$ denote the Stirling numbers of first kind (see [13] for a description of these numbers).

Proposition 3.2. *The number of cyclic trees on $[n]$ such that the root is smaller than its sons is*

$$\sum_{k=0}^{n}\left(\frac{n!}{(n-k)!}s(n-1,k) - \frac{(n-2)!}{(n-1-k)!}s(n,k)\right).$$

Proof. We use the notations $C(x)$, $C_0(x)$ and $c_{n,0}$ for this family of trees. Like previously, we look at the series R. We have $r_k = (k-1)!$. It follows that

$$\begin{aligned} S(x) &= \sum_{k\geq 2}\frac{x^k}{k(k-1)} \\ \Rightarrow \frac{\partial^2 S}{\partial x^2}(x) &= \sum_{k\geq 0} x^k = \frac{1}{\ln(1-x)} \\ \Rightarrow S(x) &= (1-x)\ln(1-x) + x. \end{aligned}$$

Using Lemma 3.1 we have

$$C_0(x) = \frac{x}{C(x)}\left((1-C(x))\ln(1-C(x)) + C(x)\right) \tag{1}$$

with (see [9, 1] for example)

$$C(x) = x(1-\ln(1-C(x))).$$

So

$$C_0(x) = C(x) + \frac{\ln(1-C(x))}{1-\ln(1-C(x))}.$$

We can apply Theorem 3.1 with

$$f(x) = 1 - \ln(1-x)$$

$$g(x) = x + \frac{\ln(1-x)}{(1-\ln(1-x))} \Rightarrow \frac{\partial g}{\partial x}(x) = 1 - \frac{(1-\ln(1-x))^{-2}}{(1-x)},$$

and we have

$$(f(x))^n \frac{\partial g}{\partial x}(x) = (1-\ln(1-x))^n - \frac{(1-\ln(1-x))^{n-2}}{(1-x)}.$$

Let

$$h(x) = \frac{(1-\ln(1-x))^{n-2}}{(x-1)},\ l(x) = \frac{(1-\ln(1-x))^{n-1}}{n-1}.$$

We have

$$h(x) = \frac{\partial l}{\partial x}(x).$$

Moreover, if

$$h(x) = \sum_{k\geq 1} h_k x^k,\ l(x) = \sum_{k\geq 1} l_k x^k,$$

then

$$h_n = (n+1) l_n.$$

So

$$[x^{n-1}]h(x) = n[x^n]l(x) = n[x^n]\frac{1}{n-1}(1-\ln(1-u))^{n-1}.$$

We know (see [13]) that

$$\left(\ln\left(\frac{1}{1-x}\right)\right)^k = \sum_{i\geq 0} k! s(i,k) \frac{x^i}{i!}$$
$$\Rightarrow (1-\ln(1-x))^n = \sum_{k=0}^{n} \binom{n}{k} k! \left\{ \sum_{i=0} s(i,k) \frac{x^i}{i!} \right\}.$$

It follows that

$$[x^{n-1}](1-\ln(1-x))^n = \sum_{k=0}^{n} \binom{n}{k} k! s(n-1,k) \frac{1}{(n-1)!}$$

$$[x^{n-1}]\frac{(1-\ln(1-x))^{n-2}}{(1-x)} = \frac{n}{n-1}[x^n](1-\ln(1-x))^{n-1} = \frac{n}{n-1}\sum_{k=0}^{n-1} \binom{n-1}{k} k! s(n,k) \frac{1}{n!}.$$

As

$$[x^n]C_0(x) = \frac{1}{n}[x^{n-1}](f(x))^n \frac{\partial g}{\partial x}(x),$$

we have

$$c_{n,0} = (n-1)! \left\{ \left(\sum_{k=0}^{n} \binom{n}{k} k! s(n-1,k) \frac{1}{(n-1)!} \right) - \left(\frac{n}{n-1} \sum_{k=0}^{n-1} \binom{n-1}{k} k! s(n,k) \frac{1}{n!} \right) \right\}$$

and we have proved the result. □

Proposition 3.3. *The number of plane trees on* $[n]$ *such that the root is smaller than its sons is*

$$\left(2\sum_{k=0}^{n-2}\frac{(n-2-k)!}{k!}\frac{n-1}{n-1-k}\right)-\frac{(2n-3)!}{(n-2)!}.$$

Proof. We use the notations $P(x)$, $P_0(x)$ and $p_{n,0}$ for this family of trees. For this family of trees, we cannot use Lemma 3.1 because the sons of the root are ordered, while the sons of the other vertices are cyclic. However, there is a natural generalisation of Lemma 3.1 for this kind of structures (the proof is almost exactly the same), which allows to say that:

$$P_0(x)=\frac{x}{O(x)}\left((1-O(x))\ln(1-O(x))+O(x)\right)$$

We recall that

$$O(x)=\frac{x}{(1-O(x))}.$$

So

$$P_0(x)=(1-O(x))^2\ln(1-O(x))+O(x)-(O(x))^2.$$

We apply Theorem 3.1 with

$$f(x)=\frac{1}{1-x}$$

$$g(x)=(1-x)^2\ln(1-x)+x-x^2\Rightarrow\frac{\partial g}{\partial x}(x)=1-(1-x)(1+2\ln(1-x))-2x$$

which gives

$$(f(x))^n\frac{\partial g}{\partial x}(x)=\left(\frac{1}{1-x}\right)^n(1-(1-x)(1+2\ln(1-x))-2x)$$

$$\Rightarrow(f(x))^n\frac{\partial g}{\partial x}(x)=\left(\frac{1}{1-x}\right)^n-\left(\frac{1}{1-x}\right)^{n-1}(1+2\ln(1-x))-2x\left(\frac{1}{1-x}\right)^n$$

$$\Rightarrow(f(x))^n\frac{\partial g}{\partial x}(x)=-x\left(\frac{1}{1-x}\right)^n-2\left(\frac{1}{1-x}\right)^{n-1}\ln(1-x).$$

As

$$[x^{n-1}]\left(\frac{1}{1-x}\right)^n=\binom{2n-1}{n-1}$$

we have

$$[x^{n-1}]\left(-x\left(\frac{1}{1-x}\right)^n\right)=[x^{n-2}]\left(\frac{1}{1-x}\right)^n=-\binom{2n-3}{n-2}.$$

Moreover,

$$[x^{n-1}]\left(\frac{1}{1-x}\right)^{n-1}\ln(1-x)=[x^{n-1}]\left(\sum_{k\geq 0}\binom{n-1+k}{k}x^k\right)\left(-\sum_{k\geq 1}\frac{x^k}{k}\right)$$

$$\Rightarrow[x^{n-1}]\left(\frac{1}{1-x}\right)^{n-1}\ln(1-x)=-\sum_{k=0}^{n-2}\binom{n-1+k}{k}\frac{1}{n-1-k}$$

□

4. Distributions of n^n

We have just seen that the number of Cayley trees on $[n+1]$ such that the root has no smaller son is n^n. We now show that these familie of trees allows to give simple combinatorial interpretations for various distributions of n^n.

Proposition 4.1.

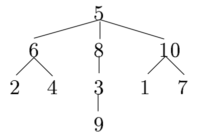
$$n^n = (n+1)^{n-1} + \sum_{k=1}^{n-1} (n-k)(n+1)^{n-k-1} n^{k-1}.$$

Proof. Here we show that the number of trees of $\mathcal{T}_{n+1,0}$ such that the root is $k+1$ for all $0 \leq k \leq n-1$ is $(n-k)(n+1)^{n-k-1}n^{k-1}$. We denote by $\mathcal{W}_{n,k}$ for all $1 \leq k \leq n-1$ the set of pairs (W, w) of words on the alphabet $[n]$ having the following properties:

- $|W| = n - k$,
- $|w| = k - 1$,
- the last letter of W is k,
- the last letter of w (if there is one) belongs to $\{k+1, k+2, \dots, n\}$,
- the letter k does not appear in w.

For example, the following pair $(W, w) = (3\ 10\ 5\ 5\ 5, 6\ 8\ 6\ 10)$ belongs to $\mathcal{W}_{10,5}$.
Clearly the cardinal of $\mathcal{W}_{n+1,k+1}$ is $(n-k)(n+1)^{n-k-1}n^{k-1}$. Hence, we give a bijection Θ from $\mathcal{W}_{n,k}$ to set of the trees belonging to $\mathcal{T}_{n,0}$ having k for root. This bijection is based on an extension of the bijective proof of Prüfer for the Cayley's formula. Let (W, w) be an element of $\mathcal{W}_{n,k}$. We construct the tree $\Theta(W, w)$ as follows.
First, let L be the set of integers of $[n]$ that does not appear in W and w (it is clear that $k \notin L$). Next, and while one of the two words W and w is not empty, let x be the maximal element of L. First remove x from L. If $x > k$ (respectively $x < k$), let y be the first letter of W (respectively w). Then remove y from W (respectively w) and we say that x is a son of y. Moreover, if y does not appear neither in W nor in w, then add y to the set L. A simple induction allows to verify that k cannot belong to the set L during all the process.

For example, for the previous pair $(W, w) = (3\ 10\ 5\ 5\ 5, 6\ 8\ 6\ 10)$ which belongs to $\mathcal{W}_{10,5}$, we have $L = \{1, 2, 4, 7, 9\}$ and the algorithm gives the following tree $\Theta(W, w)$.



The inverse map proceeds as follows. Let T be a tree of $\mathcal{T}_{n,0}$ having k for root, and W and w two empty words. While T is not reduced to one vertex, let x be its greatest leaf and y its father. Then remove x from T, and if $x > k$ then add y to the end of W else add y to the end of w. □

Remark 4.1. Using the well-known matrix-tree theorem (see [10, 18]), we have another proof of Proposition 4.1. Indeed, with the matrix-tree theorem, we can show that, for a fixed k, if G is the $n \times n$ matrix having -1 everywhere excepted on the diagonal where the k first elements are $n-1$ and the $n-k$ last are n, then the number of trees of $\mathcal{T}_{n+1,0}$ having $k+1$ for root is $det(G)$, and the result follows from a computation of this determinant.

Proposition 4.2.

$$n^n = \sum_{k=1}^{n} \binom{n+1}{k+1} k\ n^{n-k-1}.$$

Proof. It comes from the distribution of the trees of $\mathcal{T}_{n+1,0}$ under the parameter "degree k of the root". The simplest proof of this result, which is very similar to its analogue for Cayley trees, uses a generalization of the Cayley's formula stating that for a fixed k and a fixed subset $\{r_1, r_2, \dots, r_k\}$ of $[n]$, the number of forests of k trees and n vertices having $\{r_1, r_2, \dots, r_k\}$ for roots is (see [25])

$$k\ n^{n-k-1}. \tag{2}$$

Hence, to build a tree T of $\mathcal{T}_{n+1,0}$ such that the root has degree k, it is sufficient to choose a set $\{r_0, r_1, \dots, r_k\} \subset [n+1]$ such that $r_0 < r_1 < \cdots < r_k$, where r_0 is the root of T, the other r_i for all $1 \le i \le k$ being its sons, and to build all the forests of k trees on $[n+1] \setminus \{r_0\}$ having the r_i for all $1 \le i \le k$ for roots. So Proposition 4.2 follows immediately from the formula (2). □

Remark 4.2. In [6], it is shown that the number of Cayley trees on $[n+1]$ such that the root has degree k is

$$(n+1)\ \binom{n}{k}\ k\ n^{n-k-1}.$$

It follows that

$$(n+1)^n = \sum_{k=1}^{n} (n+1)\ \binom{n}{k}\ k\ n^{n-k-1},$$

which leads to a surprising relation between this distribution of $(n+1)^n$ and the distribution of n^n given in the previous proposition:

$$n^n = \sum_{k=1}^{n} \frac{n+1}{k+1}\ \binom{n}{k}\ k\ n^{n-k-1}.$$

Proposition 4.3.

$$n^n = (n+1)^{n-1} + \sum_{k=1}^{n-1} \binom{n}{k-1}\ k^{k-2}\ (n-k)^{n+1-k}.$$

Proof. It suffices to distribute the trees of $\mathcal{T}_{n+1}$ under the parameter "number k of vertices of the subtree rooted in 1". □

Remark 4.3. In the same way, the distribution of Cayley trees under the same parameter leads to

$$(n+1)^n = \sum_{k=1}^{n+1} \binom{n}{k-1}\ k^{k-2}\ (n+1-k)^{n+1-k},$$

which gives another surprising relations between two distributions of n^n and $(n+1)^n$.

Now, we do not prove a new distribution of n^n but we give a new combinatorial interpretation of one given in [15], which leads to a new identity (as far as we know) about Stirling numbers. We recall that we denote by $s(n,m)$ the Stirling numbers of the first kind.

Proposition 4.4. *The number of trees of $\mathcal{T}_{n+1,0}$ having k decreasing edges for all $0 \le k \le n-1$ is*

$$\sum_{m=k+1}^{n} (-1)^{m+k+1}\ \binom{m}{k+1}\ s(n+1, n+1-m)\ n^{n-m}.$$

Proof. In order to prove this result, we need to introduce the notion of descent in an endofunction. An *endofunction* on $[n]$ is a mapping from $[n]$ into $[n]$. A *descent* in an endofunction f on $[n]$ is an integer $i \in [n]$ such that $i > f(i)$. In [15], it is shown (although under a different form, using weighted paths in matrices filled with 0 and 1) that the number of endofunctions on $[n]$ with k descents is

$$\sum_{m=k}^{n}(-1)^{m+k}\binom{m}{k}\; s(n+1,n+1-m)\; n^{n-m}.$$

Moreover Joyal in [14] shows that there is a bijection between the endofunctions on $[n]$ and the Cayley trees on $[n+1]$ where $n+1$ is a leaf such that the number of descents of an endofunction is equal to the number of decreasing edges minus 1 in the corresponding Cayley tree.

The proposition is a direct consequence of these two results, of our correspondence ϕ given in Lemma 2.1, and of Remark 2.1. □

Proposition 4.5. *The number of Cayley trees of $\mathcal{T}_{n,k}$ having k decreasing edges is $s(n, k+1)$.*

Proof. It suffices to see that a Cayley tree of $\mathcal{T}_{n,k}$ having k decreasing edges (and so k edges incident to the root) is equivalent to a forest on $[n]$ of $k+1$ increasing trees, and then the result is a consequence of the well known following facts:

- the number of increasing Cayley trees on $[n]$ (trees with no decreasing edge) is equal to the number of one-cycle permutations on $[n]$ that is $(n-1)!$,
- the number of permutations on $[n]$ having c cycles is $s(n, c)$ (see [13] for example).

□

Now, if we set $k = 0$ in the two previous propositions, we have a simple combinatorial proof of the following identity about Stirling numbers of the first kind.

Corollary 4.1.

$$s(n+1,1) = n! = \sum_{m=1}^{n}(-1)^{m+1}\; m\; s(n+1,n+1-m)\; n^{n-m}.$$

In a paper about transitive factorizations of permutations as a product of transpositions, Goulden and Jackson introduce a new distribution of n^n [11, Proposition 3.2] (see also [24]). Let $k \geq 1$ and $m \geq 1$ be two positive integers. If

$$T_{k,m} = \frac{k^{k+1}}{k!}\sum_{i=1}^{m}\frac{i^i}{i!}\;\frac{(m-i)^{m-i}}{(m-i)!}\;\frac{1}{k+m-i}$$

then

$$(k+m)^{k+m} = (k+m)!\;(T_{k,m} + T_{m,k}). \tag{3}$$

Goulden and Jackson in [11] prove this identity, but there is no combinatorial interpretation of this formula. Here we give a bijective proof on the particular case $k = 1$.

Proposition 4.6.

$$n^n = n(n-1)^{n-1} + \sum_{k=1}^{n-1}\binom{n}{k}\; k^k\; (n-1-k)^{n-1-k}.$$

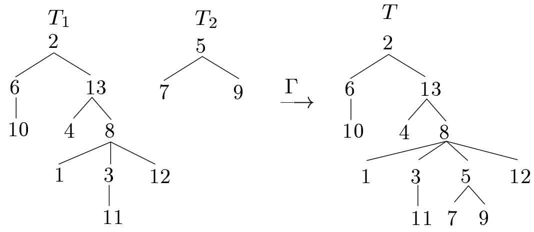
Proof. We give a distribution of the trees of $\mathcal{T}_{n+1,0}$ which corresponds exactly to the previous formula.

First, the number of trees of $\mathcal{T}_{n+1,0}$ such that $n+1$ is a leaf is clearly $n(n-1)^{n-1}$.

Now, let $\mathcal{C}_{n+1,k}$ be the set of pairs (T_1, T_2) of trees such that

- the roots of T_1 and T_2 have no smaller son,
- T_1 contains the vertex $n+1$,
- the sets V_1 and V_2 of vertices of T_1 and T_2 are disjoint and verify $V_1 \cup V_2 = [n+1]$; so exists k belonging to $[n-1]$ such that $|V_1| = k+1$ (and also $|V_2| = n-k$).

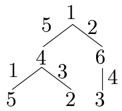
Then there is a bijection Γ from the set $\cup_{k\geq 1}\mathcal{C}_{n+1,k}$ to the set of the trees of $\mathcal{T}_{n+1,0}$ in which $n+1$ is not a leaf. In order to describe this bijection, we need to introduce the notion of maximal decreasing path of a vertex. The *maximal decreasing path* of the vertex y in T is the sequence of vertices $(x_0, x_1, x_2, \dots, x_j)$ such that $x_0 = y$, x_1 is the greatest son of x_0 smaller than x_0, x_2 is the greatest son of x_1 smaller than x_1, …, and x_j is the greatest son of x_{j-1} smaller than x_{j-1}.
Let (T_1, T_2) be a pair of trees belonging to $\mathcal{C}_{n+1,k}$ and consider $(x_0, x_1, x_2, \dots, x_h)$ the maximal decreasing path of $x_0 = n+1$ in T_1. If r is the root of T_2 and x_p is the vertex of the maximal decreasing path of $n+1$ such that $x_p > r > x_{p+1}$ (possibly $x_p = x_h$), the tree $T = \Gamma(T_1, T_2)$ is obtained by grafting T_2 on x_p.



With this construction, the inverse of Γ is clear. To obtain the second tree T_2, it suffices to consider the last vertex of the maximal decreasing path from $n+1$ in T. This vertex becomes the root of the tree T_2. □

5. Factorizations of the n-cycle

A minimal factorization of the cycle $(1, 2, \dots, n)$ is a sequence of $n-1$ transpositions $(\sigma_1, \sigma_2, \dots, \sigma_{n-1})$ such that $\sigma_1 \times \sigma_2 \times \cdots \times \sigma_{n-1} = (1, 2, \dots, n)$. Dénes [7] has shown that there are n^{n-2} such factorizations. The first direct bijective proof of this result is due to Moszkowski [19] (see also [12]). In [8], it was shown that the factorizations of any cycle of length n (not only the cycle $(1, 2, \dots, n)$) are in bijection with the family $\mathcal{C}_n$ of labelled trees on n vertices, rooted in 1 and having edges labelled on $[n-1]$. For example, to the factorization $((4,5), (1,6), (2,4), (3,6), (1,4))$ of the cycle $(1, 3, 6, 4, 5, 2)$ corresponds the following tree (to the transposition $\sigma_i = (x, y)$ corresponds an edge labelled i between the two vertices x and y).



In this section we give a new description of the bijection of Moszkowski which has the following properties:

- it is only based on trees operations;
- it appears clearly to be a particular case of two more general bijections due to Bousquet [2, 3] for the minimal factorizations of the cycle $(1, 2, \dots, n)$ as a product of permutations and to Poulalhon [21] for the minimal transitive factorizations of permutations as a product of transpositions;
- it allows to give an interpretation of n^n in terms of factorizations of the cycle $(1, 2, \dots, n+2)$ as a product of transpositions.

We now present this new description of the bijection of Moskowski between trees and factorizations.

From trees to factorizations. We describe a procedure converting a Cayley tree on $[n]$ rooted in 1 into a tree of $\mathcal{C}_n$ corresponding to a factorization of $(1, 2, \dots, n)$. This procedure is based on the following property (which is easy to verify).

Property 5.1. *Consider a tree of $\mathcal{C}_n$ corresponding to a factorization of the cycle $(1, 2, \ldots, n)$ and let $i_1, i_2, \ldots, i_k$ be the labels of the edges on the path from the vertex x to the vertex $x+1$ where (x_{j-1}, x_j) denotes the edge labelled i_j (with $1 \leq j \leq k \geq 1$, $x_0 = x$ and $x_k = x+1$). Then $i_1 < i_2 < \cdots < i_k$, and for every vertex x_j with $1 \leq j \leq k$ the smallest edge incident to x_j with a label greatest than i_j is i_{j+1}.*

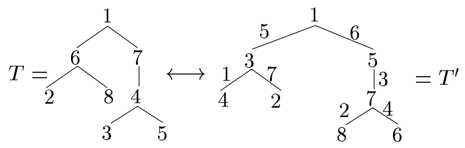
The process to convert a Cayley tree T on $[n]$ rooted in 1 consists in two steps (labelling the edges and changing the labels of the vertices) described as follows.

First, give the label $y-1$ to each edge (x, y) where x is the father of y in T.

Next, let c be a counter with initial value 2. Perform a depth first search in T such that, for each vertex x:

- firstly visit in increasing order all its sons greater than x,
- next change the label of x into c and add one to c,
- lastly visit in increasing order all its sons smaller than x.

The depth first search ends at the vertex 1 which keeps the label 1 (and not the value $n+1$ of the counter c) and gives a tree T' belonging to $\mathcal{C}_n$. It follows from the description of this algorithm that the tree so obtained corresponds to a factorization of the cycle $(1, 2, \ldots, n)$. Moreover, it is clear that two different trees on $[n]$ rooted in 1 produce two different trees of $\mathcal{C}_n$.



From factorizations to trees. Consider a tree of $\mathcal{C}_n$ corresponding to a factorization of $(1, 2, \ldots, n)$. To obtain the corresponding Cayley tree rooted in 1 it suffices to replace the label of each vertex y by one plus the label of the only edge (x, y) where x is the father of y.

Hence, we can say that the number of factorizations of $(1, 2, \ldots, n)$ with only one transposition having the form $(1, k)$ is the number of Cayley trees on $[n]$ rooted in 1 such that the root has only one son, which is equivalent to the number of rooted Cayley trees on $[n-1]$ that is $(n-1)^{n-2}$. Moreover, if we consider such factorizations and we look at the value of k we have the following result.

Proposition 5.1. *Let $f_{n,k}$ with $2 \leq k \leq n$ be the number of factorizations of $(1, 2, \ldots, n)$ with only one transposition having the form $(1, k)$. Then*

$$f_{n,2} = f_{n,n} = (n-2)^{n-2},$$

$$f_{n,k} = \sum_{l=1}^{n-1} \sum_{r=1}^{l} \binom{n-1}{l} \binom{n-1-l}{k-r-1} (r-1)^{k-2-r} (l-r)(n-k)^{n-1-k-l+r} \quad \forall\ 3 \leq k \leq n-1.$$

Proof. This result follows from the description of the correspondence between factorizations and trees given above: $f_{n,k}$ is the number of trees on $[n-1]$ having root r such that the number of vertices belonging to the subtrees having for root a son of r greater than r is $k-2$. We deduce that the image of 1 is 2 if all the sons of the root are smaller than the root, and from Lemma 2.1, we have $f_{n,2} = (n-2)^{n-2}$. In the same way, the image of 1 is n if all the son of the root are greater than it and by symmetry we have $f_{n,n} = (n-2)^{n-2}$.

For $k \neq 2$ and $k \neq n$, the formula is constructed as follows:

- choose a set of l vertices representing the root and its sons;
- for each vertex of this set, we consider that it can be the root of the tree (it has then $r-1$ sons smaller than it and $l-r$ greater sons);

- then we insert a forest with $l-r$ trees and $k-2$ vertices under the sons of the root greater than it, the remaining vertices being inserted under the sons of the root smaller than it (see [25]).

$\square$

6. Enumerating ordered alternating trees

An alternating tree T is a tree such that for every path $x_1, x_2, x_3, x_4, \ldots$ in T we have $x_1 < x_2 > x_3 < x_4 > \cdots$ or $x_1 > x_2 < x_3 > x_4 < \cdots$. Postnikov in [20] gives the formula for the number of Cayley alternating trees, and Chauve, Dulucq and Rechnitzer in [5] give its first bijective proof, with the following result.

Proposition 6.1. [5] *The number of ordered alternating trees on* $[n+1]$ *is* $2n^n$.

The proof given in [5] is based on generating functions. Here we give a new proof of this result with a bijection Ψ from the trees of $\mathcal{T}_{n,0}$ to the ordered alternating trees on $[n]$ such that the root has no smaller son (every path from the root begins with an increasing edeg). We are indebted to Robert Cori for his help in this problem.

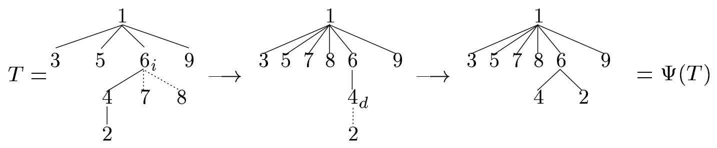
Proof of Proposition 6.1. We introduce the notions of i-wrong and d-wrong vertices. Let T be an ordered tree and y a vertex of T different of the root, having x for father:

- y is said to be *d-wrong* if x is greater than y and y is greater than at least one of its sons,
- y is said to be *i-wrong* if x is smaller than y and y is smaller than at least one of its sons.

Now, we describe the bijection Ψ. Let T be a tree of $\mathcal{T}_{n,0}$. The bijection Ψ acts in two steps. First, for each vertex x of T, the sons of x are ordered, from left to right, in increasing order. Then T is an ordered tree.
Next, we perform a breadth-first search in T and during this search, let x be the first wrong vertex of T visited (if there is no wrong vertex, T is an alternating tree). Then, the tree T is modified by the following process:

- if x is d-wrong, all the subtrees of T having for roots the sons of x smaller than it are moved, with the same order, between x and its left brother (at the left of x if it has no left brother);
- if x is i-wrong, all the subtrees of T having for roots the sons of x greater than it are moved, with the same order, between x and its right brother (at the right of x if it has no right brother);
- when a wrong vertex has been visited, the breadth-first search starts again from the first moved vertex of T.

We illustrate this bijection, with an example, where we mark with an i the i-wrong vertices and with a d the d-wrong vertices, and where the edges to move are dotted.



The inverse of this map is clear, and, using Lemma 2.1, we have a bijective proof of Proposition 6.1. $\square$

7. Concluding remarks

The main goal of this paper is the study of the family $\mathcal{T}_n$ of trees such that the root has no smaller son, which has interesting enumerative properties. This family allows to give combinatorial proofs of new identities involving n^n or identities having no combinatorial explanation (as in Proposition 4.6). However, the problem of a combinatorial proof of the identity (3) remains open.

Moreover, it would be interesting to continue the study of this family of trees from a combinatorial point of view. For example, the inversion polynomial [16] of $\mathcal{T}_n$ seems to have some properties close to the analogous polynomial for the family of Cayley trees without restriction (this point will be the subject of a note which is in preparation).

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