COMBINATORIAL OPERATORS FOR KRONECKER POWERS OF REPRESENTATIONS OF $\mathfrak{S}_n$

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ABSTRACT. We present combinatorial operators for the expansion of the Kronecker product of irreducible representations of the symmetric group $\mathfrak{S}_n$. These combinatorial operators are defined in the ring of symmetric functions and act on the Schur functions basis. This leads to a combinatorial description of the Kronecker powers of the irreducible representations indexed with the partition $(n-1,1)$ which specializes the concept of oscillating tableaux in Young's lattice previously defined by S. Sundaram. We call our specialization *Kronecker tableaux*. Their combinatorial analysis leads to enumerative results for the multiplicity of any irreducible representation in the Kronecker powers of the form $\chi^{(n-1,1)\,\otimes k}$.

1. INTRODUCTION

The subject of the present work is the investigation of the Kronecker product, sometimes called inner tensor product, of irreducible representations of the symmetric group $\mathfrak{S}_n$. Given two linear representations

$$\begin{array}{c} A:\mathfrak{S}_n \to Aut(V) \\ \sigma \mapsto A(\sigma) \end{array} \qquad \begin{array}{c} B:\mathfrak{S}_n \to Aut(W) \\ \sigma \mapsto B(\sigma) \end{array}$$

which associate linear operators to permutations $\sigma \in \mathfrak{S}_n$, the Kronecker product of A and B, denoted $A\otimes B$, is the representation of $\mathfrak{S}_n$ defined by

$$\begin{array}{c} A\otimes B:\mathfrak{S}_n \to Aut(V\otimes W) \\ \sigma \mapsto A(\sigma)\otimes B(\sigma) \end{array}$$

which is the action on the tensor product $V\otimes W$ of vector spaces V and W by means of the tensor product $A(\sigma)\otimes B(\sigma)$ of the linear operators $A(\sigma)$ and $B(\sigma)$.

The irreducible representations of $\mathfrak{S}_n$ are the representations which are indecomposable as direct sums of representations. They are indexed with the partitions λ of n:

$$A^\lambda : \mathfrak{S}_n \to Aut(V), \sigma \mapsto A^\lambda(\sigma).$$

The Kronecker product $A^\lambda \otimes A^\mu$ of two irreducible representations of $\mathfrak{S}_n$ is in general not an irreducible representation of $\mathfrak{S}_n$ and the fundamental problem of expanding it as a direct sum of irreducible representations

$$A^\lambda \otimes A^\mu = \sum_{\alpha\vdash n} t^\alpha_{\lambda,\mu} A^\alpha$$

goes back to the foundation of the representation theory. This problem was studied by Murnaghan [11, 12], Littlewood [9] and more recently by Lascoux [8], Garsia and Remmel [7], Thibon *et al.* [15], and others (see [1] and references therein).

To obtain the decomposition coefficients $t^\alpha_{\lambda,\mu}$, one can use the characters of the corresponding representations. The character of a representation A of $\mathfrak{S}_n$ is the map χ^A which sends permutations $\sigma\in\mathfrak{S}_n$ to the traces of $A(\sigma)$:

$$\begin{array}{c} \chi^A:\mathfrak{S}_n \to \mathbb{C} \\ \sigma \mapsto tr(A(\sigma)) \end{array}$$

The fact that the character of the Kronecker product $A\otimes B$ of two representations is obtained by multiplying the traces of the linear operators $A(\sigma)$ and $B(\sigma)$:

$$\begin{array}{c} \chi^{A\otimes B}:\mathfrak{S}_n \to \mathbb{C} \\ \sigma \mapsto tr(A(\sigma))tr(B(\sigma)) \end{array} \qquad (1)$$

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is a straightforward consequence of the definition of tensor product of linear operators. Hence one can use property (1) of characters and the character table, indexed by integer partitions of n (see Table 1 for example) to compute the characters of a Kronecker product of two irreducible representations. Let us recall that the character values $\chi^A(\sigma_1)$, $\chi^A(\sigma_2)$ on two permutations $\sigma_1, \sigma_2 \in C_\mu$ in the same conjugacy class C_μ are equal. Therefore we use the notation χ^λ_μ for the value of the irreducible character χ^λ on any element of the conjugacy class C_μ.

Example 1. Let Table 1 be the table of the irreducible characters of $\mathfrak{S}_4$.

$\lambda\backslash\mu$	(4)	$(3,1)$	$(2,2)$	$(2,1^2)$	(1^4)
(4)	1	1	1	1	1
$(3,1)$	-1	0	-1	1	3
$(2,2)$	0	-1	2	0	2
$(2,1^2)$	1	0	-1	-1	3
(1^4)	-1	1	1	-1	1

TABLE 1. Irreducible characters χ^λ_μ of S_4.

The character of the Kronecker product $A^{(3,1)} \otimes A^{(3,1)}$ of the irreducible representation $A^{(3,1)}$ with itself is obtained by multiplying each element of the row-vector $(-1, 0, -1, 1, 3)$ in Table 1 with itself and we obtain

$$\chi^{(3,1)\otimes(3,1)} = (1, 0, 1, 1, 9)$$

Since Table 1 contains the row-vectors of all possible irreducible characters and that $(1, 0, 1, 1, 9)$ is obviously not one of these rows, it is immediate that the character represented by $(1, 0, 1, 1, 9)$ is not irreducible. But we observe that the identity $\chi^{(3,1)\otimes(3,1)} = \chi^{(4)} + \chi^{(3,1)} + \chi^{(2,1,1)} + \chi^{(2,2)}$ is true by adding the rows of Table 1 corresponding to the partitions in the right hand side. $\diamond$

More generally, the problem of finding the coefficients $t^\alpha_{\lambda,\mu}$ has a solution when one accepts to use the character table $[\chi^\lambda_\mu]$ of $\mathfrak{S}_n$:

$$t^\alpha_{\lambda,\mu} = \chi^\lambda \otimes \chi^\mu|_{\chi^\alpha} = \sum_{\gamma\vdash n} \frac{|C_\gamma|}{n!} \chi^\lambda_\gamma \chi^\mu_\gamma \chi^\alpha_\gamma \tag{2}$$

Identity (2) follows from the orthonormality of the characters χ^λ with respect to the standard scalar product in the group algebra of $\mathfrak{S}_n$ and from (1). But since the coefficients $t^\alpha_{\lambda,\mu}$ are positive integers, we find equation (2) unsatisfactory and the goal of this paper is to contribute to other avenues for computing the coefficients $t^\alpha_{\lambda,\mu}$.

Our presentation is in three steps. First we define operators on the ring of symmetric functions which reproduce tensor product of irreducible representations when they act on Schur functions. Second we specialize these operators to the tensor powers $\chi^{(n-1,1)^{\otimes k}}$, for $k \geq 0$, and we develop a combinatorial model to represent these tensor powers. Finally we obtain some enumerative results for the multiplicity of any irreducible representation in the Kronecker powers of the form $\chi^{(n-1,1)\otimes k}$.

2. COMBINATORIAL OPERATORS

2.1. **Symmetric functions.** Let $\mathbb{Q}[\mathfrak{S}_n]$ be the group algebra of the symmetric group over the field $\mathbb{Q}$ of rational numbers and let Z_n be the center of this group algebra. The irreducible characters of S_n can be seen as elements of $\mathcal{Z}_n$ when one writes

$$\chi^\lambda = \sum_{\sigma\in\mathfrak{S}_n} \chi^\lambda(\sigma)\sigma$$

and the pointwise multiplication of two elements $a = \sum_{\sigma\in\mathfrak{S}_n} a_\sigma\sigma$ and $b = \sum_{\sigma\in\mathfrak{S}_n} b_\sigma\sigma$ of $\mathbb{Q}[\mathfrak{S}_n]$ is defined by

$$a \cdot b = \sum_{\sigma\in\mathfrak{S}_n} (a_\sigma b_\sigma)\sigma$$

In example 1 we have seen that pointwise multiplication of characters gives the character of Kronecker product of the corresponding representations: $\chi^\lambda \cdot \chi^\mu = \chi^{\lambda\otimes\mu}$.

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a set of indeterminates, $\Lambda = \Lambda_{\mathbb{Q}}[\mathbf{x}]$ the ring of symmetric functions in $x_1, x_2, \ldots$ over the field $\mathbb{Q}$ and Λ^n the restriction to homogeneous symmetric functions of degree n. Two important sets of symmetric functions are the homogeneous symmetric functions and the Schur symmetric functions. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$, one defines $h_\lambda(\mathbf{x}) = \prod_{i=1}^m h_{\lambda_i}(\mathbf{x})$, where $h_r(\mathbf{x})$ is the r'th homogeneous symmetric function, and one denotes by $s_\lambda(\mathbf{x})$ the Schur symmetric function associated to λ (see [10]). The sets $\{h_\lambda(\mathbf{x})\}_{\lambda\vdash n}$ and $\{s_\lambda(\mathbf{x})\}_{\lambda\vdash n}$ are linear basis of Λ^n. The Frobenius map $\mathcal{F} : \mathcal{Z}_n \to \Lambda^n$ is a vector space isomorphism which sends the irreducible characters χ^λ to the Schur functions s_λ : $\mathcal{F}(\chi^\lambda) = s_\lambda$. Schur functions can be expanded as determinants of homogeneous functions:

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1\le i,j\le n} \tag{3}$$

where $h_0 = 1$ and $h_r = 0$ if $r < 0$. The Littlewood-Richardson coefficients (denoted by LR) are defined as the coefficients in the expansion of the ordinary product of two or more Schur functions in the basis of Schur functions:

$$s_{\nu^{(1)}} s_{\nu^{(2)}} \cdots s_{\nu^{(r)}} = \sum_\mu LR^\mu_{\nu^{(1)},\nu^{(2)},\ldots,\nu^{(r)}} s_\mu$$

The adjoint operator to multiplication by s_γ in Λ with respect to the standard scalar product in Λ is denoted $s_\gamma^\perp$ and its action on the Schur function s_λ is
as follows:

$$s_\gamma^\perp s_\lambda = s_{\lambda/\gamma} = \begin{cases} \sum_\alpha LR^\lambda_{\gamma,\alpha} s_\alpha & \text{if } \gamma \subseteq \lambda \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$\langle s_\gamma f, g\rangle = \langle f, s_\gamma^\perp g\rangle \qquad \forall\, f, g \in \Lambda \tag{5}$$

Now let us define in Λ^n the operation $f \odot g$ corresponding to pointwise multiplication by means of the Frobenius map:

$$f \odot g = \mathcal{F}(\mathcal{F}^{-1}(f) \cdot \mathcal{F}^{-1}(g)) \qquad \forall\, f,\, g,\, \in \Lambda^n$$

We shall call this operation *inner product* of symmetric functions and we have in particular

$$s_\lambda \odot s_\mu = \sum_\alpha t^\alpha_{\lambda,\mu} s_\alpha \tag{6}$$

where the coefficients $t^\alpha_{\lambda,\mu}$ are the same than in equation (2). We can expand the inner product $h_\lambda \odot s_\mu$ in the basis $\{s_\alpha\}$ as follows (see also [10, 1.7 example 23 (d)]):

$$h_\lambda \odot s_\mu(\mathbf{x}) = \sum_{\{\nu^{(1)}\vdash\lambda_1, \nu^{(2)}\vdash\lambda_2,\ldots,\nu^{(k)}\vdash\lambda_k\}} LR^\mu_{\nu^{(1)}\nu^{(2)}\cdots\nu^{(k)}} s_{\nu^{(1)}} \cdots s_{\nu^{(k)}} \tag{7}$$

$$= \sum_{\{\nu^{(2)}\vdash\lambda_2,\ldots,\nu^{(k)}\vdash\lambda_k\}} (s_{\nu^{(2)}} \cdots s_{\nu^{(k)}})(s^\perp_{\nu^{(2)}} \cdots s^\perp_{\nu^{(k)}}) s_\mu. \tag{8}$$

To prove identity (7) in the language of λ-rings, it suffices to notice that $(h_\lambda \odot s_\mu)[\mathbf{X}] = h_\lambda[\mathbf{XY}]|_{s_\mu[\mathbf{Y}]}$. Then (7) follows from the fact that

$$h_\lambda[\mathbf{XY}] = \sum_\mu \left(\sum_{\{\nu^{(1)}\vdash\lambda_1,\ldots,\nu^{(k)}\vdash\lambda_k\}} LR^\mu_{\nu^{(1)}\cdots\nu^{(k)}} s_{\nu^{(1)}}[\mathbf{X}] \cdots s_{\nu^{(k)}}[\mathbf{X}] \right) s_\mu[\mathbf{Y}]$$

Identity (8) follows from (7) and (4).

2.2. The operators $U_{\overline{\lambda}}$.

Definition 1. Let $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$ be a partition of n, with $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_m$, and $\overline{\lambda}$ the truncated partition of λ defined by $\overline{\lambda} = (\lambda_2, \ldots, \lambda_m)$. One denotes by $U_{\overline{\lambda}}$ the operator from Λ^n to Λ^n defined as follows:

a): Expand the determinant

$$
(9) \qquad \begin{vmatrix} 1 & 1 & \dots & 1 \\ h_{\lambda_2-1} & h_{\lambda_2} & \dots & h_{\lambda_2+r-2} \\ \vdots & \vdots & \dots & \vdots \\ h_{\lambda_r-r+1} & h_{{}_r-r+2}\dots & \dots & h_{\lambda_r} \end{vmatrix}
$$

b): Replace each term $h_\alpha = h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_m}$ in the expansion of (9) by

$$
\sum_{\nu^1 \vdash \alpha_1, \ldots, \nu^m \vdash \alpha_m} (s_{\nu^1} \cdot s_{\nu^m})(s^\perp_{\nu^1} \cdots s^\perp_{\nu^m})
$$

to obtain the operator $U_{\overline{\lambda}}$.

Theorem 1. *For any partitions λ and μ of n,*

$$
\begin{aligned} U_{\overline{\lambda}} s_\mu &= s_\lambda \odot s_\mu \\ &= \sum_{\alpha \vdash n} t^\alpha_{\lambda,\mu} s_\alpha \end{aligned}
$$

Proof. This is a straightforward consequence of equations (3), (6) and (8). □

Example 2. The Kronecker product $\chi^{(n-1,1)} \otimes \chi^\mu$ is obtained by applying the operator $U_{(1)}$ on s_μ which is obtained by expanding the determinant in definition 2.1 a) and then writing the h_λ in terms of Schur functions using definition 2.1 b):

$$
\begin{vmatrix} 1 & 1 \\ h_0 & h_1 \end{vmatrix} = h_1 - 1 \Rightarrow U_{(1)} = s_{(1)} s^\perp_{(1)} - 1
$$

Similarly the operator $U_{(2)}$ needed for the computation of $\chi^{(n-2,2)} \otimes \chi^\mu$ is obtained with the determinant

$$
\begin{vmatrix} 1 & 1 \\ h_1 & h_2 \end{vmatrix} = h_2 - h_1 \Rightarrow U_{(2)} = s_{(2)} s^\perp_{(2)} + s_{(1,1)} s^\perp_{(1,1)} - s_{(1)} s^\perp_{(1)}
$$

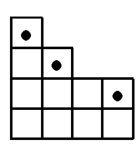
◇

The fact that the computation of $\chi^\lambda \otimes \chi^\mu$ is independent of the largest part of λ was already observed by Murnaghan ([12]) and also described by Thibon in [16], but the definition of the operators $U_{\overline{\lambda}}$ seems to be new.

3. A combinatorial model for $\chi^{(n-1,1)\otimes k}$ and some consequences

We are now ready to present a combinatorial model for the multiplicity of any irreducible representation in any Kronecker power $\chi^{(n-1,1)\otimes k}$, in terms of paths in Young's lattice. From example 2, it is immediate that the expansion of the Kronecker power $\chi^{(n-1,1)\otimes k}$ is obtained by the application of the operator ${U_{(1)}}^k = (s_{(1)} s^\perp_{(1)} - 1)^k$ on the Schur function $s_{(n)}$. In the remaining of this section we develop a combinatorial interpretation of this process, followed by some enumerative consequences.

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be an integer partition of n. We recall that the unique *Ferrers diagram* associated to λ is formed of m stacked rows of cells, ordered from bottom to top in increasing order (in the french notation) and such that the i^{th} row contains λ_i cells. A cell of a diagram λ located at the right end of a row and having no cell above it is called a *corner* of λ. For example, in the following Ferrers diagram, corresponding to the partition $(4, 4, 2, 1)$, the corners are indicated by $\bullet$.



The unique corner of a Ferrers diagram located on a longest row is called the *first corner* of the diagram.

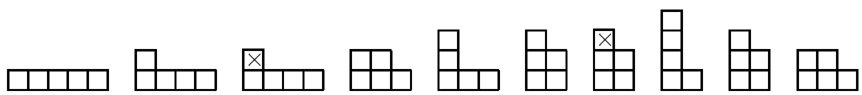
It follows immediately from the definition of the operator $s^\perp_{(1)}$ that, for a Ferrers diagram λ, $s^\perp_{(1)}(s_\lambda)$ is the sum of the Schur functions indexed by the Ferrers diagrams obtained by removing a single corner from λ. Symmetrically, $s_{(1)}(s_\lambda)$ is the sum of the Schur functions indexed by the Ferrers diagrams obtained by adding to λ a new cell that becomes a corner of the new diagram. Hence $s_{(1)}s_{(1)}^{\perp}(s_\lambda)$ is the sum of the Schur functions indexed with the Ferrers diagrams that are obtained from λ by first removing a corner from λ, which gives a diagram denoted λ', then adding a corner to λ'. One says that every diagram, or equivalently partition, $\lambda' \neq \lambda$ indexing a Schur function occurring in the sum $s_{(1)}s_{(1)}^{\perp}(s_\lambda)$ *differs from* λ *by the position of a corner*. It should be noticed that in the sum $s_{(1)}s_{(1)}^{\perp}(s_\lambda)$ the multiplicity of s_λ is at least 1. Hence, as $U_{(1)} = s_{(1)}s_{(1)}^{\perp} - 1$, one can define $U_{(1)}$ as $s_{(1)}s_{(1)}^{\perp}(s_\lambda)$ minus one occurrence of s_λ.

Example 3. $U_{(1)}(s_{(3,3,1)}) = s_{(3,3,1)} + s_{(4,2,1)} + s_{(3,2,1,1)} + s_{(3,2,2)} + s_{(4,3)}$, and the partitions $(4,2,1)$, $(3,2,1,1)$, $(3,2,2,2)$ and $(4,3)$ differ from $(3,3,1)$ by the position of a corner. The multiplicity of $s_{(3,3,1)}$ is 1 because there are two corners in $(3,3,1)$, and thus only two ways to obtain $(3,3,1)$ from itself by removing then replacing a corner, one of these occurrences being not tamen into account due to the -1 term in the definition of $U_{(1)}$. ◇

We now define the main combinatorial object that we will need in order to encode the action of $U_{(1)}$ on a Schur function s_μ iterated k times.

Definition 2. Given a positive integer k and partitions λ and μ of same weight, a *Kronecker tableau* K of length k, initial shape μ and final shape λ is a sequence $\mu^0 = \mu, \mu^1, \dots, \mu^k = \lambda$ of Ferrers diagrams where, for every pair of consecutive diagrams μ^i and μ^{i+1}, *either* μ^{i+1} *differs from* μ^i *by the position of a corner, or* $\mu^{i+1} = \mu^i$ *and one corner of* μ^{i+1}*, other than its first corner, is distinguished.* One denotes by $KT^k_{\mu,\lambda}$ the set of *Kronecker tableaux* of length k, initial shape μ and final shape λ.

Example 4. The following Kronecker tableau – where a distinguished corner is indicated by a $\times$ – belongs to $KT^9_{(5),(3,2)}$



Proposition 1. *Let k and n be two positive integers and λ a partition of n. Then*

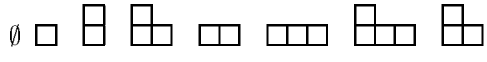
$$\chi^{(n-1,1)^{\otimes k}}|_{\chi^\lambda} = |KT^k_{(n),\lambda}|. \tag{10}$$

Proof. The definition of $U_{(1)}$ as an operator on Schur functions can be translated in the combinatorial framework of partitions and Ferrers diagrams, due to the fact that Schur functions are indexed by partitions. Hence, the sum of Schur functions ${U_{(1)}}^k(s_n)$ can be seen as a formal sum of Ferrers diagrams. The number of occurrences of a diagram λ in this formal sum of diagrams is then given by the number of ways to obtain λ from (n) by iterating k times the combinatorial operation associated to $U_{(1)}$. The identity follows immediately from this fact and from the definition of Kronecker tableaux, where the restriction that the first corner of a diagram can not be distinguished in the next diagram accounts for the -1 part of $U_{(1)} = s_{(1)}s_{(1)}^{\perp} - 1$. □

The above proposition establishes a link between the multiplicity of the irreducible character χ^λ in some Kronecker power and sequences of Ferrers diagrams seen as paths in Young's lattice (the lattice of Ferrers diagrams ordered by inclusion). We rely on this fact to obtain enumerative results about the multiplicity of any given irreducible representation in the Kronecker power $\chi^{(n-1,1)\otimes k}$. The main tool we use is a combinatorial construction defined for a family of paths in Young's lattice called *oscillating tableaux* and introduced by Sundaram, in a different algebraic context, in [14] (see also the work of Delest, Dulucq and Favreau [4, 6] for a purely combinatorial point of view).

Briefly, oscillating tableaux are paths in Young's lattice, that is sequences of Ferrers diagrams, starting at $\emptyset$ and such that *two consecutive diagrams differ by the addition or removal of exactly one corner.*

Example 5. Here is an oscillating tableau of length 7 and final shape $(2,1)$, that contains five additions of corner and two removal of corner (steps 4 and 7).



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In Lemmas 1 and 2 below, we consider a class of Kronecker tableaux that can be related to oscillating tableaux. This allows to use a variant of the combinatorial construction defined in [14, 4] that will be central in the proof of our main enumerative result, Proposition 2.

Lemma 1. *Let n and k be positive integers and λ a partition of n such that $n \geq k+\lambda_2$. There is a bijection between Kronecker tableaux of $KT^k_{(n),\lambda}$ and sequences $\nu^0, \ldots, \nu^k$ of k Ferrers diagrams such that $\nu^0 = \emptyset$, $\nu^k = \overline{\lambda}$ and, for every pair ν^i and ν^{i+1} of consecutive diagrams, either ν^{i+1} is obtained from ν^i by the addition or removal of one corner, or ν^{i+1} differs from n^i by the position of a corner, or $\nu^{i+1} = \nu^i$ and ν^{i+1} has one distinguished corner.*

Proof. Let K be a Kronecker tableau of $KT^k_{(n),\lambda}$ such that $n \geq k + \lambda_2$. Due to this last condition, the first corner of every Ferrers diagram of K, except possibly the last diagram, is on its first row. Then, by removing the first row of every diagram of K one obtains the sequence $\nu^0, \ldots, \nu^k$. Conversely, consider a sequence of $k+1$ Ferrers diagrams $\nu^0 = \emptyset, \ldots, \nu^k = \overline{\lambda}$. By adding, for every diagram ν^i, a first row of length $n - |\nu^i|$, one obtains a Kronecker tableau of $KT^k_{(n),\lambda}$. □

The combinatorial construction we describe below relies partly on the *Robinson-Schensted-Knuth (RSK) insertion and deletion algorithms*, and we first recall some basic facts about *standard tableaux* and these algorithms (see [13] for example for details on these algorithms).

- Given a positive integer n and a Ferrers diagram α with at most n cells, a *partial standard tableau* of shape α and labels in $[n]$ is a labelling of the cells of α with distinct integers chosen from $\{1, \ldots, n\}$, such that the labels are increasing in rows (from left to right) and columns (from bottom to top).
- Let S be a partial standard tableau. Given an integer x, the RSK insertion algorithms inserts x into S, creating a tableau S' whose shape differs from the shape of S by the addition of a corner and labels are the labels of S plus x. Given a corner of S, the RSK deletion algorithm removes this corner and moves some labels of cells of S, this process ending when a label is ejected from the first row of the tableau.

Lemma 2. *Let n and k be two positive integers and λ a partition of n such that $n \geq k+\lambda_2$. There is a bijection between the set $KT^k_{(n),\lambda}$ and the set of pairs (T, π), where T is a partial standard tableau of shape $\overline{\lambda}$ with labels in $[n]$ and π is a permutation of n such that: every non fixed point cycle of π is decreasing, every fixed point of π is also the label of a cell of T, and every label of T is the greatest element of a cycle in π.*

Proof. Let $K = \mu^0, \ldots, \mu^k$ be a Kronecker tableau of length k, initial shape (n) and final shape λ, such that $n \geq k + \lambda_2$. Let $\nu^0, \ldots, \nu^k$ be the sequence of Ferrers diagrams corresponding to K obtained by the construction of Lemma 1.

One can associate to $\nu^0, \ldots, \nu^k$ a sequence of partial standard tableaux $(T_0 = \emptyset, \ldots, T_k = T)$ on $[n]$ and a permutation π, such that the shape of T_i is ν^i for every i. We proceed as follows. Start with setting π as the identity permutation on $\{1, \ldots, k\}$, and for i from 1 to k:
1. If ν^i is obtained from ν^{i-1} by the addition of a corner, then add to T_{i-1} this corner, labelled with i, to obtain T_i.
2. If ν^i is obtained from ν^{i-1} by the removal of a corner, then delete this corner from T_{i-1}, using the RSK deletion algorithm. If j is the integer ejected from T_{i-1} by the RSK deletion algorithm, then multiply π by the transposition (i, j).
3. If ν^i differs from ν^{i-1} by the position of a corner, or $\nu^i = \nu^{i-1}$ and ν^{i-1} has a distinguished corner (therefore ν^i and ν^{i-1} have the same weight), then delete this corner from T_{i-1} using again the RSK deletion algorithm and add the corner needed to obtain ν^i and label it with i. If j is the ejected label then multiply π by the transposition (i, j).

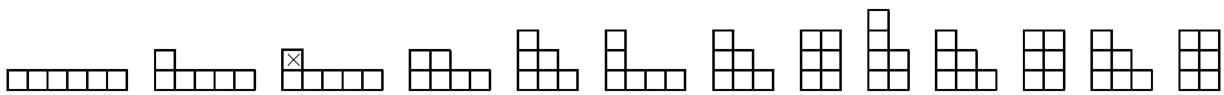
The fact that in the permutation π all non fixed point cycles are decreasing follows from the fact that in every transposition (i, j) considered in the construction above one has $i > j$.

The reverse construction starts with a partial standard tableau $(T_k = T)$ with shape ν^k and a permutation π of the set $\{1, \ldots, k\}$ with each non fixed point cycle in decreasing order such that each entry of T is the greatest element of a cycle of π (including the fixed points). Then perform the following steps, for i from k to 1:
1. If no cell of T_i is labelled with i, there exists $j < i$ such that $\pi(i) = j$. Then insert the integer j into the

tableau T_i using the RSK insertion algorithm to obtain T_{i-1} and define ν^{i-1} as the shape of T_{i-1}.
2. If a cell of T_i is labelled with i, then remove the cell labelled i: by induction this cell is a corner and this removal gives a partial standard tableau denoted U.
2.a. If furthermore there exists $j < i$ such that $\pi(i) = j$, then insert the integer j into the tableau U, using the RSK insertion algorithm, to obtain T_{i-1}, and define ν^{i-1} as the shape of T_{i-1}, distinguishing the corner added during this insertion if it takes the same position than the corner removed from T_i.
2.b. Otherwise, after removing i from T_i, multiply π by the transposition (i, j), and define ν^{i-1} as the shape of T_{i-1}.

The fact that these two constructions define a bijection follows immediately from its close relationship with the construction on oscillating tableaux defined by Sundaram [14] and Delest, Dulucq and Favreau [4, 6]. □

Example 6. The following Kronecker tableau belonging to $KT^{12}_{(6),(2,2,2)}$



corresponds to the sequence of partial standard tableaux $\nu^0, \ldots, \nu^k$

$$\emptyset \quad \boxed{1} \quad \boxed{2} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array} \quad \begin{array}{cc} 4 & \\ 2 & 3 \end{array} \quad \begin{array}{c} 4 \\ 2 \end{array} \quad \begin{array}{cc} 4 & \\ 2 & 6 \end{array} \quad \begin{array}{cc} 4 & 7 \\ 2 & 6 \end{array} \quad \begin{array}{cc} 8 & \\ 4 & \\ 2 & 7 \end{array} \quad \begin{array}{cc} 8 & \\ 4 & 7 \end{array} \quad \begin{array}{cc} 8 & 10 \\ 4 & 7 \end{array} \quad \begin{array}{cc} 8 & \\ 4 & 10 \end{array} \quad \begin{array}{cc} 8 & 12 \\ 4 & 10 \end{array}$$

and to the pair

$$T = \begin{array}{|c|c|}\hline 8 & 12 \\ \hline 4 & 10 \\ \hline\end{array}\ , \ \pi = (1, 2, \ldots 12).(2, 1).(5, 3).(8, 6).(9, 2).(11, 7) = (4)(5, 3)(8, 6)(9, 2, 1)(10)(11, 7)(12).$$

◇

To conclude this section, we derive from the above bijection an explicit formula and a generating function for the coefficients $\chi^{(n-1,1)}{}^{\otimes k}|_{\chi^\lambda}$ when $n \geq k + \lambda_2$.

Proposition 2. *Let k and n be two positive integers and λ a partition of n such that $n \geq k + \lambda_2$. Then*

$$\chi^{(n-1,1)^{\otimes k}}|_{\chi^\lambda} = f^{\overline{\lambda}} \sum_{m_1=0}^{n-\lambda_1} \left(\binom{k}{m_1} \sum_{m_2=n-\lambda_1-m_1}^{\lfloor (k-m_1)/2 \rfloor} \binom{m_2}{n-\lambda_1-m_1} p_2(k-m_1, m_2) \right), \tag{11}$$

where $f^{\overline{\lambda}}$ is the number of standard tableaux of shape $\overline{\lambda}$ and $p_2(k-m_1, m_2)$ is the number of set partitions of a set of $k - m_1$ distinct integers into m_2 parts of size at least 2.

Proof. From Proposition 1, it is sufficient to enumerate the number of Kronecker tableaux of length k, initial shape (n) and final shape λ. As $n \geq k + \lambda_2$, it follows from Lemma 2 that this reduces to the enumeration of some couples (T, π) where T is a partial standard tableau of shape $\overline{\lambda}$ and π is a permutation on the set $\{1, \ldots, k\}$. Formula (11) follows if one denotes by m_1 the number of fixed points of π and m_2 the number of cycles of size at least 2 in π. □

Remark 1. For two given integers n and k, the number $p_2(n, k)$ is known as an associated Stirling number of second kind (reference A008299 in [17], see also [3, p. 222]). Such numbers are defined by the following recurrence: $p_2(n, k) = 0$ if $n < 2k$ and $p_2(n, k) = kp_2(n-1, k) + (n-1)p_2(n-2, k-1)$ if $n \geq 2k$. The computation of $p_2(n, k)$ can also be done by extracting the coefficient of $y^k x^n/n!$ in $e^{yp(x)}$ where $p(x) = e^x - x - 1$.

Corollary 1. *Let ℓ be a positive integer, $\overline{\lambda} = (\ell_2, \ldots, \ell_m)$ an integer partition of ℓ. Then*

$$\sum_{k \geq \ell} \chi^{(n_k-1,1)^{\otimes k}}|_{\chi^{\overline{\lambda}+(n_k-\ell)}} \frac{x^k}{k!} = \frac{f^{\overline{\lambda}}}{\ell!} e^{p(x)} (e^x - 1)^\ell, \tag{12}$$

where, for every $k \geq \ell$, n_k is any integer satisfying $n_k \geq k + \lambda_2$ and $\overline{\lambda} + (n_k - \ell)$ is the integer partition obtained by adding the part $n_k - \ell$ to $\overline{\lambda}$.

Proof. For clarity, we denote $\overline{\lambda}^* = (\overline{\lambda} + (n_k - \ell))$ and $p(x) = e^x - x - 1$. It follows from the fact that $n_k \geq k + \lambda_2$, lemma 1, propositions 1 and 2, that

$$\begin{aligned}
\sum_{k\geq \ell} \chi^{(n_k-1,1)^{\otimes k}}|_{\chi^{(\overline{\lambda}^*)}} \frac{x^k}{k!} &= f^{\overline{\lambda}} \sum_{k\geq \ell} \sum_{m_1=0}^{\ell} \left(\frac{x^k}{k!} \binom{k}{m_1} \sum_{m_2=\ell-m_1}^{\lfloor (k-m_1)/2 \rfloor} \binom{m_2}{\ell - m_1} p_2(k-m_1, m_2) \right) \\
\Longleftrightarrow \sum_{k\geq \ell} \chi^{(n_k-1,1)^{\otimes k}}|_{\chi^{(\overline{\lambda}^*)}} \frac{x^k}{k!} &= f^{\overline{\lambda}} \sum_{m_1=0}^{\ell} \frac{x^{m_1}}{m_1!} \left(\sum_{k\geq \ell} \sum_{m_2=\ell-m_1}^{\lfloor (k-m_1)/2 \rfloor} \binom{m_2}{\ell - m_1} p_2(k-m_1, m_2) \frac{x^{k-m_1}}{(k-m_1)!} \right) \\
\Longleftrightarrow \sum_{k\geq \ell} \chi^{(n_k-1,1)^{\otimes k}}|_{\chi^{(\overline{\lambda}^*)}} \frac{x^k}{k!} &= f^{\overline{\lambda}} \sum_{m_1=0}^{\ell} \frac{x^{m_1}}{m_1!} \left(\sum_{m_2\geq \ell-m_1} \binom{m_2}{\ell - m_1} \sum_{q\geq 0} p_2(q, m_2) \frac{x^q}{q!} \right) \\
\Longleftrightarrow \sum_{k\geq \ell} \chi^{(n_k-1,1)^{\otimes k}}|_{\chi^{(\overline{\lambda}^*)}} \frac{x^k}{k!} &= f^{\overline{\lambda}} \sum_{m_1=0}^{\ell} \frac{x^{m_1}}{m_1!} \left(\sum_{m_2\geq \ell-m_1} \binom{m_2}{\ell - m_1} \frac{p(x)^{m_2}}{m_2!} \right) \\
\Longleftrightarrow \sum_{k\geq \ell} \chi^{(n_k-1,1)^{\otimes k}}|_{\chi^{(\overline{\lambda}^*)}} \frac{x^k}{k!} &= f^{\overline{\lambda}} \sum_{m_1=0}^{\ell} \frac{x^{m_1}}{m_1!} \frac{p(x)^{\ell-m_1}}{(\ell-m_1)!} \left(\sum_{m\geq 0} \frac{p(x)^m}{m!} \right) \\
\Longleftrightarrow \sum_{k\geq \ell} \chi^{(n_k-1,1)^{\otimes k}}|_{\chi^{(\overline{\lambda}^*)}} \frac{x^k}{k!} &= f^{\overline{\lambda}} \sum_{m_1=0}^{\ell} \frac{x^{m_1}}{m_1!} \frac{p(x)^{\ell-m_1}}{(\ell-m_1)!} e^{p(x)} \\
\Longleftrightarrow \sum_{k\geq \ell} \chi^{(n_k-1,1)^{\otimes k}}|_{\chi^{(\overline{\lambda}^*)}} \frac{x^k}{k!} &= \frac{f^{\overline{\lambda}}}{\ell!} e^{p(x)} (e^x - 1)^{\ell}.
\end{aligned}$$

□

4. Conclusion

We presented in this note a combinatorial interpretation of the multiplicity of any irreducible representation in a Kronecker power $\chi^{(n-1,1)^{\otimes k}}$ in terms of sequences of Ferrers diagrams (Kronecker tableaux) that leads, when n is large enough with respect to k and $\overline{\lambda}$, to an enumeration formula and a generating function. Moreover, we now have a combinatorial model for the expansion of $\chi^{\mu^{\otimes k}}$ for any μ given by the differential operators $U_{\overline{\mu}}$. However, at this point, the problem of transforming this model into enumerative results in terms of a relationship between n, k, μ and λ is still open.

It should be noticed that our approach could be extended to the more general case of the computation of $\chi^{(n-1,1)^{\otimes k}} \otimes \chi^{\mu}|_{\chi^{\lambda}}$ for arbitrary μ. This would require the use of a generalization of the oscillating tableaux of Sundaram which already exists and is called *skew oscillating tableaux* in [5] (see also [2]). However, this construction leads to an intricate expression for the enumeration of Kronecker tableaux of initial shape μ with more than one part, and we were not able to find a compact generating function similar to Corollary 1.

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