Generalized Motzkin numbers and skew tableaux

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Abstract. We give an interpretation in terms of skew tableaux of the first three columns of a triangle of generalized Motzkin numbers given in [1].

1. Introduction

The Motzkin numbers, m_n, given by the following formula (see [1] for example)

$$m_n = \sum_{k \geq 0} \binom{n}{2k} C_k$$

where C_k is the kth Catalan number, are known to count the standard tableaux of height at most 3. In [1], Donaghey and Shapiro propose a generalization of these numbers, related to some classes of paths in the discrete plane. Here, we give an interpretation in terms of skew standard tableaux of the first three columns of this triangle.

We denote by $ST^k(\lambda/\alpha)$ the set of skew standard tableaux having shape λ/α and height at most k, and by $f_k^{\lambda/\alpha}$ the number of such tableaux.

2. The Generalized Motzkin Numbers Triangle

Consider a path in the North-East quadrant of the discrete plane, having n steps which can be North steps (called N steps), East steps (E steps) and stationary steps (called S steps), and such that each these paths don't cross the $x = y$ line and end at the point $(k, k-j)$, with $j \geq 0$. We denote by $m(n,j)$ the number of such paths. Then we can define the triangle of $m(n,j)$, introduced in [1].

$n \backslash j$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	2	2	1				
3	4	5	3	1			
4	9	12	9	4	1		
5	21	30	25	14	5	1	
6	51	76	69	44	20	6	1

The Motzkin numbers correspond to the first column of this triangle: $m_n = m(n,0)$.

We can also interpret this triangle in terms of the generalized ballot problem. Indeed $m(n,j)$ counts the number of ways to count the votes during an election with two candidates (called A and B) such that A receives k votes and B $k-j$, the l remaining votes ($n = 2k - j + l$) being "invalid votes", and such that at every time of the counting, B has at most as votes as A. From [2], we know that if $l = 0$, the number $a_{k,k-j}$ of such counting are given by

$$a_{k,k-j} = \frac{j+1}{k+1}\binom{2k-j}{k}$$

which gives that

$$m(n,j) = \sum_{k \geq j} \binom{n}{2k-j} a_{k,k-j}.$$

Remark 2.1. The numbers of this triangle admit clearly thefollowing recurrence : $m(n,j) = 0$ if $j < 0$ or $j > n$, and $m(n,j) = m(n-1,j-1) + m(n-1,j) + m(n-1,j+1)$.

3. Generalized Motzkin numbers and skew tableaux

It is well known that Motzkin numbers count the number of standard tableaux of height at most 3.

Theorem 3.1.

$$m_n = \sum_{\lambda \vdash n} f_3^{\lambda}$$

Now we recall a classical proof of this result, which is a bijection from the paths counted by $m(n,0)$ to the standard tableaux of size n and height 3.

The Robinson-Schensted correspondence ([3]) gives a bijection between the standard tableaux of size n and height at most 3 and the involutions on $[n]$ having a longest decreasing subsequence of length at most 3. Hence, it remains to give a bijection from such involutions to the paths counted by $m(n,0)$. We represent an involution under the from of a product of cycles of length 1 (fixed points) or 2 (cycles), ordered by the increasing order of their greatest element. For example

$$\pi = 4.9.3.1.5.10.7.8.2.6 = (3)(1\ 4)(5)(7)(8)(2\ 9)(6\ 10),$$

which is an involution on [10] having a longest decreasing subsequence of length 3 (10.8.6 for example) corresponds to the following standard tableau

$$T = \begin{array}{|c|c|c|c|} \cline{1-3} 4 & 9 & 10 & \multicolumn{1}{c}{} \\ \cline{1-3} 3 & 5 & 7 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 6 & 8 \\ \hline \end{array} .$$

Now, to compute the path $P(\pi)$ corresponding to an involution π, it suffices to read the representation of π as a word from left to right and to associate to each read element $\pi(i)$ of π

- a S step if $\pi(i)$ is a fixed point,
- a N step if $\pi(i)$ is the lowest element of a two elements cycle,
- a E step if $\pi(i)$ is the greatest element of a two elements cycle.

We denote by Ψ this bijection (the inverse is clear :to the k^{th} N-step, we associate the k^{th} E-step). For example, the involution π displayed previously corresponds to the path

$$P = E.E.S.N.S.E.S.S.N.N.$$

Now we show that, using the same technic, we can interpret the second column of the triangle of the generalized Motzkin numbers in terms of standard skew tableaux.

Corollary 3.1. *If $\alpha = (2)$, then*

$$m(n+1,1) = \sum_{\lambda/\alpha \vdash n} f_3^{\lambda/\alpha}$$

We describe a bijection Φ_1 from the standard skew tableaux of $ST^k(\lambda/\alpha)$ to the paths counted by the column $m(n+1,1)$, which can be decomposed into three steps.

Step 1. Let $T \in \bigcup_{\lambda/\alpha \vdash n} ST^3(\lambda/\alpha)$. There is an immediate bijection from the standard skew tableaux of shape λ/α (with $\alpha = (2)$ of size n to the standard tableaux of size $n+2$, shape λ and such that the bottom left corner is the pattern $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline \end{array}$. We denote by T' the image of T by this bijection. For example

$$T = \begin{array}{|c|c|c|} \cline{1-2} 3 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 1 & 4 & \multicolumn{1}{c}{} \\ \hline & & 2 \\ \hline \end{array} \Longrightarrow T' = \begin{array}{|c|c|c|} \cline{1-2} 5 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & 6 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 4 \\ \hline \end{array} .$$

Step 2. Now we use the bijection Ψ and we associate to T' the corresponding path $\Psi(T')$ which has $n+2$ steps, from $(0,0)$ to (k,k), never crosses the line $x = y$ and such that its first two steps are $S.S$ or $S.E$ or

$E.E$ (because of the $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$ pattern in the bottom left corner). In our example, we have $P' = \Psi(T') = S\ E\ E\ E\ N\ S\ N\ N$.

Step 3. In a third time, we show that the paths like P' (i.e. the paths having $n+2$ steps, from $(0,0)$ to (k,k), which never cross the line $x = y$ and such that their first two steps are $S.S$ or $S.E$ or $E.E$) are in bijection with the paths P having $n+1$ steps, from $(0,0)$ to $(k, k-1)$, which never cross the line $x = y$. This bijection acts in two times:

- first if the first two steps of P' are $S.S$, we replace them by $N.$, if they are $S.E$, we replace them by $S.$ and if they are $E.E$, we replace them by E, and we denote P'' the path so obtained,
- in a second time, we reverse the word corresponding to P'' and we exchange the E steps and the N steps, and we have the path P we want.

In our example, we have $P'' = S.E.E.N.S.N.N$ and $P = E.E.S.E.N.N.S$. It is not difficult to see that this construction is a bijection, which proves the Corollary. Moreover, using the same technic (in a slightly more complicated way), we have the following corollary.

Corollary 3.2. *If* $\alpha = (4,2)$*, then*

$$m(n+2,2) = \sum_{\lambda/\alpha \vdash n} f_3^{\lambda/\alpha}$$

We desribe the main differences between the proof of this corrolary ands the proof of the previous (the principle is exactly the same).

Step 1. The left corner of the tableaux we consider now is the pattern

$$\begin{array}{|c|c|cc} \hline 5 & 6 & \multicolumn{2}{c}{} \\ \hline 1 & 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array} \ .$$

Hence we have, for example

$$T = \begin{array}{|c|c|c|c|c|} \cline{1-2} 3 & 5 & \multicolumn{3}{c}{} \\ \cline{1-4} & & 1 & 4 & \multicolumn{1}{c}{} \\ \hline & & & & 2 \\ \hline \end{array} \Longrightarrow T' = \begin{array}{|c|c|c|c|c|} \cline{1-2} 9 & 11 & \multicolumn{3}{c}{} \\ \cline{1-4} 5 & 6 & 7 & 10 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 & 4 & 8 \\ \hline \end{array} \ .$$

Step 2. Like in the description of the bijection Φ_1, this left corner pattern induces that the paths that are in correspondence with tableaux having this pattern have first six steps that are restricted to 9 *starting patterns* that are :

$$\begin{array}{l} E.E.E.E.N.N \\ E.E.E.E.S.S \\ E.E.E.E.N.S \\ S.E.E.E.N.N \\ S.E.E.E.S.S \\ S.E.E.E.N.S \\ S.S.E.E.N.N \\ S.S.E.E.S.S \\ S.S.E.E.N.S \end{array}$$

In our example, the path corresponding to T' is

$$S.E.E.E.N.N.S.E.N.S.N.$$

Step 3. Finally, we need to associate to each of these starting patterns of length 6 a pattern of length 2.

$$
\begin{aligned}
&E.E.E.E.N.N \longrightarrow S.S\\
&E.E.E.E.S.S \longrightarrow E.E\\
&E.E.E.E.N.S \longrightarrow E.S\\
&S.E.E.E.N.N \longrightarrow S.N\\
&S.E.E.E.S.S \longrightarrow S.E\\
&S.E.E.E.N.S \longrightarrow E.N\\
&S.S.E.E.N.N \longrightarrow N.N\\
&S.S.E.E.S.S \longrightarrow N.E\\
&S.S.E.E.N.S \longrightarrow N.S
\end{aligned}
$$

Then we apply to each path corresponding to a tableau of height at most 3 having the right left corner pattern the following transformation : first we replace its starting pattern by the corresponding pattern of length 2, then we exchange the E steps and the N steps and we reverse the word corresponding to the path. In our example, we have the final path

$$E.S.E.N.S.E.S$$

which is clearly a path from $(0,0)$ to $(k,k-2)$ of length $n+2$ (here $n=5$ and $k=3$).

Then the question which is interesting is the following.

Question 3.1. Is there an interpretation in terms of skew standard tableaux of the other columns of the generalized Motzkin numbers triangle.

References

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