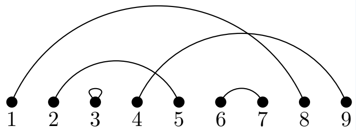
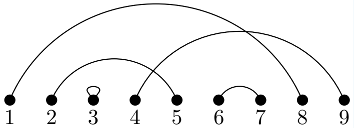
Combinatorics of Arc Diagrams, Ferrers Fillings, Young Tableaux and Lattice Paths

Jacob Post

July 22, 2009

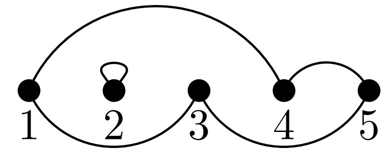


Overview



Arc Diagrams

- Permutations ($\sigma \in S_n$)



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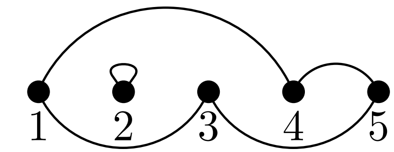
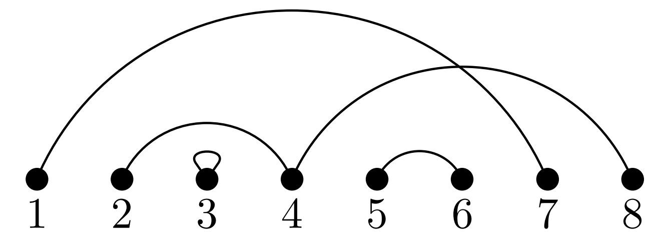
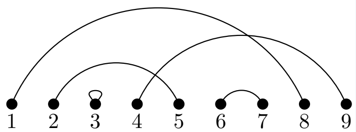


- Set Partitions ($\nu \in P_n$)





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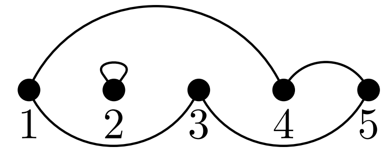

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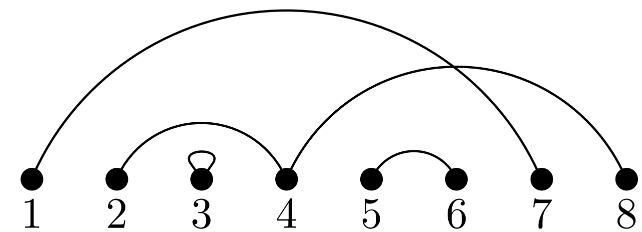
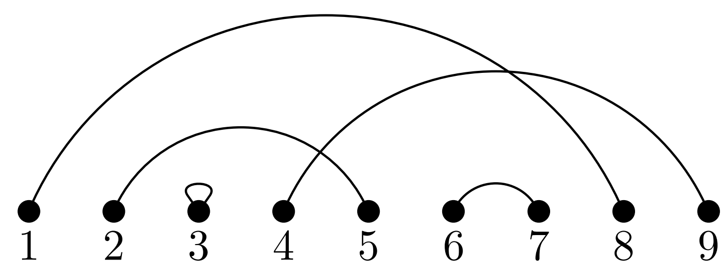


- Involutions ($\pi \in I_n$)



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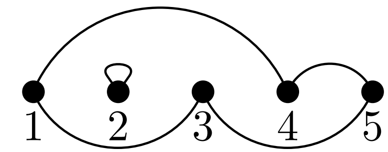

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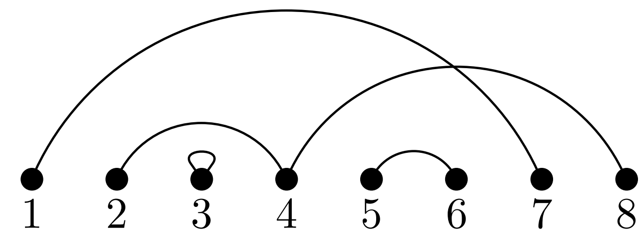

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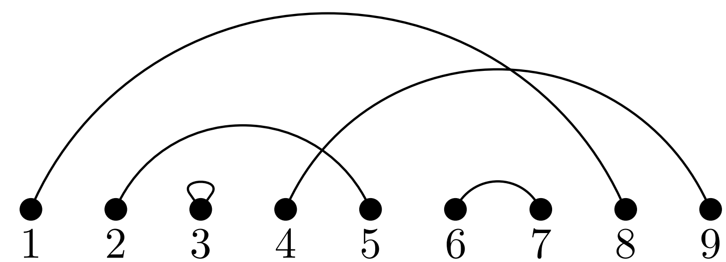
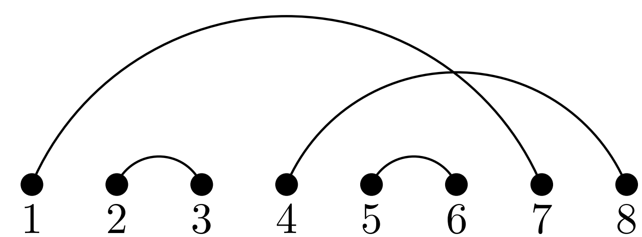
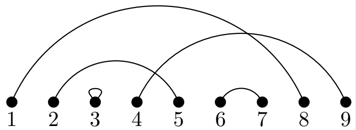


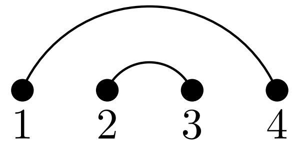
- Matchings ($\mu \in M_n$)





Nestings and Crossings

- Nesting



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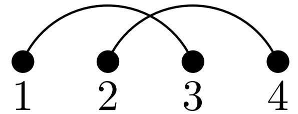

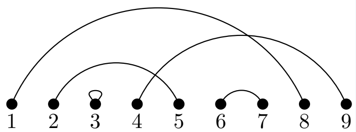

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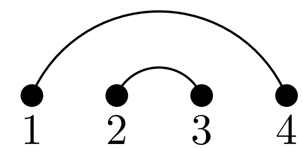

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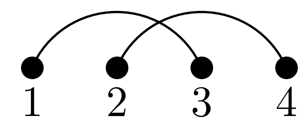
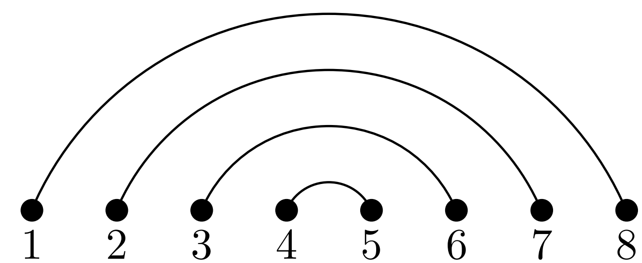


- k-nesting



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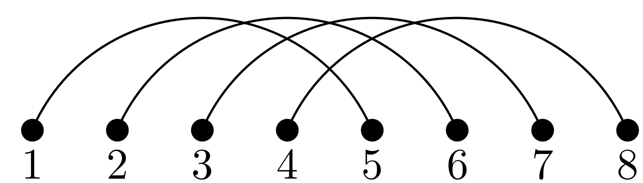
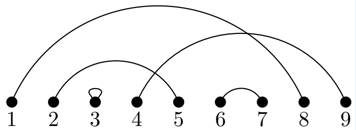
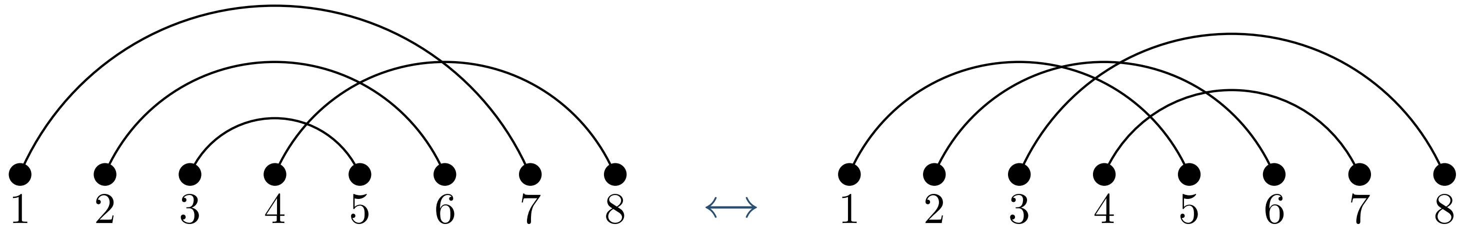
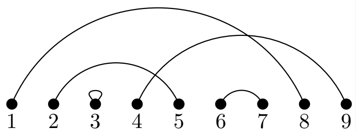




Equidistribution of Nestings and Crossings

- **Theorem (Chen et.al., 2006)**: Fixing objects of size n, maximal i-crossing and maximal j-nesting $\leftrightarrow$ maximal j-crossing and maximal i-nesting.





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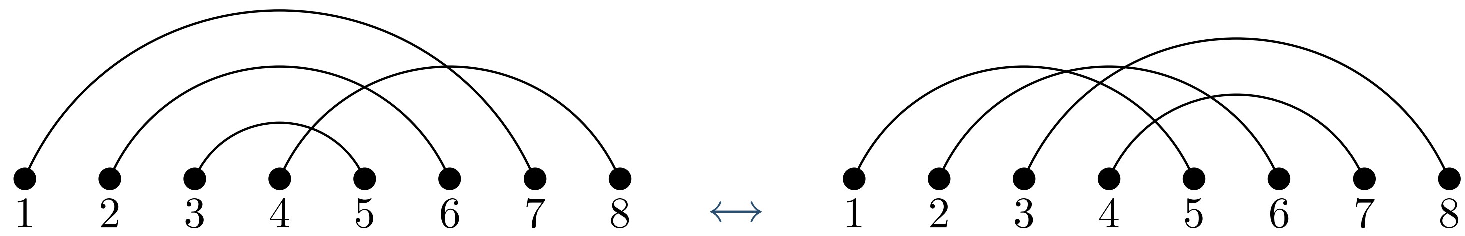
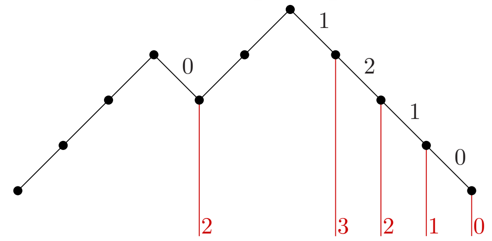
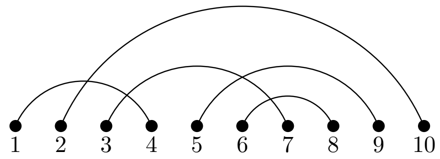
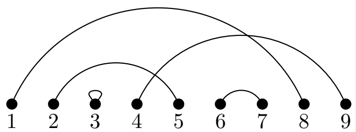


- **Theorem (Kasraoui and Zeng, 2006)**: Fixing objects of size n, i crossings and j nestings $\leftrightarrow$ j crossings and i nestings.

- **Theorem**: Weighted Dyck paths of length $2n$ are in bijection with matchings on $[2n]$.

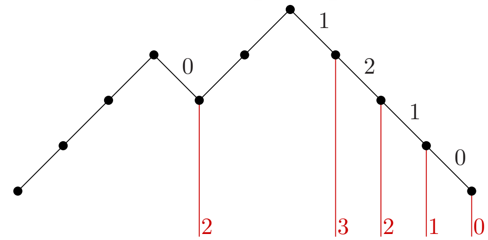


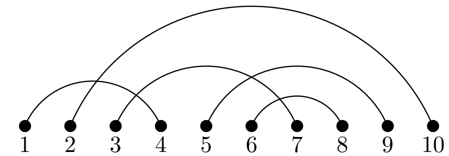




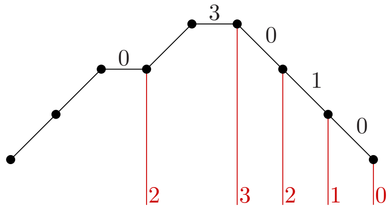
Weighted Dyck/Motzkin Paths

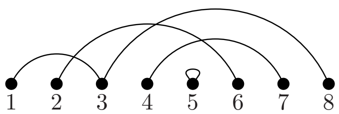
- **Theorem**: Weighted Dyck paths of length $2n$ are in bijection with matchings on $[2n]$.

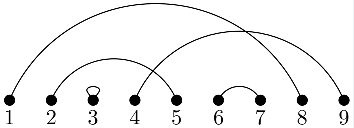




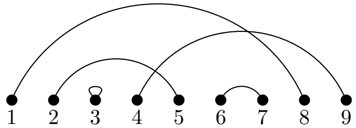
- **Theorem**: Weighted Motzkin paths of length n are in bijection with set partitions on n.







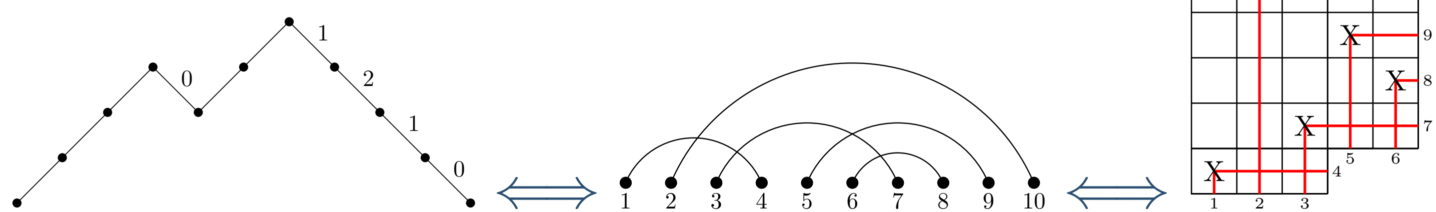
- Generalization of permutation matrices: fill each row & column of a Ferrers shape with exactly 1 'x'.

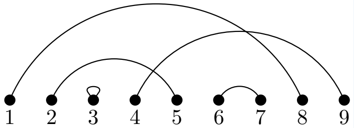


Strict 0-1 Ferrers Fillings

- Generalization of permutation matrices: fill each row & column of a Ferrers shape with exactly 1 ‘x’.

- **Theorem (Krattenthaler, 2006)**: Strict 0-1 Ferrers filling with n ‘x’s are in bijection with matchings on $[2n]$.





Standard Young Tableaux

- Filling of a Ferrers shape with $[n]$ increasing downwards and rightwards.

1	2	4
3	6	7
5	10	11
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- **Theorem (Robinson-Schensted, 1934, 1961)**: Pairs of size n SYT with the same shape are in bijection with $\sigma \in S_n$.

P, Q =

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5		

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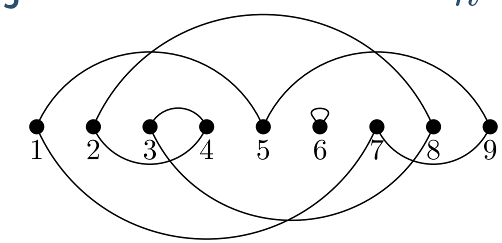
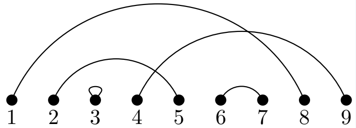
7		





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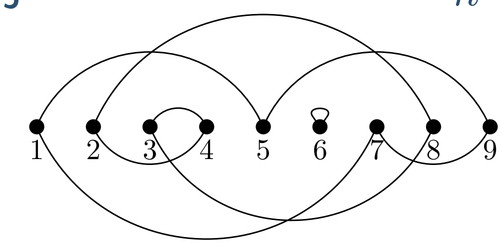
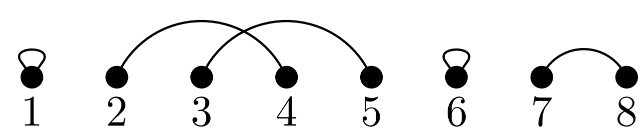
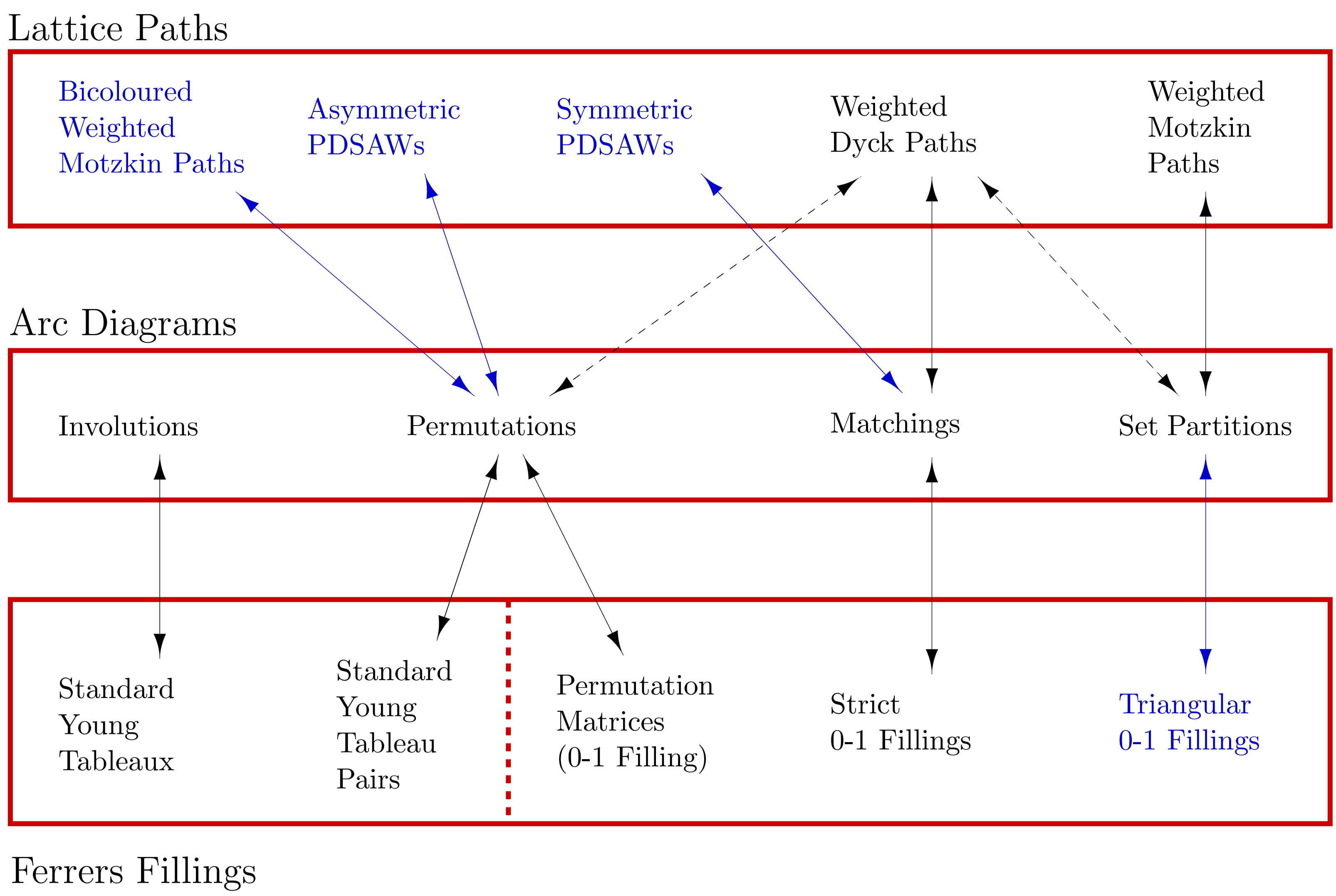


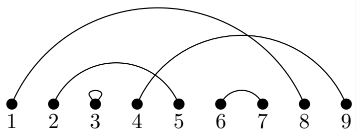
- The case of involutions: $\sigma \sim P, Q \iff \sigma^{-1} \sim Q, P$. Therefore, SYT of size n are in bijection with involutions on $[n]$.

1	2	3	6	7
4	5	8		



Object Summary

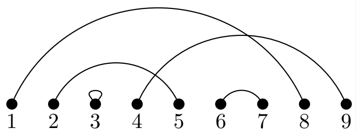




Knuth Transformation Examples

- Knuth transformation:

$$15276483 \sim \begin{array}{|c|c|c|c|}\hline 1&2&3&8\\\hline 4&6\\\cline{1-2} 5\\\cline{1-1} 7\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&2&4&7\\\hline 3&5\\\cline{1-2} 6\\\cline{1-1} 8\\\cline{1-1}\end{array} \quad \longleftrightarrow \quad 15276843 \sim \begin{array}{|c|c|c|c|}\hline 1&2&3&8\\\hline 4&6\\\cline{1-2} 5\\\cline{1-1} 7\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\\hline 3&5\\\cline{1-2} 7\\\cline{1-1} 8\\\cline{1-1}\end{array}$$



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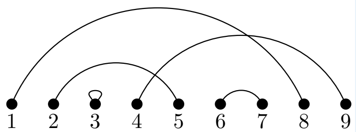
- Dual Knuth transformation:

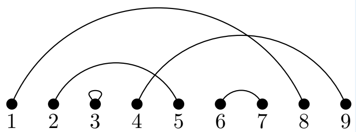
$$15276843 \sim \begin{array}{|c|c|c|c|}\hline 1&2&3&8\\\hline 4&6\\\cline{1-2} 5\\\cline{1-1} 7\\\cline{1-1}\end{array}\ \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\\hline 3&5\\\cline{1-2} 7\\\cline{1-1} 8\\\cline{1-1}\end{array} \leftrightarrow 14276853 \sim \begin{array}{|c|c|c|c|}\hline 1&2&3&8\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1} 7\\\cline{1-1}\end{array}\ \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\\hline 3&5\\\cline{1-2} 7\\\cline{1-1} 8\\\cline{1-1}\end{array}$$



Knuth Transformations

- Knuth transformations:
$$\begin{array}{ccc} \ldots\ b\ a\ c\ \ldots & \leftrightarrow & \ldots\ b\ c\ a\ \ldots \\ \ldots\ a\ c\ b\ \ldots & \leftrightarrow & \ldots\ c\ a\ b\ \ldots \end{array}$$
- Alters contents of tableau Q, but not the shape.

- Dual Knuth transformations:
$$\begin{array}{ccc} \ldots\ i\ \ldots\ i{-}1\ \ldots\ i{+}1\ \ldots & \leftrightarrow & \ldots\ i{+}1\ \ldots\ i{-}1\ \ldots\ i\ \ldots \\ \ldots\ i{-}1\ \ldots\ i{+}1\ \ldots\ i\ \ldots & \leftrightarrow & \ldots\ i\ \ldots\ i{+}1\ \ldots\ i{-}1\ \ldots \end{array}$$
- Alters contents of tableau P, but not the shape.



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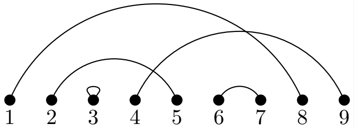
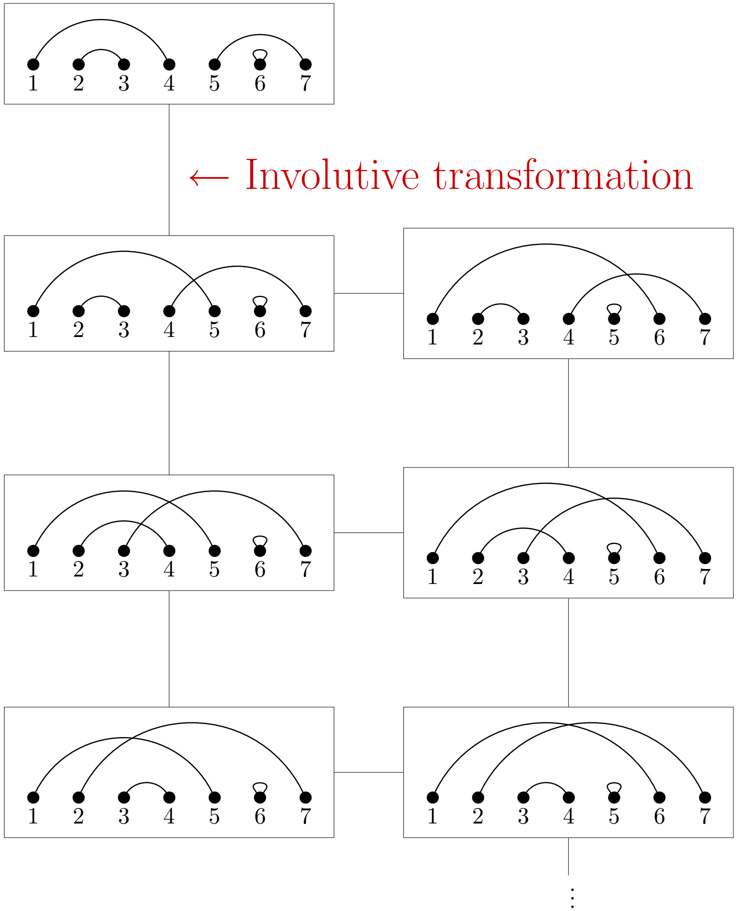
- Alters contents of tableau P, but not the shape.

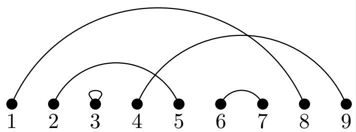
- **Theorem (Knuth)**: Any pair of permutations with the same tableau shape can be transformed into one another through a sequence of transformations.



Involutive transformations:

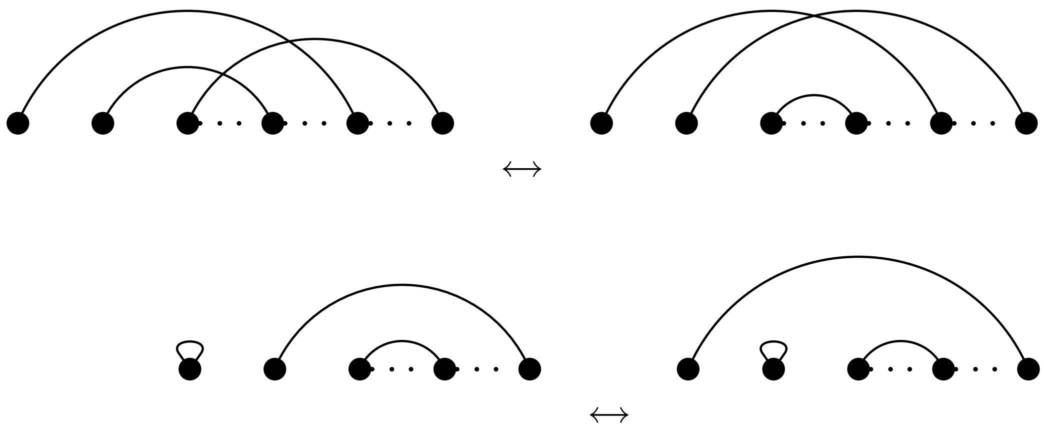
- Local changes between at most 3 arcs.
- Connects involutions with same shape.
- Can be used for induction.



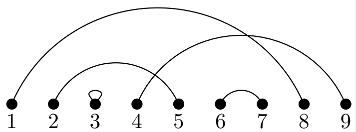


Involutive Transformations

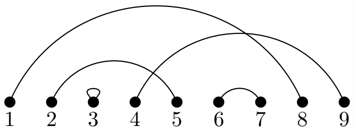
- Involutive transformations: Applying both a Knuth transformation and dual Knuth transformation.
- **New**: An enumeration of all involutive transformations in terms of arc diagrams:



Etc.



- Maximal decreasing structure: maximal length of i disjoint decreasing subsequences (i varies).
- Example: $\sigma = 5416327$
  - $d_1 = 4$ $\quad 5416327$



Greene's Result

- Maximal decreasing structure: maximal length of i disjoint decreasing subsequences (i varies).
- Example: $\sigma = 5416327$
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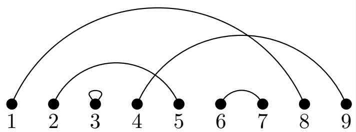
  - $d_2 = 6$ $\quad 5416327$



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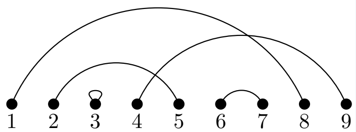
  - $d_2 = 6$ — 5416327
  - $d_3 = 7$ — 5416327



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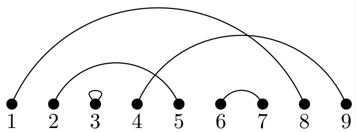
  - $d_3 = 7$ — 5416327
  - $\mathrm{mds}(\sigma) = 4, 2, 1$



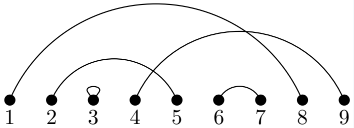
Greene's Result

- Maximal decreasing structure: maximal length of i disjoint decreasing subsequences (i varies).
- Example: $\sigma = 5416327$
  - $d_1 = 4$ — 5416327
  - $d_2 = 6$ — 5416327
  - $d_3 = 7$ — 5416327
  - $\mathrm{mds}(\sigma) = 4, 2, 1$
- **Theorem (Greene, 1974)**: Maximal decreasing structure of a permutation corresponds to its shape.
- σ's associated Young tableaux have column heights $4, 2, 1 = \mathrm{mds}(\sigma)$:

$\sigma \sim$

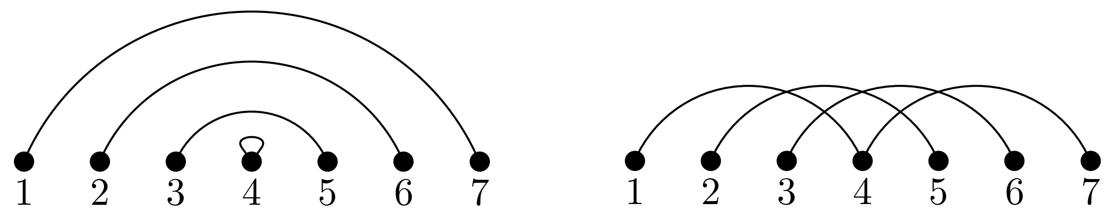
1	2	7
3	6	
4		
5		

1	4	7
2	5	
3		
6		



$\frac{1}{2}$-Nestings

- Two conventional interpretations:
  - Strong: Singletons do not contribute to k-nestings.
  - Weak: Singletons contribute fully to k-nestings.
  - Similar for transitory vertices and crossings.



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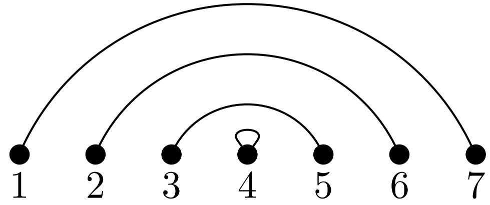
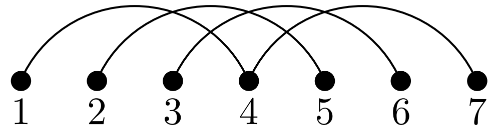
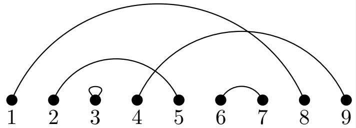




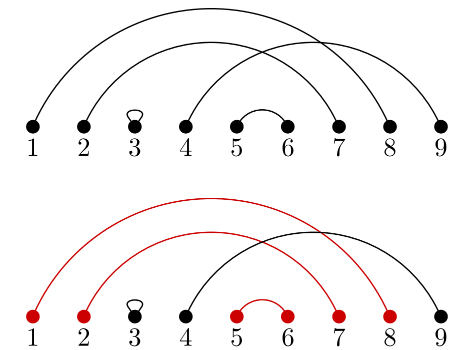
- Alternative $\frac{1}{2}$-nesting interpretation: Singletons contribute $\frac{1}{2}$ to k-nestings.



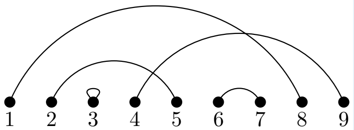
Maximal Nesting Structures

- Maximal nesting structure is analogous to maximal decreasing structure, using k-nestings ($\frac{1}{2}$-nesting interpretation).

- Example:



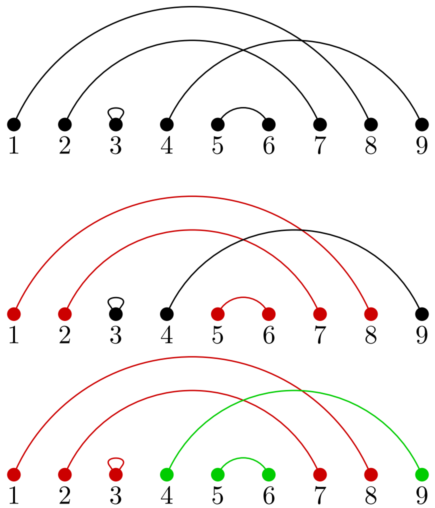
  - $m_1 = 3$

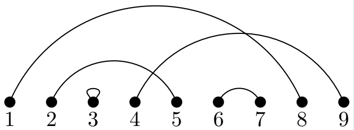


- Maximal nesting structure is analogous to maximal decreasing structure, using k-nestings ($\frac{1}{2}$-nesting interpretation).

- Example:

    - $m_1 = 3$
    - $m_2 = 4.5$





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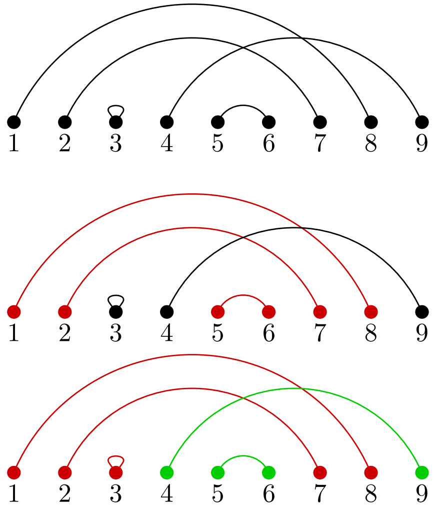
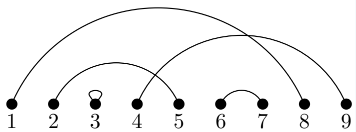


  - $m_1 = 3$
  - $m_2 = 4.5$
  - $\text{mns}(\pi) = 3, 1.5$
  - $2\text{mns}(\pi) = 6, 3$



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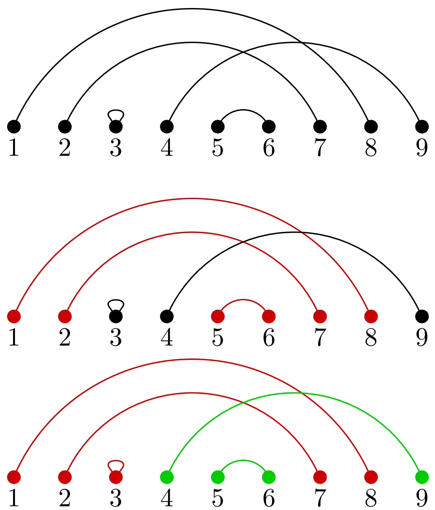
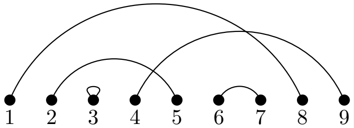
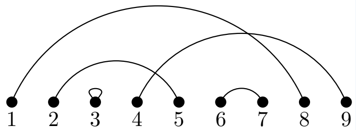


  - $m_1 = 3$
  - $m_2 = 4.5$
  - $\mathrm{mns}(\pi) = 3, 1.5$
  - $2\mathrm{mns}(\pi) = 6, 3$
- **Theorem (New)**: For involutions π, $2\mathrm{mns}(\pi) = \mathrm{mds}(\pi)$. I.e. the MNS corresponds to the associated tableau's shape.
- The associated Young tableau has column heights $6, 3 = 2\mathrm{mns}(\pi)$.



Odd Column Property

- **Theorem (Schensted, 1961)**: An involution with m singletons has a tableau shape with m odd columns.

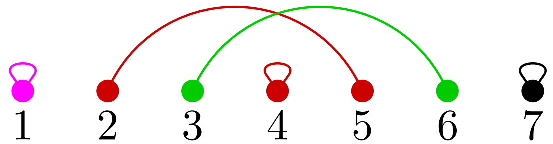


Odd Column Property

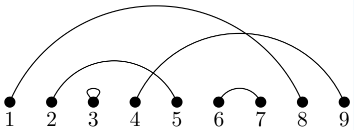
- **Theorem (Schensted, 1961)**: An involution with m singletons has a tableau shape with m odd columns.

- **Theorem (New)**: If the first i columns of the tableau have c odd columns, then any maximal set of i k-nestings will include c singletons.

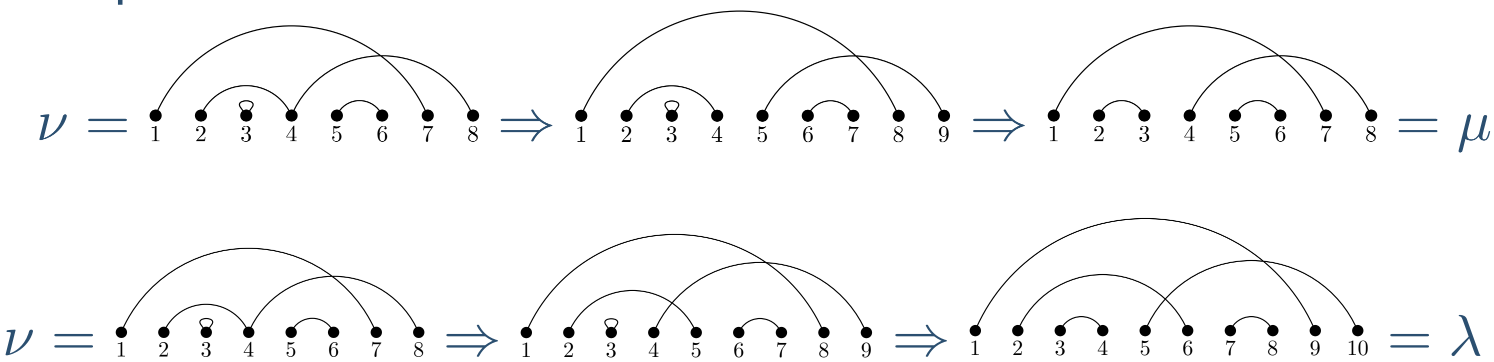
- Example:



1	2	3	7
4	6		
5			

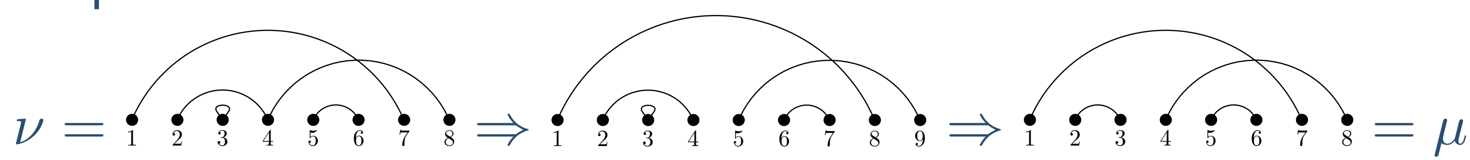


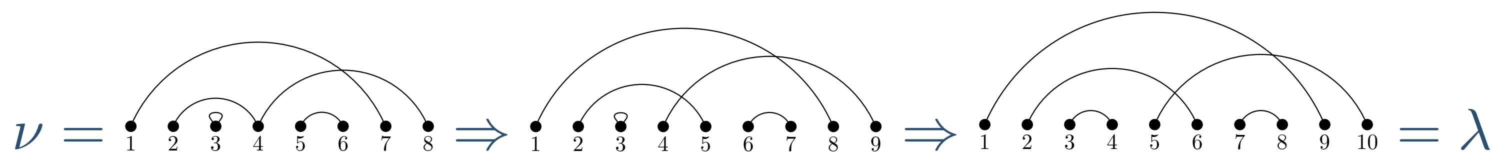
- Example for set partitions, using strong and weak interpretations:



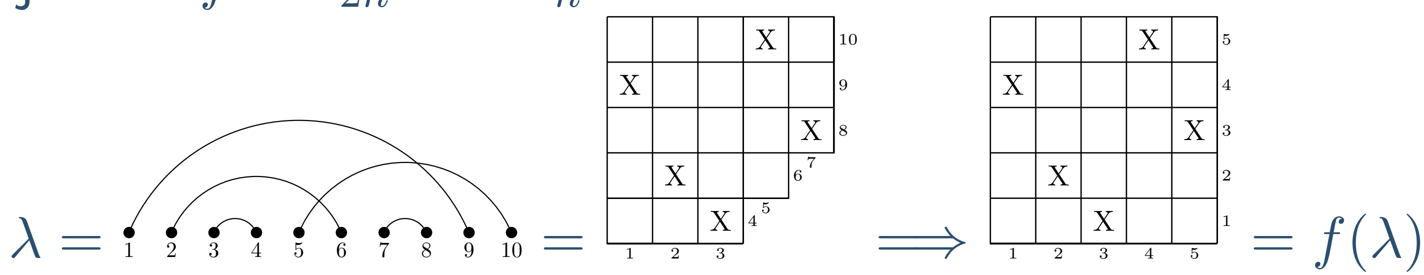
Extending to Set Partitions

- Example for set partitions, using strong and weak interpretations:





- Surjection $f : M_{2n} \Longrightarrow S_n$



Arc Diagrams, Nesting and Crossing
Other Objects
Shape Preserving Transformations
Maximal Nesting Structures
Greene's Result
$\frac{1}{2}$-Nestings
Maximal Nesting Structures
Odd Column Property
Extending to Set Partitions
Nesting Summary
Set Partition Bicolouring Bijection
Bicolouring Bijection Principle
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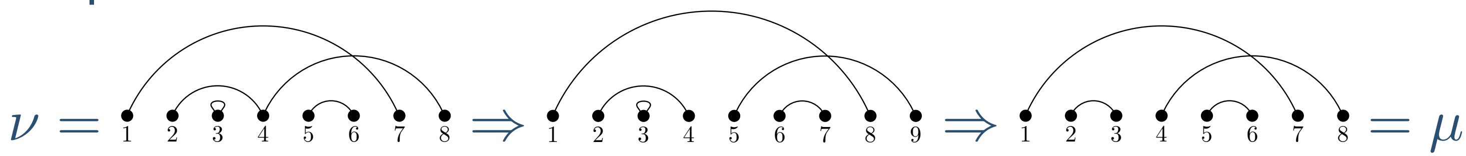

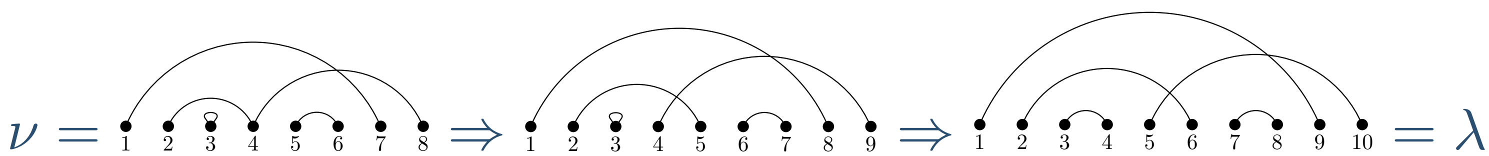

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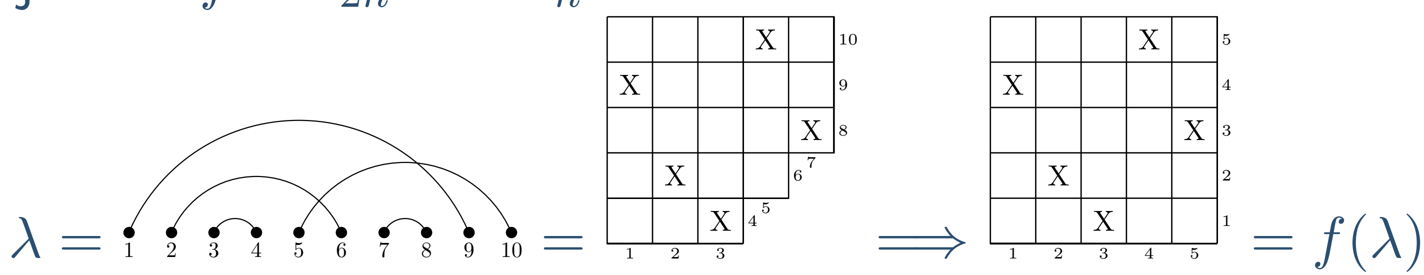
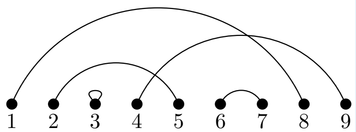
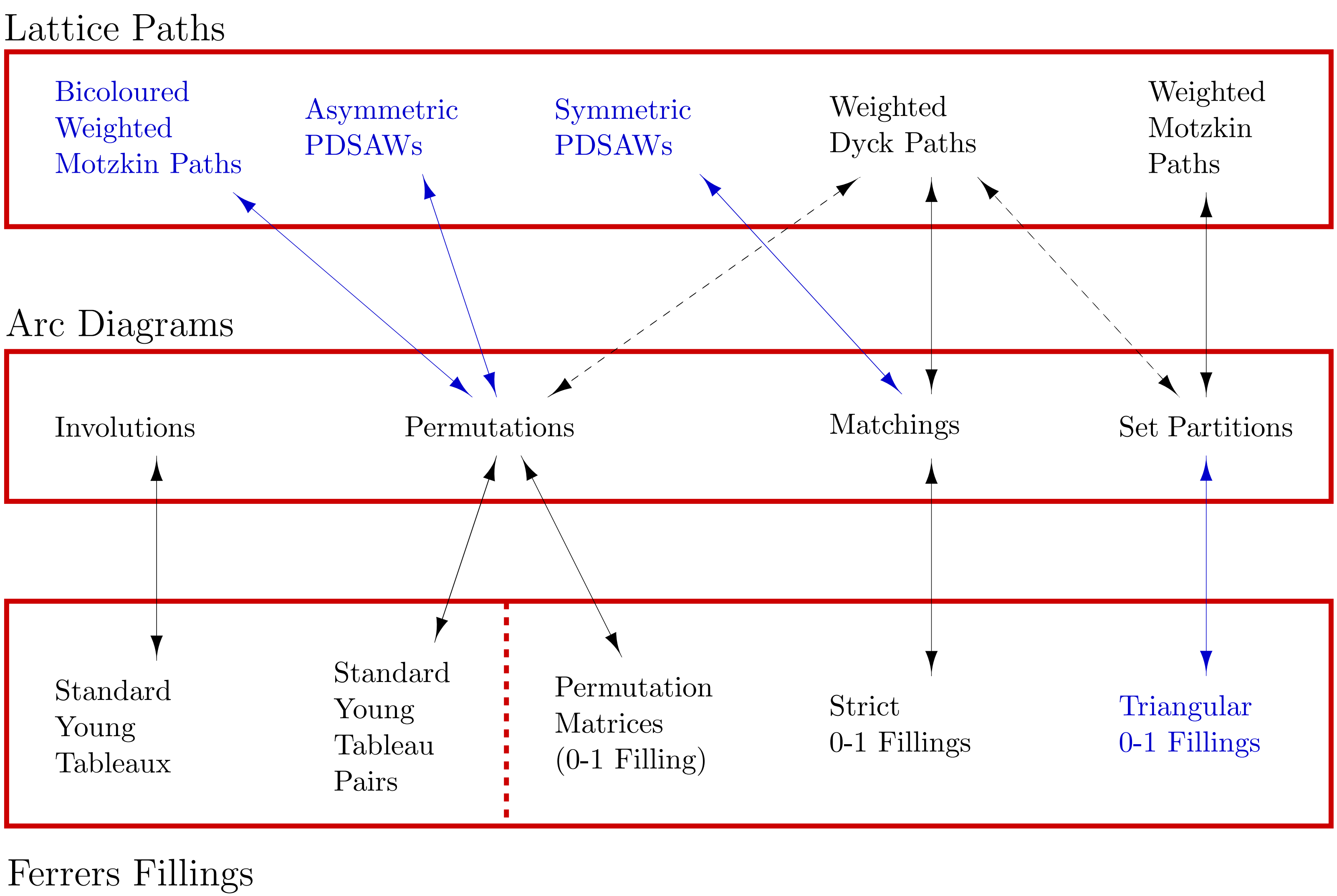


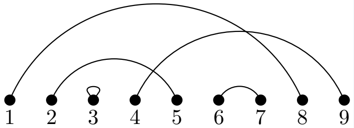
- **Theorem (New, Chen et.al., 2006)**:
  - $\lfloor \mathrm{mns}(\nu) \rfloor = \mathrm{mns}(\mu) = \mathrm{mds}(f(\mu))$ (Strong)
  - $\lceil \mathrm{mns}(\nu) \rceil = \mathrm{mns}(\lambda) = \mathrm{mds}(f(\lambda))$ (Weak)



Nesting Summary

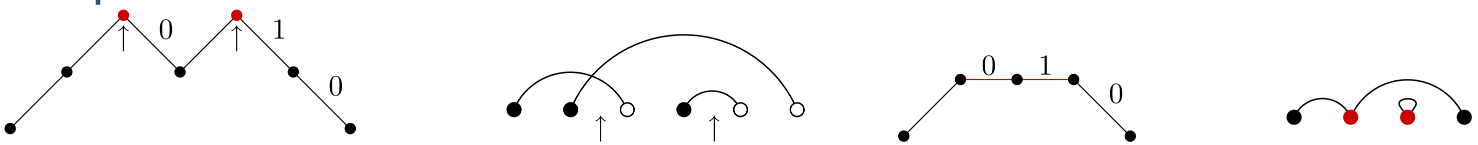
- Direct Greene-like result on the MNS of involutions and their shape.
- Clarifies the connection between the MNS of set partitions and permutation shapes.
- Involutive transformations give a tool for manipulating arc diagrams.
- Clarifies Reifegerste's work on Knuth transformations in terms of tableaux.


Lattice Paths
Bicoloured Weighted Motzkin Paths
Asymmetric PDSAWs
Symmetric PDSAWs
Weighted Dyck Paths
Weighted Motzkin Paths
Arc Diagrams
Involutions
Permutations
Matchings
Set Partitions
Standard Young Tableaux
Standard Young Tableau Pairs
Permutation Matrices (0-1 Filling)
Strict 0-1 Fillings
Triangular 0-1 Fillings
Ferrers Fillings



Weighted Dyck/Motzkin Path Bijections

- **Theorem**: Weighted Dyck paths with bicoloured peaks $\leftrightarrow$ set partitions.



- Preserves number of arcs and weak nesting/crossing.

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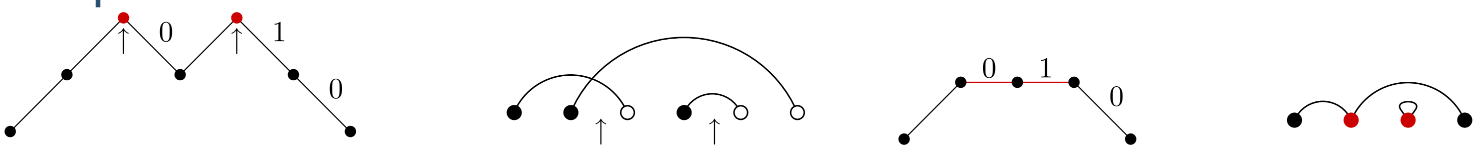
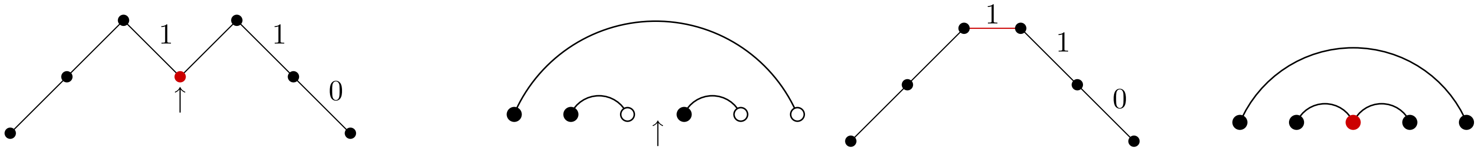
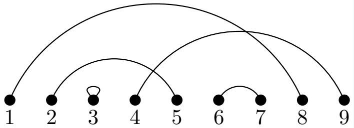


- Preserves number of arcs and weak nesting/crossing.

- **Theorem**: Weighted Dyck paths with bicoloured valleys ↔ singleton free set partitions.



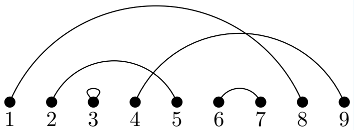
- Preserves number of arcs and strong nesting/crossing.



Weighted Dyck Path and Set Partition Statistics

- $\left\{\!\!\left\{ n \atop k \right\}\!\!\right\}$: # of singleton free set partitions on $[n]$ with k partitions.
- $\left\{ n \atop k \right\}$: # of set partitions on $[n]$ with k partitions (singletons allowed).

- $\left\langle\!\!\left\langle n \atop k \right\rangle\!\!\right\rangle$: Second-order Eulerian numbers.



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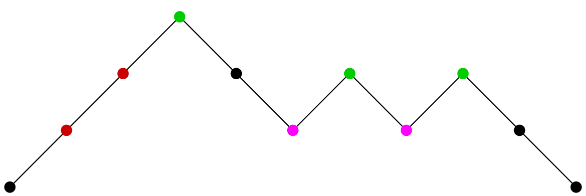
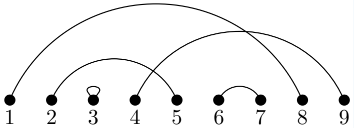
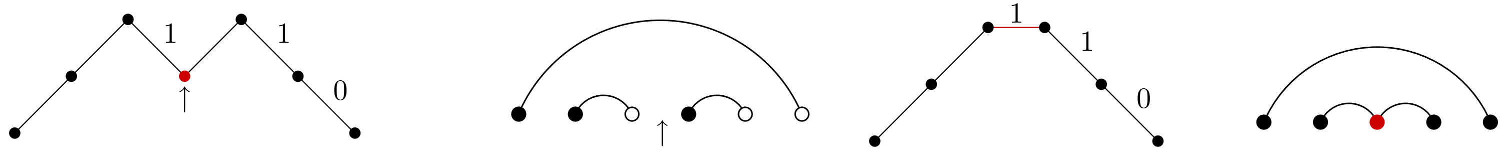
- $\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle$: Second-order Eulerian numbers.
- **Theorem (New?)**: Among all weighted Dyck paths of semilength n there are
  - $\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle$ with k strong rises
  - $\left\langle\!\!\left\langle {n \atop n-k} \right\rangle\!\!\right\rangle$ with k peaks
  - $\left\langle\!\!\left\langle {n \atop n-k-1} \right\rangle\!\!\right\rangle$ with k valleys





Identities of Stirling Numbers of the Second Kind

- A closer look at the bicoloured valley bijection:



Each red valley reduces # of partitions and vertices by 1.

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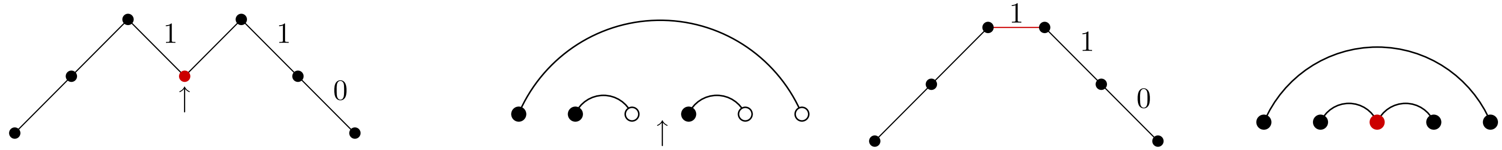
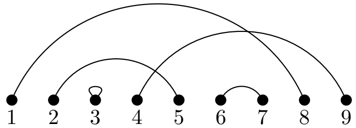


Each red valley reduces # of partitions and vertices by 1.

- Therefore, a singleton free set partition on $n+k$ with k partitions must be in bijection with a WDP of length $2n$ and $n-k$ red valleys.



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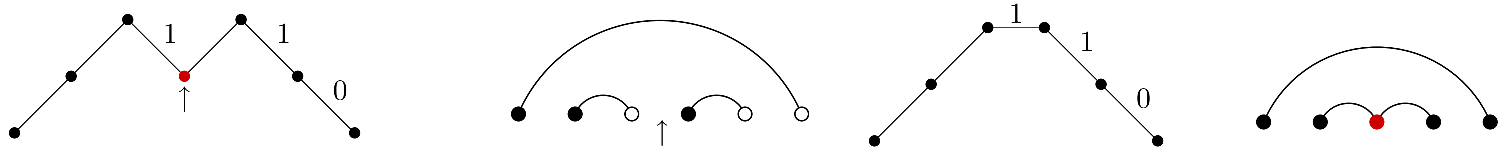

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- **Theorem (Smiley, 2001)**: $\left\{\!\!\left\{ {n+k \atop k} \right\}\!\!\right\} = \sum_j \binom{j}{n-k} \left\langle\!\!\left\langle {n \atop n-j-1} \right\rangle\!\!\right\rangle$

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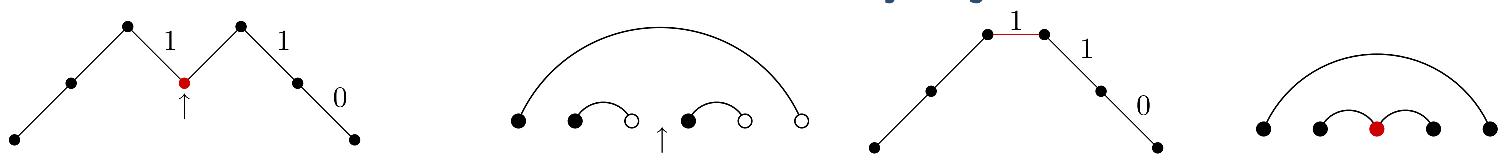

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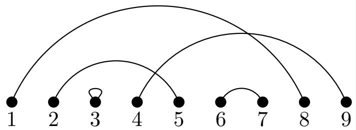
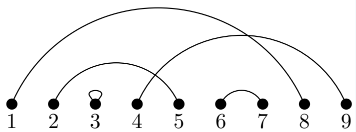
- **Theorem (Carlitz, 1965)**: $\left\{ {n \atop n-k} \right\} = \sum_j \binom{n+j}{2k} \left\langle\!\!\left\langle {k \atop k-j-1} \right\rangle\!\!\right\rangle$



Generalization to Sets of Structures

- Required “components”:
  - A *structure*: sets, sequences, cycles, . . .
  - Can create *matching like objects*.
  - A bicoloured *feature* bijection between matching like objects and singleton free sets of structures.
- Example:
  - Sets (a structure)
  - Weighted Dyck paths (matchings)
  - Valleys (features)



Generalized Results

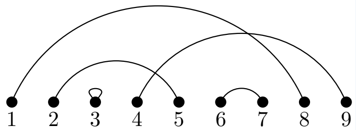
- Notations:
  - $V^0(n,k)$: # of singleton free sets of k structures on $[n]$.
  - $V(n,k)$: # of sets of k structures on $[n]$ (singletons allowed).
  - $B(n,k)$: # of matching-like objects on $[2n]$ with k features.

- **Theorem (New)**:

$$V^0(n+k,k) = \sum_j \binom{j}{n-k} B(n,j)$$

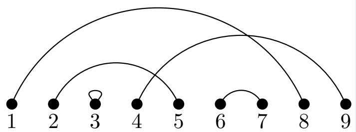
$$V(n,n-k) = \sum_j \binom{n+j}{2k} B(k,j)$$

$$B(n,k) = \sum_i (-1)^{n-k+i} \binom{n-i}{k} V^0(n+i,i)$$



Stirling Numbers of the First Kind

- Permutations are sets of cycles:
  - The structures are cycles.
  - Again we use weighted Dyck paths.
  - The features are strong rises.



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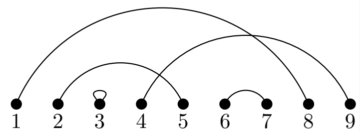
- **Theorem**: Weighted Dyck paths with bicoloured strong rises $\leftrightarrow$ derangements.

$$[1,3] \times [2,7][4,10][5,9] \times [6,8]$$
$$[1,2,6][3,9][4,8] \times [5,7]$$
$$[1,2,5][3,8][4,7,6]$$



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- **Theorem**:

$$\left[\!\left[{n+k \atop k}\right]\!\right] = \sum_j \binom{j}{n-k} \left\langle\!\!\left\langle {n \atop j} \right\rangle\!\!\right\rangle$$

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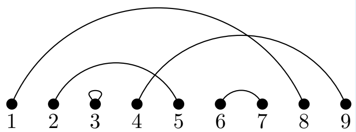


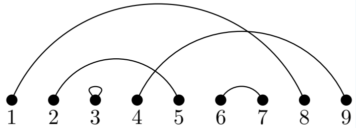
Bicolouring Bijection Principle Summary

- Handles many statistics.
- Combinatorial interpretations of identities.
- Aids discovery of new bijections.

Other Results

- **Theorem (New)**: Semilabled structured trees and semilabeled structured series-reduced forests are in bijection with sets of structures, transporting many statistics.

- Extends results of Diaconis and Holmes, Erdös and Székely.



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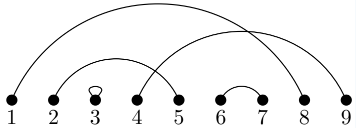
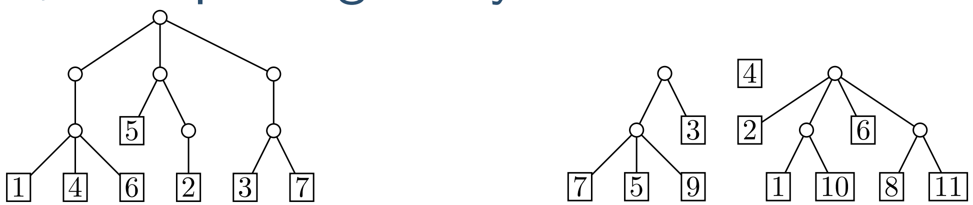
- Extends results of Diaconis and Holmes, Erdös and Székely.
- 4 new variations of the RSK algorithm (complementing 4 known variations).



Other Results

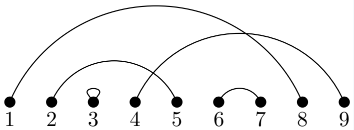
- **Theorem (New)**: Semilabled structured trees and semilabeled structured series-reduced forests are in bijection with sets of structures, transporting many statistics.



- Extends results of Diaconis and Holmes, Erdös and Székely.

- 4 new variations of the RSK algorithm (complementing 4 known variations).

- Exploration of Knuth graphs: lattices of standard Young tableaux of a fixed shape where edges are determined by possible involutive transformations.
- Builds on the work of Reifegerste.



Thank You!