Math 320 (Winter 2006)
Introduction to Real Analysis II
Assignment 2 Solutions

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Q1. We have solved in Chapter 12, Exercise 17 from Rudin. That is, we have created a perfect set E whose elements are of the form $0.x_1x_2x_3x_4\cdots$, where $x_i = 4$ or 7 (this should also have been done in lecture). Note that E can also contain rational numbers like $x = 0.\overline{7}$.

Also, note that $E^* = \{\ x + k \mid x \in E\}$ for any $k \in \mathbb{R}$ is also perfect. So we select $k \in \mathbb{Q}^c$ such that $x + k \in \mathbb{Q}^c$ for all $x \in E$.

Let $k = 0.101001000100001\ldots$, where the number of zeros between each successive pair of 1's increase by one.

Let $x \in E$ and suppose $k \in \mathbb{Q}^c$ (the case $x \in \mathbb{Q}$ is trivial). Since x has an infinite number of 4's and 7's, $x + k$ will have an infinite number of 5's or 8's, or both.

Without lost of generality, suppose $x + c$ has an infinite number of 5's. If $x + k \in \mathbb{Q}$, then $x + k = 0.d_1d_2\cdots d_k\overline{r_1r_2\cdots r_n}$, where d_i are leading digits and r_i are repeated digits of whatever length. Since $x + k$ has an infinite number of 5's, one of the r_i's is a 5. This also requires 5 to occur every n digits.

However, after some point, there will be more than n 0's between each successive 1's, a contradiction.

Therefore, $x + k \in \mathbb{Q}^c$, and so E^* has no rationals.

Q2. (a) If A and B are closed, then $\overline{A} = A$ and $\overline{B} = B$. So

$$\overline{A} \cap \overline{B} = \overline{A} \cap B = A \cap \overline{B} = \emptyset.$$

(b) Since $\overline{A} \subset B^c$ because B^c is closed, then $\overline{A} \cap B = \emptyset$. By symmetry, we also have $A \cap \overline{B} = \emptyset$.

(c) Note that A is open. It suffices to show that B is also open.

For any $q \in B$, if $r := d(p, q) - \delta$, then $N_r(q) \subset B$.

(d) Fix $p, q \in X$, where X is a connected metric space.

Assume X is countable. Note that the open interval $(0, d(p, q))$ is uncountable. Then there is some $\delta \in (0, d(p, q))$ such that there is no s with $d(s, p) = \delta$.

However, X would then be a disjoint union of two sets $\{\ x \in X \mid d(x,p) < \delta\ \}$ and $\{\ x \in X \mid d(x,p) > \delta\ \}$, contradicting the connectedness of X.

Q3. Let $K = \{1/n\} \cup \{0\}$, and let $\{V_\alpha\}$ be an open cover.

Since 0 is a limit point, for any $\epsilon > 0$, there is an n such that $\epsilon > 1/(n+1)$. Then we have $d(0, 1/(n+1)) < \epsilon$, and so $\{V_{\alpha_i}\}_{i=1}^{n} \cup \{N_{1/(n+1)}(0)\}$ is a finite subcover of K.

Q4. Consider

$$A = \left\{ \frac{1}{n} + \frac{1}{m} \mid m, n \in \mathbb{Z}^+ \right\} \cup \{0\}.$$

Note that A is compact by Heine-Borel Theorem.

Then as $m \to \infty$, the set of limit points is $A' = \{1/n\} \cup \{0\}$, in which case A' is countable.

Q5. Consider $\{(1/n, 1)\}$ for $n \in \mathbb{Z}^+$, then there is an open covering which cannot be reduced to a finite subcover.

Q6. Consider the set

$$A = \{p \in \mathbb{R} \mid 2 < p^2 < 3\} \cap \mathbb{Q} = ((-\sqrt{3}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{3})) \cap \mathbb{Q}.$$

Then A is open in $\mathbb{R}$ and in $\mathbb{Q}$.

Also, $A = (\sqrt{2}, \sqrt{3}) \cap \mathbb{Q} = [\sqrt{2}, \sqrt{3}] \cap \mathbb{Q}$ is closed in $\mathbb{Q}$. A is also bounded.

However, A is not compact because we can find a sequence of rational numbers in $(\sqrt{2}, \sqrt{3})$ converging to $\sqrt{2}$, but $\sqrt{2} \notin E$.

Q7. Let U be open and let $V = U \cap \mathbb{Q}$ be the set of all rationals in U. V is countable, so we can list all the elements (ie. $a_1, a_2, \dots$).

If $U = \mathbb{R}$, then we are done. Otherwise, for each a_i, there exists a largest unique neighbourhood $B_{r_i}(a_i)$ centering at a_i, such that $B_{r_i}(a_i) \subset U$.

We wish to show that $U = \bigcup_i B_{r_i}(a_i)$. Given any $x =\in U$, by Q6 in Homework 1, there are two cases :: (1) $x = a_i$ for some i, or (2) x is a limit point of the rationals.

In case (1), we simply have $x \in B(a_i)$.

In case (2), let $B_r(a_i) \subset U$ be a neighbourhood of x contained in U. Also, let $a_i \in B_{r/4}(x)$.

By the triangle inequality, we have $B_{r/2}(a_i) \subset B_r(x) \subset U$, and so $B_{r/2}(a_i) \subset B_{r_i}(a_i)$ by the maximality property. On the other hand, $x \in B_{r/2}(a_i)$, and so $x \in B_{r_i}(a_i)$. Therefore, $U = \bigcup_i B_{r_i}(a_i)$.

Note that $\{B_i\}$ may not all be disjoint. In which case, we take the union of the non-disjoint intervals (union of two overlapping open intervals will still be open).

Q8. For any given $\delta = 1/i$, $i \in \mathbb{Z}^+$, denote $x_{i,j}$ to be the j-th element picked for that particular δ.

Then for any given i, consider the set $A = \{\ x_{i,j} \mid d(x_{i,j}, x_{i,k} \geq \delta,\ j \neq k\ \}$, where all the elements are at least δ apart.

Then A can only be finitely many of such elements, since for each i, there are only N_j elements. Otherwise, there would be infinitely many elements with radius of at least δ, and so A would contain no limite points.

Consider the set $E = \{\ x_{i,j} \mid i \in \mathbb{Z}^+,\ 1 \leq j \leq N_i\ \}$. This is countable by Theorem 2.13. It suffice to show E is also dense.

For any $\epsilon > 0$ and $x \in X$, take i such that $1/i < \epsilon$. We need to show that there is an $y \in E$ such that $d(x, y) < \epsilon$.

Then there exists $k \leq N_i$ with $d(x_{i,k}, x) < 1/i$, or else we could continue this process by choosing $x_{i,N_{i+1}} = x$. Hence, $y = x_{i,k}$.